Lecture III: classification of small examples.

Before we start:

Q Why do we need to limit ourselves to 'small' cases?

* the $R^G \subset R$ examples show the problem contains the problem of classifying finite groups
* the planar algebra isn't always a complete invariant of the subfactor

Q What should 'small' mean?

There are a number of obvious options:

* 'rank' or number of simple bimodules
* 'global dimension', $\displaystyle\sum_{\substack{N-N \\ \text{bimodules}}} (\dim V)^2$

  (for $R^G \subset R$, this is $|G|$.)
* 'index'

Today we'll focus solely on index, essentially for historical reasons.

(Even at relatively small index things get hard fast: any finite quotient of $\mathbb{Z}/2 * \mathbb{Z}/3$ gives an index 6 subfactor $R^{\mathbb{Z}/2} \subset R \rtimes \mathbb{Z}/3$.)

Theorem (Ocneanu, et al.)

All subfactors with index <4 have principal graph an ADE diagram. Moreover, there are the following numbers of realizations of each:

A_n	D_{2n}	D_{2n+1}	E_6	E_7	E_8
1	0	1	2	0	2

Theorem (Popa, et al.)

All subfactors with index 4 have principal graphs which are simply-laced affine Dynkin diagrams.

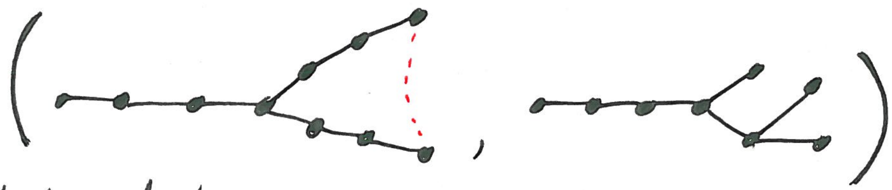
$A_n^{(1)}$	$D_n^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$	A_∞	D_∞
n	$n-2$	1	1	1	1	1

In fact, all of these are realized as Vec_G^ω for G a finite subgroup of $SU(2)$, and $\omega \in H^3(G, \mathbb{C}^*)$.

<u>Theorem</u> (arXiv:1007.1730 Morrison–Snyder
arXiv:1007.2240 Morrison–Penneys–Peters–Snyder
arXiv:1109.3190 Izumi–Jones–Morrison–Snyder
arXiv:1010.3797 Penneys–Tener)

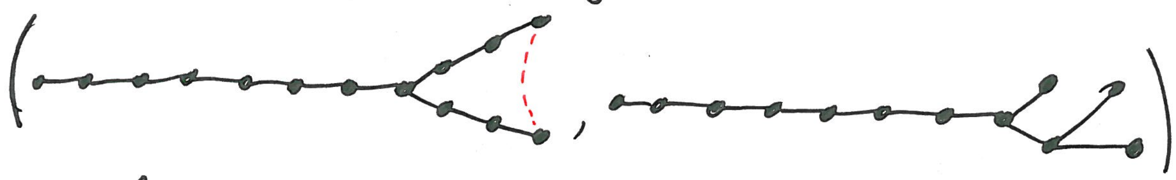
An extremal subfactor with index between 4 and 5 has standard invariant TL (with principal graph A_∞) or one of the following 10 cases:

- the Haagerup planar algebra with index $\frac{5+\sqrt{13}}{2}$, principal graph



