SMALL INDEX SUBFACTORS, PLANAR ALGEBRAS, AND FUSION CATEGORIES

The goal of these lectures is to describe recent work on the classification of small index subfactors.

We've recently written a survey paper on the subject, The classification of subfactors of index at most 5, `arXiv:1304.6141`.

We will study subfactors from the point of view of their "standard invariant", a unitary pivotal 2-category, with two objects $\mathcal{C}$ and $\mathcal{D}$, along with a chosen generating 1-morphism $X : \mathcal{C} \to \mathcal{D}$.

In particular $\mathrm{End}\,(\mathcal{C})$ is a semisimple pivotal category, and in many interesting cases has finitely many simple objects, so is an example of a fusion category.

Fusion categories constitute an interesting middle ground in category theory.

The 'classical' examples are the representation theories of finite groups, or the representation theory of the $U_q(\mathfrak{g})$ quantum group with q a root of unity.

Why, generally speaking, do we study categories?

(1) Sometimes (following Grothendieck), because it is best to study a class of objects 'from the outside', looking only at the relationships between objects (morphisms, tensor products, and so on) but not at their 'gory details' (e.g. the points of a topological space).
(2) Other times, because the most convenient axiomatization of an algebraic gadget is through category theory. For example, we can say a group is a category with one object and all morphisms invertible.

Fusion categories are 'categorical' only in the second sense, I think.

So why am I talking about them here? After all, if a geometric group theorist showed up here, prefacing his talks with "Recall a group is a category with one object and all morphisms invertible...", you'd rightfully be annoyed.

Fusion categories, however, are very much higher categories. Admittedly not that high, with $n = 2$, but it's enough. In order to study them, we need to use 2-dimensional topology in a pervasive way — exemplified by the notion of a planar algebra, and its associated toolkit.

Dimension 2 is still radically simpler than higher n, but I think it's nice to see how far the interaction between algebra and topology successfully takes us, even if this is only a 'warm-up' case.

Returning towards subfactors, we'll see that the study of small index subfactors both gives us new, apparently exotic, examples of fusion categories, and provides a natural motivation for building the tools to classify and construct examples.

(In fact, the subjects of fusion categories and of (standard invariants of) small index subfactors are philosophically extremely close. Following Müger, the standard invariant captures exactly a pair of fusion categories, along with a (categorified) Morita equivalence between them.)

Small index subfactors. An important measure of the 'size' of a subfactor is its index, $(\dim X)^2$. The classification of subfactors with index at most 4 has been well understood since the mid 90s. At index 6 and above certain wild features arise making classification intractable.

Building on earlier work of Haagerup we've now completed the classification for the interval $(4, 5]$. This has led to two surprises —

(1) The classification is surprisingly sparse, with only 10 examples, coming in 5 related pairs.
(2) Some of these examples lie in families, or arise as special cases of general constructions, but others are seemingly 'sporadic' or 'exotic'. (The proofs that they even exist seem rather unenlightening at this point!)

Lecture I: Subfactors, the standard invariant, and planar algebras.

For completeness, let's mention the definition of a subfactor, before immediately turning to its representation theory.

- A factor is a von Neumann algebra with trivial centre.
- A factor is type I if it has minimal projections; these are $M_n(\mathbb{C})$ or $B(\mathcal{H})$.

  A factor is type II if it has no minimal projection, but there is a trace.
  It is type II_1 if that trace can be normalized so $tr(1) = 1$. and type II_∞ otherwise

  A factor is type III otherwise.
  We'll restrict our attention to II_1 factors.
- The prototypical example is the 'hyperfinite II_1 factor'

$$R = \lim_{n\to\infty} \bigotimes_{k=1}^{n} M_2(\mathbb{C}).$$

- A subfactor $A \subset B$ is 'merely' an inclusion of factors.

  The index $[B:A]$ is the dimension of $L^2(B)$ as an A-module

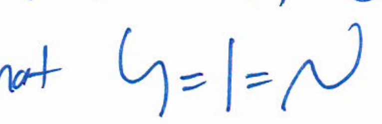
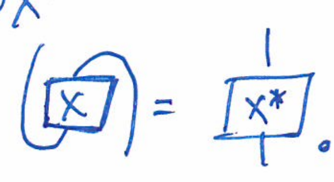
Given a subfactor $A \subset B$, we can analyze its bimodules, which come in 4 flavours, A-A, A-B, B-A and B-B.

We can tensor bimodules together, so as well as the obvious examples ${}_AA_A$, ${}_AB_B$, ${}_BB_A$ and ${}_BB_B$ we can form ${}_AB \otimes_B B_A$ (which is just ${}_AB_A$), ${}_BB \otimes_A B_B$, and so on.

(We can work with either algebraic or L^2-bimodules here; the factor trace gives an inner product on algebraic bimodules, and L^2-completion gives an equivalence of categories.)

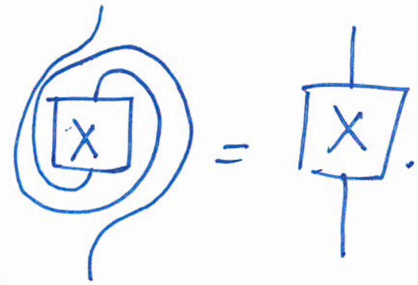
We only consider those bimodules $\otimes$-generated by ${}_AB_B$ and ${}_BB_A$.

