Lecture IV: constructing exotic examples

Today we'll tackle the problem of constructing subfactors 'with our bare hands'.

Given a principal graph (or the entire combinatorial fusion data) can we determine how many (if any!) subfactors realize it?

3 'general purpose' approaches:

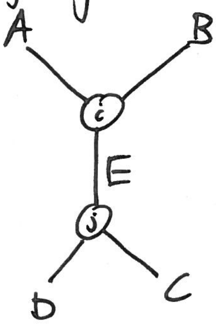
(A) Look for 6-j symbols satisfying the pentagon equations.
Gauge orbits of solutions $\xleftrightarrow{1:1}$ planar algebras

(B) Look for subalgebras of the graph planar algebra.
Maybe we can identify planar equations satisfied by some element? Even if we can locate such elements, how do we characterize the subalgebra they generate? Show it has the right principal graph?

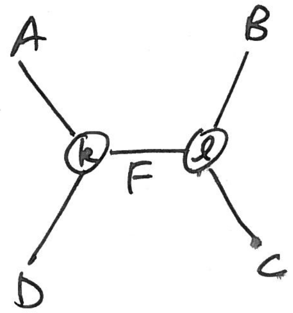
(C) Find bi-unitary connections, and identify which give ~~the~~ flat algebras with the right principal graph.

Solving the pentagon equations is generally totally impractical.

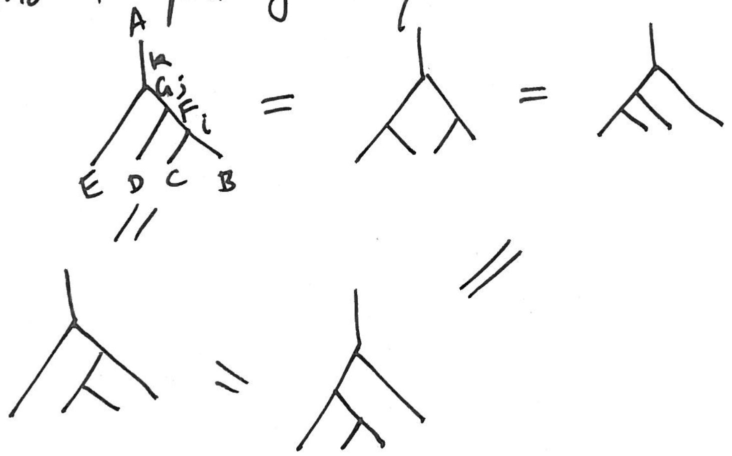
The 6j-symbols give the ~~equations~~ identities



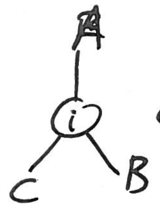
$= \sum F^{AB\;E\;ij}_{DC\;F\;kl}$



and the pentagon equations are



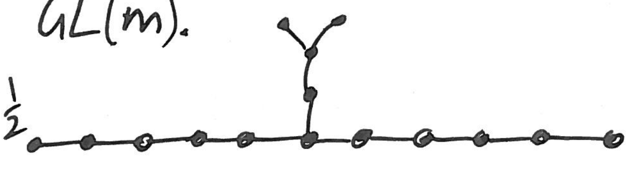
For each trivalent vertex space



$\in \mathrm{Hom}(1 \to A \otimes B \otimes C)$

we have a gauge group $GL(m)$.

E.g. for the even part of Asaeda-Haagerup



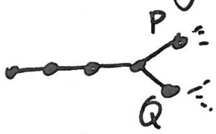
there are 465976 equations in 22208 unknowns

and a 356 dimensional gauge group

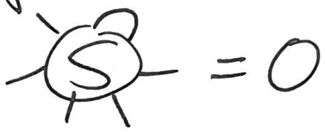
The tools of applied algebraic geometry are hopelessly inadequate.

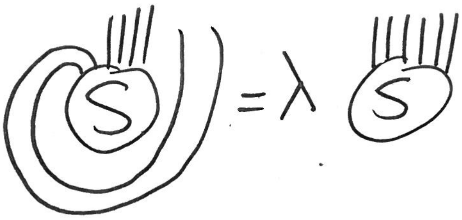
To search in the GPA, we need some equations.

For Haagerup, Asaeda-Haagerup or extended Haagerup, (or generally magic numbers $10^{n-1}1\ldots$)

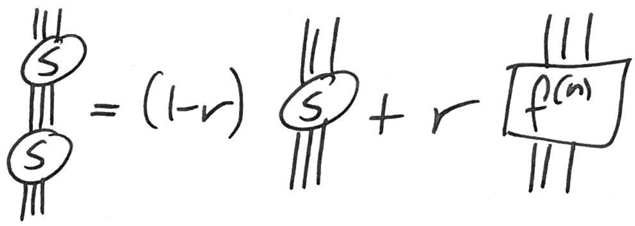


the planar algebra must contain an element $S \in P_n$ satisfying:

① 

② 

and

③ 

where $r = \frac{\dim P}{\dim Q} > 1$

(since S and $f^{(n)}$ must form a 2d algebra with idempotents with traces $\dim P$ and $\dim Q$)

Using Jones/Snyder's triple point obstruction, we usually know λ.