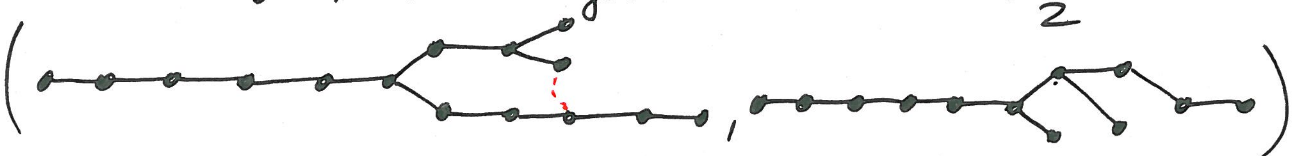
and its dual

- the extended Haagerup planar algebra



and its dual

- the Asaeda–Haagerup planar algebra with index $\frac{5+\sqrt{17}}{2}$

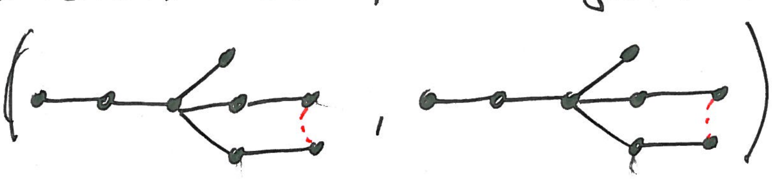


and its dual

- the 3311 GHJ planar algebra with index $3+\sqrt{3}$

and its dual

- and ~~the~~ Izumi's 2221 planar algebra with index $\frac{5+\sqrt{21}}{2}$



and its complex conjugate.

Our general approach is:

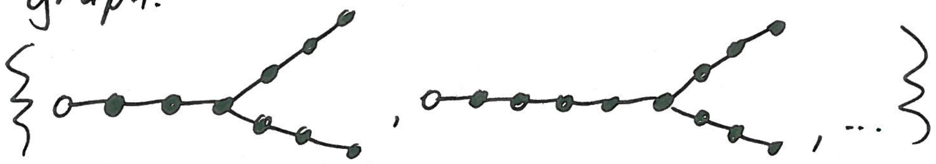
1. Enumerate possible principal graphs
2. Eliminate as many as possible using combinatorial or number theoretic constraints
3. Directly classify the planar algebras realizing each of the remaining graphs.

Enumerating principal graphs

Solely using combinatorial constraints, we cannot expect to get down to finitely many principal graphs up to some index limit L.

We need to introduce infinite families to organize the enumeration.

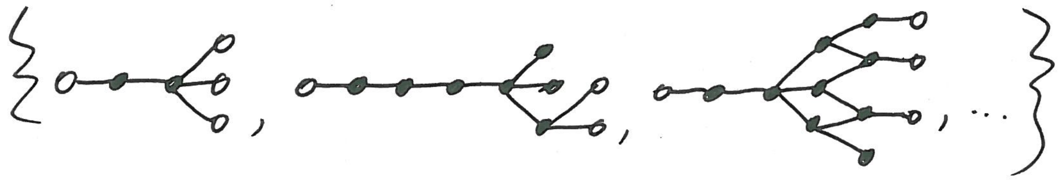
A vine represents all graphs obtained by 'translating' the given principal graph.

Example: 

(we always translate by even amounts, to preserve parity.)

(We always work with a principal graph pair, with dual data, but I'll often only draw one of the graphs for brevity.)

A *weed* represents all graphs obtained by translating or 'extending' the given principal graph.

Example 

Clearly the weed  represents all irreducible principal graphs.

Now, a 'classification statement' (Γ_0, L, V, W) is the theorem

"Every subfactor principal graph represented by the weed Γ_0 with index in $(4, L)$ is in fact represented by one of the vines in V or one of the weeds in W."

We can begin with the trivial classification statement

$$(\text{'•—•}, L, \{\}, \{•—•\})$$

but we'd like to reach a classification statement in which all the vines and weeds are 'manageable'

As it will turn out, all vines are 'manageable', in the sense that ~~we can~~ only finitely many of the represented graphs have cyclotomic index, and we can effectively determine these.

What do we do with an 'unmanageable' weed?

<u>Meta-theorem</u>: $(\Gamma_0, L, V, W \cup \{B\}) \Rightarrow (\Gamma_0, L, V \cup \{B\}, W \cup W_B)$

where W_B denotes all the depth 1 extensions of B staying below the index limit L, and further satisfying associativity.

<u>Lemma</u> W_B is always a finite set, as adding too many edges increase the index above L.

The challenge now is to skillfully apply the meta-theorem.