<u>Theorem</u> (the standard invariant)
These bimodules form a unitary pivotal 2-category.

- "Unitary" means there is an antilinear map $\bar{\ }: \mathrm{Hom}^2(X,Y) \to \mathrm{Hom}^2(Y,X)$ so $\langle a|b \rangle = tr(\bar{a}b)$ is a positive definite inner product.
  (Unitary implies semisimplicity!)
- "Pivotal" means there is a $\otimes$-reversing functor $*: \mathcal{C} \to \mathcal{C}$ with $** = 1$, and evaluation and coevaluation maps
  $\cap : X^* \otimes X \to \mathbb{1}$, $\cup : \mathbb{1} \to X \otimes X^*$
  so that  and .

Sketch

- We produce the (co)evaluation maps $B \otimes_A B \to B$ and $B \to B \otimes_A B$ using 'Pimsner-Popa bases'.
- The difficult step (Burns) is $** = 1$,



- The antilinear $\bar{\ }$ is adjoint of linear operators.

The A-A bimodules and B-B bimodules give a pair of pivotal categories called the <u>even parts</u>.

When these have finitely many simple objects (and so are fusion categories), we say the subfactor is <u>finite depth</u>.

An important invariant of a subfactor is its principal graph: Γ

* vertices for each simple bimodule (so four classes, A-A, A-B, B-A, B-B)
* edges between P_A and Q_B for the dimension of $\mathrm{Hom}(P \otimes_A B, Q)$. (equivalently, $\mathrm{Hom}(P_A, Q \otimes_B B_A)$)

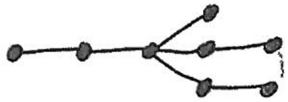
We also record the involution $*$ corresponding to duals of bimodules.

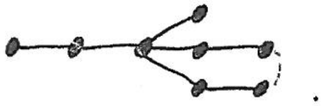
The graph has two components, for the left-A and left-B modules. Each component is bipartite for right A and B modules.

We draw the trivial bimodules $*$ at the left, with vertices arranged according to distance from $*$.

The k-th A-B bimodule at depth d is dual to the k-th B-A bimodule at depth d.

We indicate dual data on the even parts with dotted red lines.

E.g. 



The index is at least the square of the graph norm (the largest eigenvalue of the adjacency matrix)

When the graphs are finite the index equals the square of the graph norm.

Examples from finite groups

We can realize any finite group as acting by outer automorphisms on the hyperfinite II_1 factor R (essentially uniquely!) (Jones 1980)

* Thus we have subfactors $R^G \subset R$, with index $|G|$.

  The fusion category of R^G-R^G bimodules is then $Rep(G)$, and the dual category of R-R bimodules generated by $R \otimes_{R^G} R$ is Vec_G.

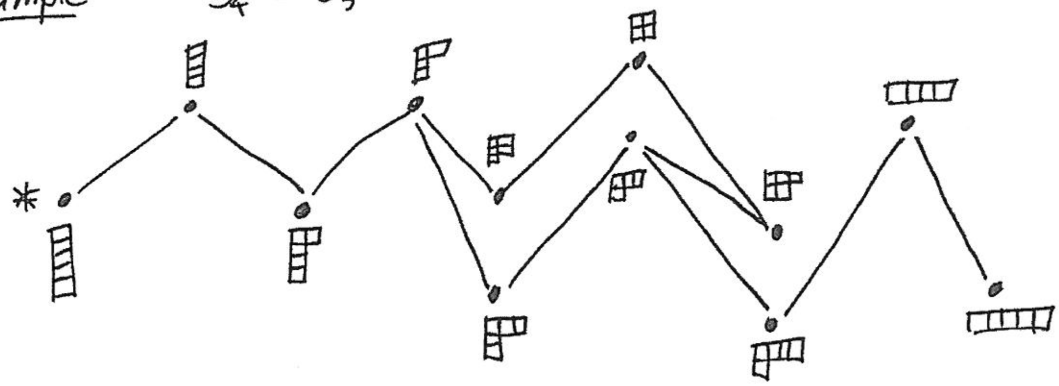
  Thus this subfactor corresponds to the Morita equivalence between $Rep(G)$ and Vec_G.

* Given $H \subset G$, we can look at $R^G \subset R^H$, with index $[G:H]$

  Now the R^G-R^G bimodules give $Rep(G)$,
  the R^G-R^H bimodules give $Rep(H)$,

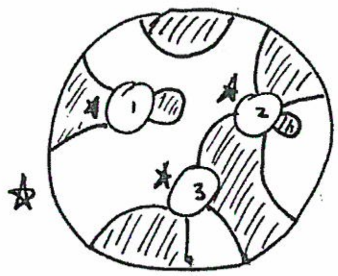
  and the functors
  $$Rep(G) \xrightarrow{-\otimes R^H} Rep(H)$$
  and $$Rep(H) \xrightarrow{-\otimes R^H} Rep(G)$$
  are restriction and induction, and the principal graph is the induction restriction graph for $H \subset G$.
  The R^H-R^H bimodules, and the dual principal graph, are harder to describe.