Thus the first step is to solve the linear equations ①&② in the graph planar algebra.

This is easy and cuts down to a small vector space (e.g. dim = 4, 16, 21 for H, AH, EH.)

We then need to solve ③. This is already very hard for EH and especially AH.

(The best strategy is to solve by hand for about half the variables, the use numerical methods to isolate the discrete solutions, then guess and check algebraic solutions)

At this point we have a candidate subalgebra

$$\mathcal{G}(S) \subset GPA(\Gamma)$$

(and indeed, if such a planar algebra exists it must be $\mathcal{G}(S)$ for one of our solutions S.)

This subalgebra is certainly unitary and spherical (inherited from the GPA)

but it is not obvious that $\mathcal{G}(S)_0 = \mathbb{C}$, or $\Gamma(\mathcal{G}(S)) = \Gamma$, or even $\mathcal{G}(S) \subsetneq GPA(\Gamma)$!

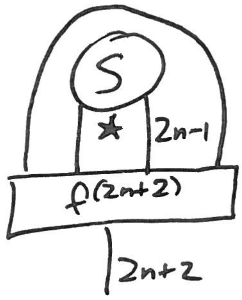
To show $\mathcal{G}(S)_0 = \mathbb{C}$, we need to identify relations satisfied by S that suffice to evaluate closed diagrams built from S.

(Often, this also lets us show $\dim(\mathcal{G}(S)_k) = \dim \mathcal{TL}_k$ for $k < n$, and indeed lets us identify $\Gamma(\mathcal{G}(S))$.)

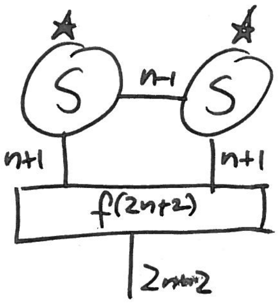
Our favourite approach to this is 'jellyfish relations' pioneered in the construction of extended Haagerup. (arXiv:0909.4099)

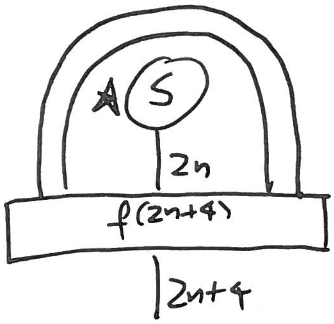
Various generalizations are proving very fruitful, and we're still discovering new consequences.

<u>Theorem</u> the 'candidate generators' for H $(n=4)$ and EH $(n=8)$ satisfy relations

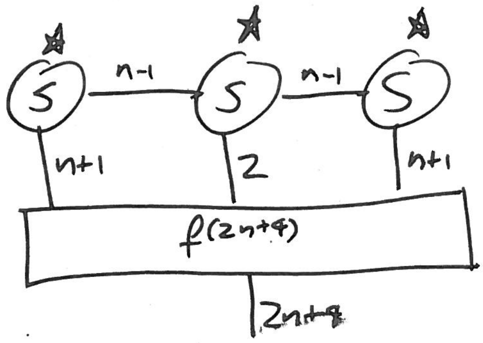


$$= i\,\frac{\sqrt{[n][n+2]}}{[n+1]}$$



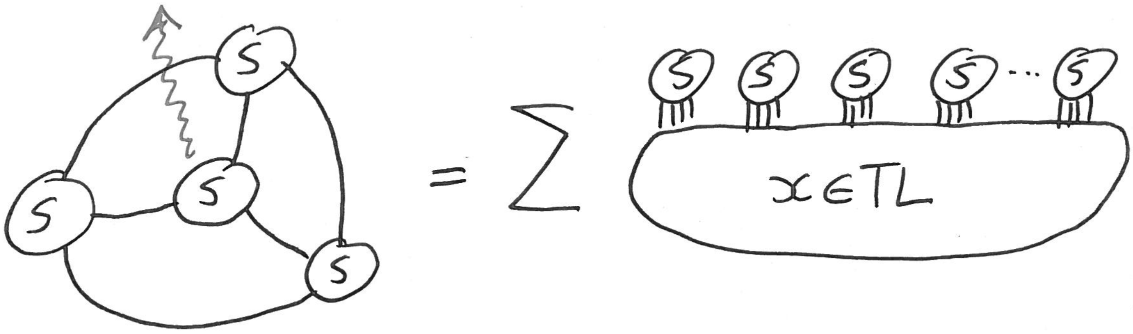


$$= \frac{[2][2n+4]}{[n+1][n+2]}$$



Theorem these relations suffice to evaluate all closed diagrams.

Proof
Float all the jellyfish to the surface:



Next, a simple argument about diagonals in polygons shows there must be a pair of S's connected by at least n strands.
The relation $S^2 = (1-r)S + rf^{(n)}$ lets us rewrite the diagram with fewer S's, all still 'at the surface'.
Eventually we're left with just Temperley-Lieb diagrams, which are easy to evaluate $\square$.

This evaluation algorithm is remarkable in that it *increases* the 'complexity' at intermediate steps.

With little additional work we can completely determine $\Gamma(G(S))$.