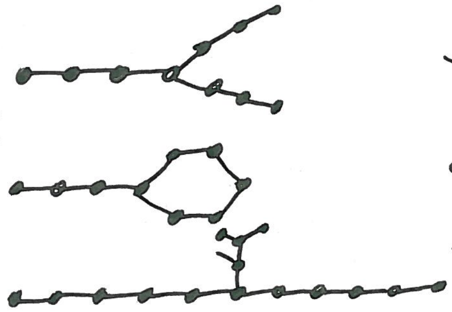
Each time we process a weed B in this way, we learn more, because each graph in W_B has been determined out to one greater depth, and hence might become more manageable.

We need to be careful the set of weeds does not grow too big, or to use a very large computer!

If we're very lucky, W_B might be empty, and eventually we have only weeds.

This is the case below index $3+\sqrt{3}$, and Haagerup's initial result there was the classification statement

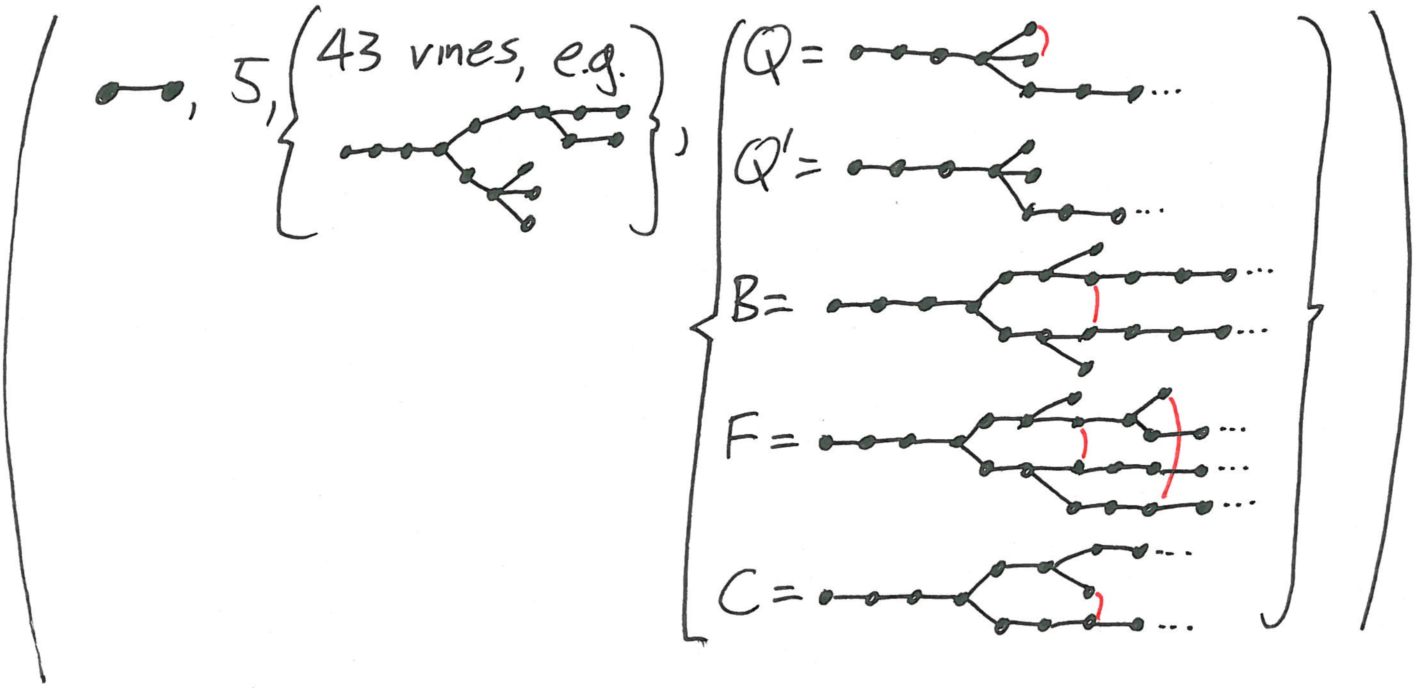
$\Big($ •—•, $3+\sqrt{3}$, $\Big\{$



$\Big\}$, $\{\}\Big)$

Up to 5, however, we have to chose a judicious place to stop. (In particular, by developing new tools to rule out persistent weeds.)