Example $S_4 \subset S_5$



The data of a standard invariant is equivalent to that of a **planar algebra** P:

- vector spaces $P_{n,\pm}$, $n \geq 0$
- for each planar tangle

T = 

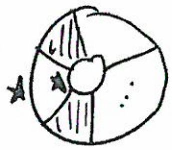
(a disc D_0 and interior discs D_i, a 1-manifold in $D_0 \setminus \cup D_i$, and shading of the regions, and a marked point on each boundary circle...)

a linear map

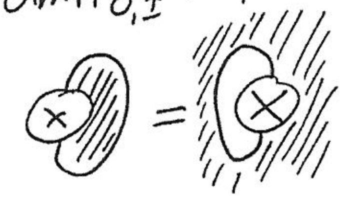
$$P(T) : P_{2,-} \otimes P_{3,+} \otimes P_{2,+} \longrightarrow P_{5,+}$$

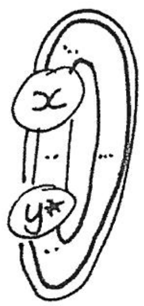
from the vector spaces associated to the inner circles to the vector space associated to the outer circle

such that:

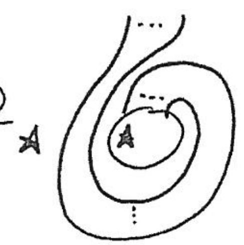
- the linear maps compose in the same way planar tangles do
- isotopic (rel ∂) tangles give the same maps
-  is the identity

A planar algebra is

- evaluable if $\dim P_{0,\pm} = 1$
- spherical if  (as multiples of the empty diagrams in $P_{0,\pm}$)
- unitary if there is an antilinear $* : P_{n,\pm} \to P_{n,\pm}$, intertwining reflection of planar tangles, such that

$\langle x, y \rangle =$  is positive definite.

A subfactor planar algebra is one which is evaluable, bosonic spherical and unitary.

- bosonic if  $= 1$ on every $P_{n,\pm}$

(actually, I just made 'bosonic' up: everyone takes it as part of the definition of a planar algebra.)

Reconstruction

Starting from a subfactor planar algebra, we can rebuild a subfactor.

$$A \subset B = \lim_{n\to\infty}\left(\Big| \; \boxed{P_{n-1,-}} \subset \boxed{P_{n,+}} \right).$$

Theorem (Popa) Starting with a finite depth subfactor of the hyperfinite II_1, this reconstructs the original subfactor.

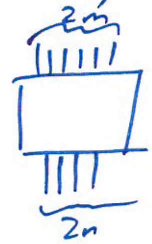
Outside of these cases, the problem is more subtle.
Popa has shown that every subfactor planar algebra is realized by some subfactor of $L(F_\infty)$.

Standard invariant from a planar algebra

Name our two objects $\mathcal{C}$ and $\mathcal{D}$.

$$\mathrm{Hom}(\mathcal{C} \to \mathcal{C}) = \mathrm{Hom}(\mathcal{D} \to \mathcal{D}) = 2\mathbb{N}$$

$$\mathrm{Hom}(\mathcal{C} \to \mathcal{D}) = \mathrm{Hom}(\mathcal{D} \to \mathcal{C}) = 2\mathbb{N}+1$$

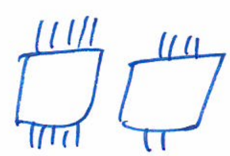
$$\mathrm{Hom}_{\mathcal{C}\mathcal{C}}(2n, 2m) = P_{n+m,+} =$$  (unshaded on both sides)

$$\mathrm{Hom}_{\mathcal{D}\mathcal{D}}(2n, 2m) = P_{n+m,-}$$

$$\mathrm{Hom}_{\mathcal{C}\text{-}\mathcal{D}}(2n+1, 2m+1) = P_{n+m+1,+}$$

$$\mathrm{Hom}_{\mathcal{D}\text{-}\mathcal{C}}(2n+1, 2m+1) = P_{n+m+1,-}$$

Composition is by stacking ,

Tensor product is by horizontal juxtaposition. 

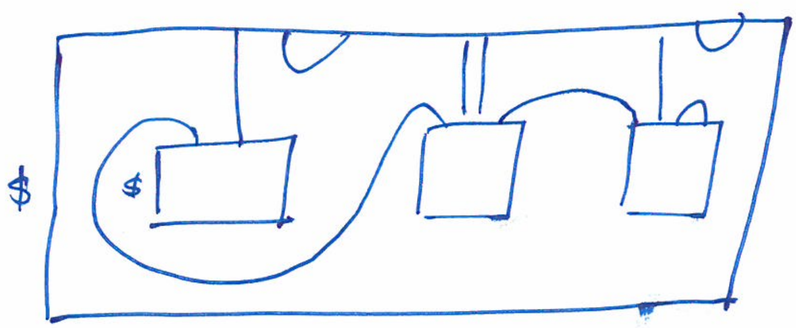
Planar algebra from a standard invariant.

We start with $\mathcal{C}$ a pivotal 2-category and $X: A \to B$ our 'favourite' 1-morphism.

Define $P_{n,+} = \mathrm{Hom}_{A-A}(1 \to \underbrace{X \otimes X^* \otimes \cdots \otimes X^*}_{2n \text{ factors}})$

and $P_{n,-} = \mathrm{Hom}_{B-B}(1 \longrightarrow \underbrace{X^* \otimes X \otimes \cdots \otimes X}_{2n \text{ factors}})$

To define the action of planar tangles, first isotope them to a standard form:



and read from bottom to top.

Lecture II: Examples, and the planar algebra toolkit.

Today we'll introduce three important tools for the analysis of planar algebras.

① The Temperley-Lieb algebra TL_δ is initial for planar algebras with index δ^2.

$$TL_\delta \hookrightarrow P$$

② Every planar algebra with principal graph Γ embeds in the graph planar algebra for Γ.

$$P \hookrightarrow GPA(\Gamma)$$

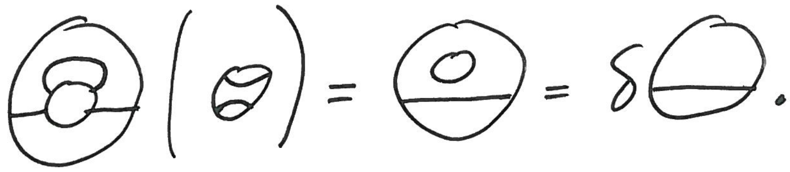
③ The annular Temperley-Lieb category acts on every planar algebra, and we can decompose the planar algebra into irreducible modules.

Temperley-Lieb

Defn $TL'_{\delta,n} = \mathbb{C}\{\text{crossingless matchings on } 2n \text{ points}\}$

e.g. $TL_{\delta,3} = \mathbb{C}\{$  $\}$

and planar tangles act by gluing, removing closed circles for a factor of δ.

Eg 

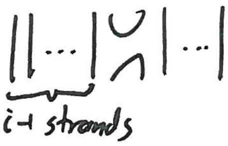
We have a map $TL'_\delta \to P$ for any planar algebra with loop value δ, because we can interpret a Temperley-Lieb diagram as a planar tangle with no inputs.

Theorem
- TL'_δ is nondegenerate except when $\delta = 2\cos\frac{\pi}{n}$.
- It is positive definite for $\delta \geq 2$.
- When $\delta = 2\cos\frac{\pi}{n}$, the radical is generated by the $(m-1)$th Jones-Wenzl idempotent, and the quotient is positive definite.

(Jones, Index for subfactors, 1983)

Let's denote $TL'_\delta/\text{radical}$ by TL_δ.

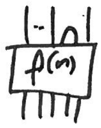
What are the <u>Jones-Wenzl</u> idempotents?

<u>Definition 1</u> In TL_n, thought of as an associative algebra, we have idempotents $e_i = \frac{1}{\delta}$  for $i = 1, \ldots, n-1$.

$$f^{(n)} = 1 - \sup(e_1, \_\_, e_n).$$

<u>Definition 2</u>

$f^{(n)}$ is the unique element of TL_n satisfying:

a)  $= 0$ for all caps on top or bottom

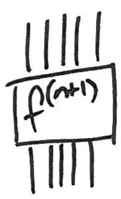
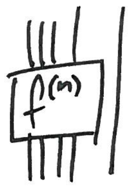
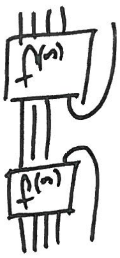
b)  $=$ 

c)  $=$ 

d)  $= [n+1]_q = \dfrac{q^{n+1} - q^{-n-1}}{q - q^{-1}}$, where $q + q^{-1} = \delta$

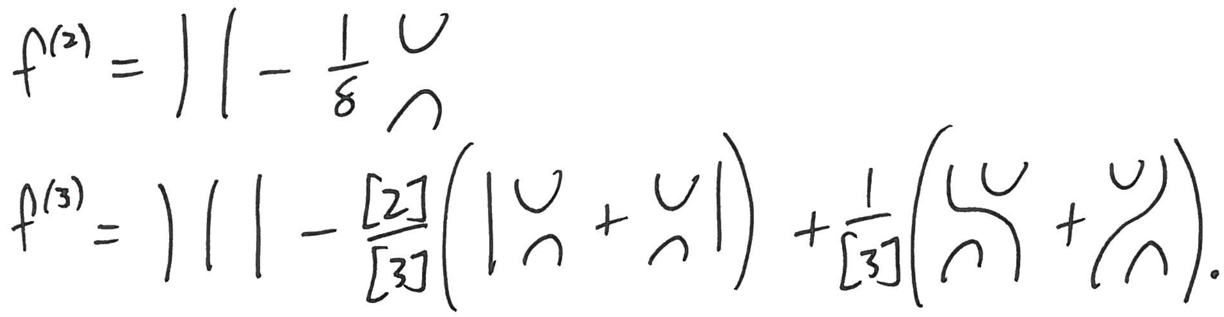
e) the coefficient of |||| in  is 1.

<u>Definition 3</u> (Wenzl) $f^{(0)} = \phi$, $f^{(1)} = 1$,

 $=$  $- \dfrac{[n]}{[n+1]}$ 

(You can remember the coefficient by taking traces.)

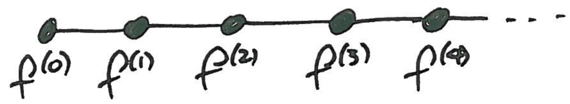
In particular we have



<u>Lemma</u> The Jones-Wenzl idempotents are minimal, and $f^{(n)} \otimes f^{(m)} \cong f^{(|n-m|)} \oplus f^{(|n-m|+2)} \oplus \cdots \oplus f^{(n+m)}$.