Thus we reach the classification statement



What to do about vines?

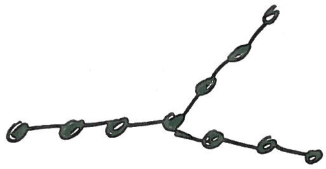
Coste and Gannon, building on work of De Boer and Goeree, showed the entries of the S-matrix of a modular category (and hence the dimensions of simple objects) are cyclotomic integers.

(Sketch: the Galois group acts faithfully on S either by permuting rows or columns; since these operations commute, the Galois group is abelian, hence the entries are cyclotomic.)

Etingof-Nikshych-Ostrik showed this implies the dimensions of simple objects in any fusion category (not necessarily modular or even braided) are also cyclotomic.

(In fact $\mathbb{Q}(\{\dim X\}_{X \in \mathcal{C}}) = \mathbb{Q}(\{\dim Y\}_{Y \in Z(\mathcal{C})})$.)

Richard Ng has recently given a constructive proof, writing the dimensions explicitly as sums of roots of unity.

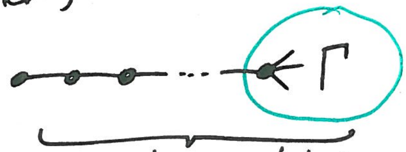
Asaeda-Yasuda used this obstruction to show the vine  only admitted two translations, by 0 and by 4.

Other vines required other ad-hoc arguments.

In fact, all dimension functions (and hence multiplicity free eigenvalues of the principal graph) must be cyclotomic.

Eventually, we found a hammer that deals with all vines.

<u>Theorem</u> (Calegari-Morrison-Snyder)

In a family of graphs $\Gamma_n =$ 

n vertices in total

either 1) $\Gamma_n = A_n$ or D_n, or

2) there is an effective constant N, and $\forall n \geq N$, the adjacency matrix of Γ_n has a multiplicity free eigenvalue λ so $\mathbb{Q}(\lambda^2)$ is not abelian.

(In fact, we can show eventually the largest eigenvalue λ has $\mathbb{Q}(\lambda^2)$ not abelian but N may now be very large.)

Penneys-Tener in 'Subfactors with index less than 5; part IV' showed how to compute these constants, and checked all the cases below N as well.

This reduces the 43 vines from part I to 28 individual cases, and some other number theoretic results in that paper get us down to just 4 cases: H, EH, AH and 2221.

What about the weeds?

We need to learn about connections.

First, given a pair of principal graphs and dual data, we can define a two-sided graph planar algebra

- two string types, called 'dexter' and 'sinister'
- P_w, for w a word in $\{d,s\}$.

$$= \left\{ \text{loops on } \begin{tikzcd} M\text{-}N \arrow[r, "-\otimes X"] & M\text{-}M \arrow[d, "X^*\otimes -"] \\ N\text{-}N \arrow[u, "X^*\otimes -"] \arrow[r, "-\otimes X"] & N\text{-}M \end{tikzcd} \right.$$

$$\left. \begin{array}{l} \text{following horizontal edges at 'dexter' steps} \\ \text{vertical edges at 'sinister' steps} \end{array} \right\}$$

- the same action of planar tangles as for the usual graph planar algebras.

A connection is merely an element $c \in P_{dsds}$.

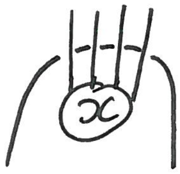
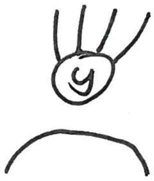
A connection is bi-invertible if $\exists c^{-1}$ so

A connection is bi-unitary if $c^{-1} = c^*$.

There is a gauge group $G^d = \prod_{\substack{\text{bimodules} \\ P,Q}} U(\dim\langle P\otimes X, Q\rangle)$, $G = G^d \times G^s$.

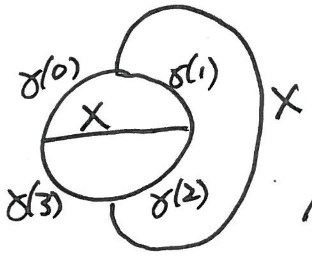
which acts of P, preserving bi-unitary connections.