<u>Proof</u> Definition 3 gives the $m=1$ case, the rest are determined by associativity.

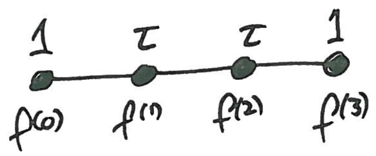
<u>Corollary</u> When $\delta \geq 2$, the principal graph of TL_δ is A_∞



<u>Lemma</u> When $\delta = 2\cos\frac{\pi}{m}$, $f^{(k)}$ is contained in the radical for all $k \geq m-1$.

<u>Corollary</u> The principal graph of $TL_{2\cos\frac{\pi}{m}}$ is A_{m-1}

E.g. $\Gamma\left(TL_{\frac{1+\sqrt{5}}{2} = 2\cos\frac{\pi}{5}}\right) =$



<u>Examples</u>: the ADE classification below index 4.

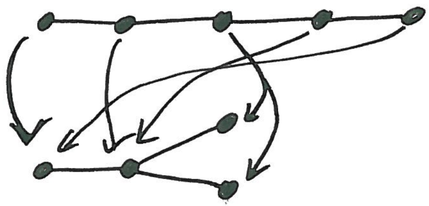
Besides the A series coming from Temperley-Lieb, there are 2 other families of planar algebras with index below 4.

- There is a unique planar algebra with principal graph D_{2n} for each n. Some constructions:

1) In A_{4n-3}, consider $X = f^{(0)} \oplus f^{(4n-4)}$

This carries the structure of an algebra object, and the fusion category of X-X bimodule objects has principal graph D_{2n}

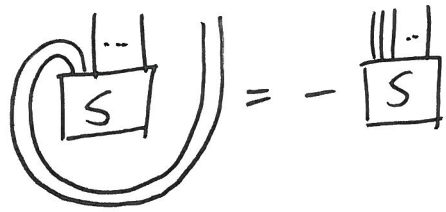
You should think of the D_{2n} principal graph as an orbifold of A_{4n-3}:

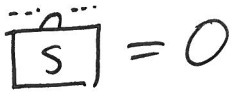


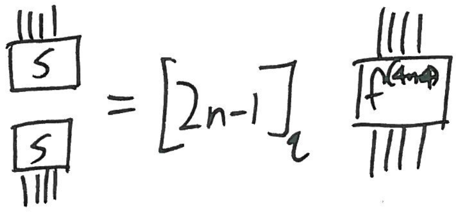
Under the map $A_{4n-3} \hookrightarrow D_{2n}$, $f^{(0)}$ and $f^{(4n-4)}$ become isomorph

while $f^{(2n-2)}$ stops being simple, and splits into two pieces.

2) An explicit planar algebra; generated by $S \in P_{2n-2}$ with relations

i) 

ii) 

iii) 

(c.f. arXiv:0808.0764)

- There are a pair of complex conjugate E_6 subfactors, and a pair of complex conjugate E_8 subfactors.

Again, they can be constructed as bimodule object categories starting from A_{11} or A_{29},
or given by explicit skein theories (Bigelow math.QA/0903.0144)
or via conformal field theory (Xu MR1617550)

Examples from finite groups

We can realize any finite group as acting by outer automorphisms on the hyperfinite II_1 factor R (essentially uniquely!) (Jones 1980)

* Thus we have subfactors $R^G \subset R$, with index $|G|$.

The fusion category of R^G–R^G bimodules is then $\mathrm{Rep}(G)$, and the dual category of R-R bimodules generated by $R \underset{R^G}{\otimes} R$ is Vec_G.

Thus this subfactor corresponds to the Morita equivalence between $\mathrm{Rep}(G)$ and Vec_G.

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and the functors

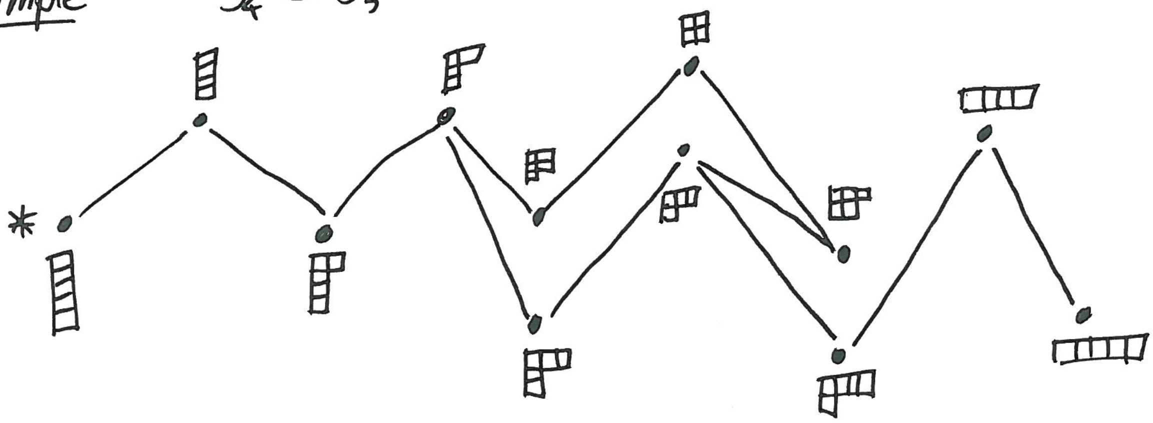
$$\mathrm{Rep}(G) \xrightarrow{-\otimes R^H} \mathrm{Rep}(H)$$

and $\mathrm{Rep}(H) \xrightarrow{-\otimes R^H} \mathrm{Rep}(G)$

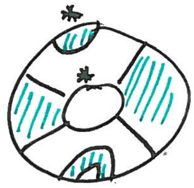
are restriction and induction, and the principal graph is the induction restriction graph for $H \subset G$.

The R^H–R^H bimodules, and the dual principal graph, are harder to describe.

Example $S_4 \subset S_5$



Annular Temperley-Lieb

The planar tangles with one input disc form a category with objects $(n \in \mathbb{N}, \pm)$ and morphisms eg.  $: (2,+) \to (5,-)$

and every planar algebra naturally becomes a representation of this category.

What are the irreducible representations?

<u>Theorem</u> For $\delta > 2$, the irreps of ATL are indexed by

* (n, ω) with $n > 0$, $\omega^n = 1$, or
* $(0, d)$ with $0 \leq d \leq \delta$.

These are realized by a cyclic vector in V_n satisfying relations

 $= 0$ and  $= \omega$  (for $n > 0$)

or  $= d$  (for $n = 0$)

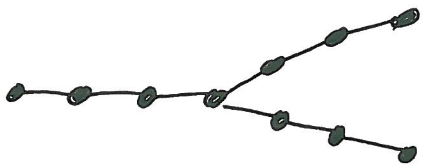
If $P = \bigoplus_n \bigoplus_z a_{n,z} V_{n,z}$, we say

P has 'magic numbers' $\left(a_n = \sum_z a_{n,z}\right)_{n \geq 0}$

If P is evaluable, $a_0 = 1$, and the corresponding irrep $V_{0,\delta}$ is the Temperley-Lieb subalgebra.

If P is k-supertransitive, TL_n exhausts P_n for $k \leq n$, so $a_1, \dots, a_k = 0$.

We know the dimensions of the irreps $V_{n,z}$, and we ~~know~~ ~~the~~ can calculate $\dim P_n$ from the principal graph, so in fact the magic numbers are computable directly from the graph. $a_n = \sum_{r=0}^{n} (-1)^{r-n} \frac{2n}{n+r} \binom{n+r}{n-r} w_r$, where w_r is the # of loops of length $2r$ based at $*$

<u>Example</u> $\Gamma =$ 

$$a_n = (1, 0, 0, 0, 1, 0, 2, \dots)$$

When the graph is finite, the generating function for a_n is rational.

Although we know $a_4 = 1$, we can't determine the rotational eigenvalue $\omega^4 = 1$ solely from the principal graph.

The graph planar algebra.

We can realize any planar algebra inside a certain combinatorially defined 'graph planar algebra'.

Given a graph Γ with dimension function d, we define

$$GPA(\Gamma,d)_n = \mathbb{C}\{\text{loops of length } 2n \text{ on } \Gamma\}^*$$

The action of planar tangles is given by

$$T(\{V_i\})(V_0)(\gamma_0) = \sum_{s \in \text{states}} \Bigg(\prod_{c \in \text{critical points}} s(c)\Bigg)\Bigg(\prod_i V_i(\gamma_i)\Bigg)$$

where

- a 'state' is a labelling of the regions in T by vertices of Γ and of strands in T by edges, extending γ_0.
- $s(c) = \sqrt{\dfrac{d(c^+)}{d(c^-)}}$ where c^+ is the label from s 'inside' the critical point and c^- is the label outside $\overset{c^-}{\frown_{c^+}}$
- γ_i is the path on Γ obtained by reading s around the i-th boundary circle of T

<u>Theorem</u> (Penneys-Jones, Morrison-Walker)
A planar algebra with principal graph Γ embeds in $GPA(\Gamma)$.

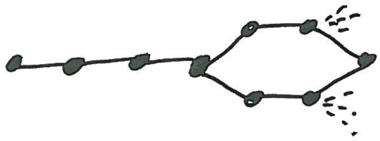
There is a beautiful explanation of this and related facts using the Turaev-Viro TQFTs.

<u>Lemma</u> Denote $GPA(\Gamma)_v$ the subspace of loops based at v.

$*$ $P \hookrightarrow GPA(\Gamma) \twoheadrightarrow GPA(\Gamma)_*$ is an isomorphism, $*$ $P \hookrightarrow GPA(\Gamma) \twoheadrightarrow GPA(\Gamma)_*$ is surje

<u>Corollary</u> A strictly k-supertransitive principal graph may not have a chain of length $> k$.

<u>Corollary</u> A 3-supertransitive principal graph with magic numbers 100010 is either the Haagerup subfactor or begins



'Near-group' examples

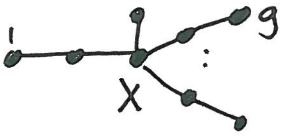
There are two interesting classes of subfactors in which the even part is 'slightly larger than a group'.

(1) A fusion ring of the form $G \cup \{X\}$, with G a finite group must have $gX = X$ and $X^2 = n'X + \sum_g g$.