Given a connection we can define the planar algebra of flat elements:

$$F_n = \left\{ x \in P_{\underbrace{dd\cdots d}_{n \text{ times}}} \;\middle|\; \exists y \in P_{ss\cdots s} : \right.$$  $=$  $\left.\right\}$

Gauge equivalent connections give equivalent planar algebras.

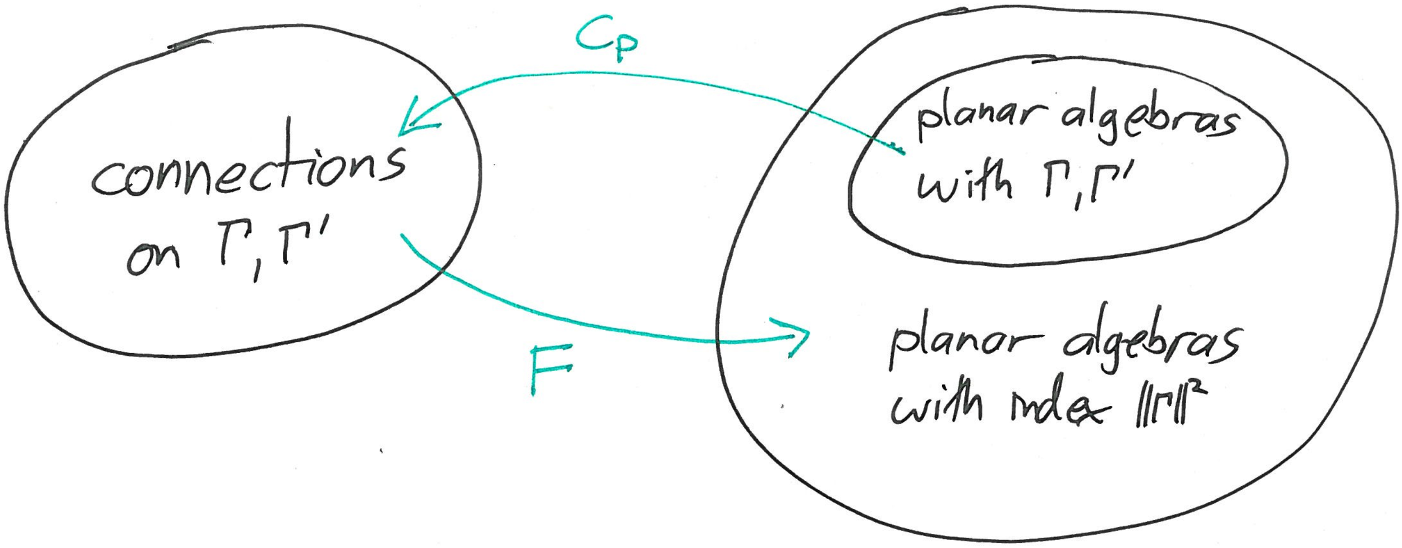
Conversely, every planar algebra P with principal graphs (Γ, Γ') gives a connection $C_P \in GPA(\Gamma, \Gamma')_{dsds}$

via $C_P(\gamma) =$ 

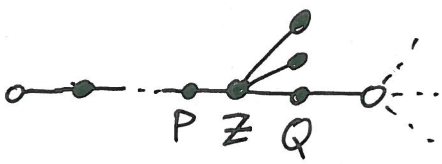
$/ \bigcirc_{\gamma(0)}$

(here the edges are labelled by idempotents/vertices of Γ, Γ', vertices are labelled by intertwiners/edges of Γ, Γ')

and the flat elements recover P.



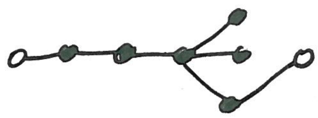
Theorem

A connection on 

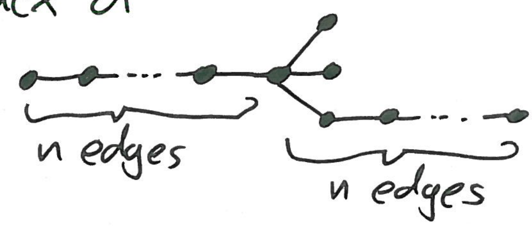
exists only if $\dim P = \dim Q$.