Examples * the even part of the E_6 subfactor with $G = \mathbb{Z}/2$, $n' = 2$.

$E_6 =$ , $\frac{1}{2}E_6 =$ 

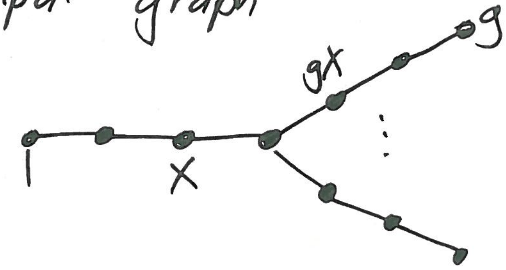
* the Tambara-Yamagami categories, $n' = 0$.

A 'near-group' subfactor with $n' = |G|$ must have principal graph 

Izumi & Xu constructed one for $G = \mathbb{Z}/3$, and Evans & Gannon many more. Existence has been reduced to a polynomial probl

When G is abelian, Izumi and independently Evans & Gannon can show $n' = |G| - 1$ or $n' \mid |G|$.

(2) Subfactors with even part fusion ring $G \cup GX$,
and principal graph



Izumi constructed examples for $G = \mathbb{Z}/3, \mathbb{Z}/5, \mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/4$.
The $\mathbb{Z}/3$ example is the Haagerup subfactor.
Evans & Gannon constructed others, for
$G = \mathbb{Z}/7$, two for $\mathbb{Z}/9$, and none for $\mathbb{Z}/3 \times \mathbb{Z}/3$.
With Penneys & Peters we've constructed one for $\mathbb{Z}/8$.
An example for $\mathbb{Z}/4 \times \mathbb{Z}/2$ might be very helpful
for explaining the mysterious Asaeda-Haagerup subfactor,
via Grossman-Snyder's analysis of the maximal atlas for AH.

<u>Theorem</u> (Izumi) If $|G|$ is odd, G is cyclic.

New subfactors from old

The GHJ construction.

Given $\mathcal{C}$ a fusion category acting on a module category M, each simple object $X \in M$, gives rise to an algebra object in $\mathcal{C}$, the internal endomorphisms $A = \underline{\mathrm{End}}(X, X)$.

Now the 1-1, 1-A, A-1 and A-A bimodule objects in $\mathcal{C}$ give a planar algebra.

Example Take E_6 (as a fusion category) as a module category over A_{11}.

Take $X =$ 

Then $A =$ 

$f^{(0)} + f^{(8)} + f^{(10)}$

and the principal graph of the GHJ subfactor is  "3311".

with index $3+\sqrt{3}$.

~~Intermediate subfactors~~

Tensor products, free products, and intermediate subfactors

The tensor product of planar algebras P and Q is defined by

$$(P \otimes Q)_n = P_n \otimes Q_n$$

with planar tangles acting pointwise in the two factors.

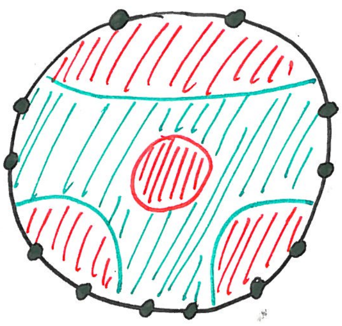
(index is multiplicative under tensor product)

The free product is harder to define!

$$(P * Q)_n = \bigoplus_{n\text{-paintings}} P_{red} \otimes Q_{blue} \Big/ \begin{matrix}\text{repainting}\\ \text{blank areas}\end{matrix}$$

Here an n-painting is a disc divided into red and blue regions, with $2n$ marked points on the boundary, such that there is a pair of red points, followed by a pair of blue points, followed by a pair of red points, and so on.

E.g.

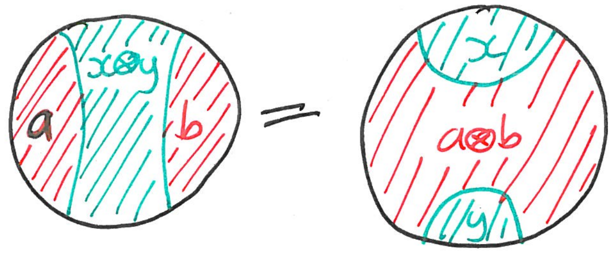


Then $P_{red} = \bigotimes_{\substack{\text{contiguous}\\ \text{red regions}\\ R}} P_{\frac{1}{2}\#R}$, where $\#R$ denotes the number of marked boundary points in the region R

and similarly for Q_{blue}.

In the example above, $P_{red} = P_1 \otimes P_1 \otimes P_1 \otimes P_0$ and $Q_{blue} = Q_3$

Finally, what is repainting?

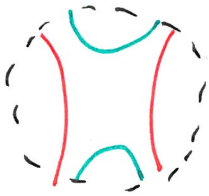


Equivalently, a 'blank' red area is considered the same as a blank blue area.

(Exercise, define amalgamated free products!)

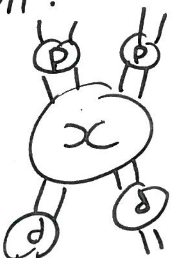
Planar tangles act by replacing strands $/ \mapsto //$.