(Proof: unitarity of the 4×4 matrix of connection entries passing through Z and Z^*) arXiv:1109.3190

Corollaries:

* a subfactor represented by the weed 

  has index equal to the index of

* in fact has principal graph nn11
* in fact has principal graph 3311

  (otherwise the index isn't cyclotomic)

There's a circle worth ~~of~~ bi-unitary connections on 3311. An analysis of the relationship between the connection and the rotational eigenvalues of the planar algebra generators picks out two points giving 3311 subfactors.

(For the others, the flat subalgebra is just Temperley-Lieb.)

We still need to eliminate the weeds B, C and F. This was done in 'Subfactors with index less than 5, part II', arXiv: 1007.2240, but there are now better arguments.

Say a principal graph is <u>stable</u> at depth n if each vertex at depth n is connected to at most one, distinct, vertex at depth $n+1$ (on both graphs).

<u>Theorem</u> (Popa, c.f. Bigelow-Penneys)
Stable at depth $n \Rightarrow$ stable at depth $n+1$.

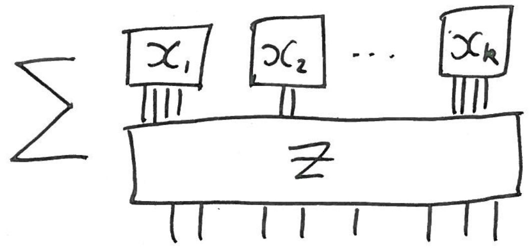
Since the weed B is stable at depth 6, it must remain stable, and these graphs are easy to rule out.

∎

Sketch of theorem

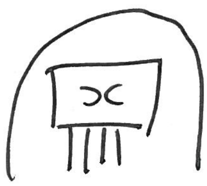
For a subset $W \subset P_\bullet$, define

$$\mathrm{trains}(W) = \left\{ \sum \right.$$



$$\left. \middle| \begin{array}{l} x_i \in W \\ Z \in TL \end{array} \right\}$$

If P is stable at depth n, for any $x \in P_{n,\pm}$



$$\in \mathrm{trains}(P_{\leq n}) \qquad (*)$$

Let Q be the subalgebra of P generated by $P_{\leq n}$.

Repeatedly applying $(*)$, every element of Q is in $\mathrm{trains}(P_{\leq n})$.

Therefore, $\Gamma(Q)$ agrees with $\Gamma(P)$ up to depth n, but is stable thereafter.

Note $\mathrm{index}\, P = \mathrm{index}\, Q$, so $\|\Gamma(P)\| = \|\Gamma(Q)\|$.

An analysis of Frobenius-Perron eigenvalues shows $\Gamma(P) = \Gamma(Q)$ (and indeed $P = Q$).

Theorem (Snyder) arXiv:

Consider an n-1 supertransitive subfactor with excess 1. (i.e both graphs begin with a triple point)

Suppose a vertex on Γ at depth n is univalent.

Let r be the ratio of dimensions of the vertices of Γ' at depth n.

Let λ be the rotational eigenvalue of the new n-box.

Then $$r + \frac{1}{r} = \frac{\lambda + \lambda^{-1} + 2}{[n][n+2]} + 2$$

Sketch

For the 3×3 matrix U of connection entries passing through the two branch points, there is a 'nice' gauge choice (using the TL-intertwiners).

In this gauge choice, we have a formula relating $\rho^{\frac{1}{2}}$ and U, acting on elements in $TL^{\perp}$.

Finally, the identity falls out of unitarity for U. □

Applying this to a weed is somewhat tricky!
We don't know r, n or the index.

However, F&C are 'manageable' in that
we can write r, $[n]$ and $[n+2]$ in terms of n and q.

Using $-2 \leq \lambda + \lambda^{-1} \leq 2$, we rule out all but finitely
many n, and for the remaining cases
see λ is not a root of unity.