Notice $(P*Q)_2$ contains a special element,

$p =$ 

which is a 'biprojection':

Moreover, we can recover P and Q from the free product and this element:

$P_n \cong \Big\{$  $\Big| \; x \in (P*Q)_n \Big\}$

$Q_n \cong \Big\{$  $\Big| \; x \in (P*Q)_n \Big\}$

and with a bit of thought also the planar structure.

In fact, in any planar algebra with a biprojection p,
we can define P and Q in this way,
and construct a map

$$P * Q \longrightarrow F$$

of planar algebras sending the canonical biprojection to p.

On the subfactor side, this situation corresponds to an intermediate subfactor.
If F is the planar algebra for the subfactor $A \subset C$,
a biprojection p gives an intermediate subfactor

$$A \subset B \subset C$$

so P and Q are the planar algebras for $A \subset B$ and $B \subset C$.

Although intermediate and composite planar algebras are extremely interesting, they don't play a significant role in the small index classification.
We'll only need:

<u>Lemma</u>: if some but not all the depth 2 bimodules have dimension 1, the sum of these and $\frac{1}{\delta} \frac{\cup}{\cap}$ is a biprojection, hence there is an intermediate subfactor, so the index is composite.

Lecture III: classification of small examples.

Before we start:

Q Why do we need to limit ourselves to 'small' cases?

* the $R^G \subset R$ examples show the problem contains the problem of classifying finite groups
* the planar algebra isn't always a complete invariant of the subfactor

Q What should 'small' mean?

There are a number of obvious options:

* 'rank' or number of simple bimodules
* 'global dimension', $\sum_{\substack{N-N \\ \text{bimodules}}} (\dim V)^2$

  (for $R^G \subset R$, this is $|G|$.)
* 'index'

Today we'll focus solely on index, essentially for historical reasons.

(Even at relatively small index things get hard fast: any finite quotient of $\mathbb{Z}/2 * \mathbb{Z}/3$ gives an index 6 subfactor $R^{\mathbb{Z}/2} \subset R \rtimes \mathbb{Z}/3$.)

<u>Theorem</u> (Ocneanu, et al.)

All subfactors with index <4 have principal graph an ADE diagram. Moreover, there are the following numbers of realizations of each:

A_n	D_{2n}	D_{2n+1}	E_6	E_7	E_8
1	0	1	2	0	2

<u>Theorem</u> (Popa, et al.)

All subfactors with index 4 have principal graphs which are simply-laced affine Dynkin diagrams.

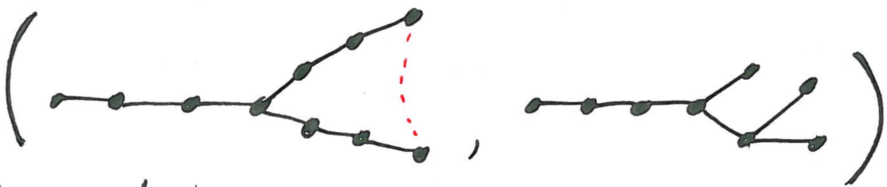
$A_n^{(1)}$	$D_n^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$	A_∞	D_∞
n	$n-2$	1	1	1	1	1

In fact, all of these are realized as Vec_G^ω for G a finite subgroup of $SU(2)$, and $\omega \in H^3(G, \mathbb{C}^*)$.

<u>Theorem</u> (arXiv:1007.1730 Morrison-Snyder
arXiv:1007.2240 Morrison-Penneys-Peters-Snyder
arXiv:1109.3190 Izumi-Jones-Morrison-Snyder
arXiv:1010.3797 Penneys-Tener)

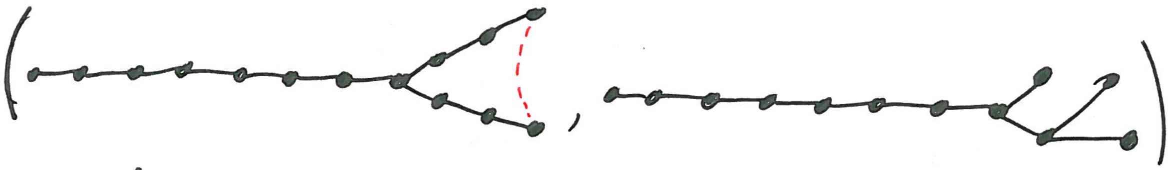
An extremal subfactor with index between 4 and 5 has standard invariant TL (with principal graph A_∞) or one of the following 10 cases:

- the Haagerup planar algebra with index $\frac{5+\sqrt{13}}{2}$, principal graph



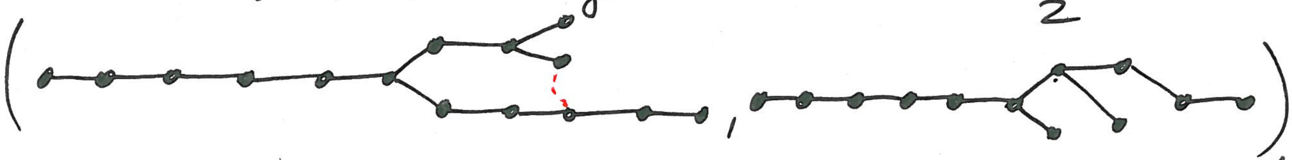
and its dual

- the extended Haagerup planar algebra



and its dual

- the Asaeda-Haagerup planar algebra with index $\frac{5+\sqrt{17}}{2}$

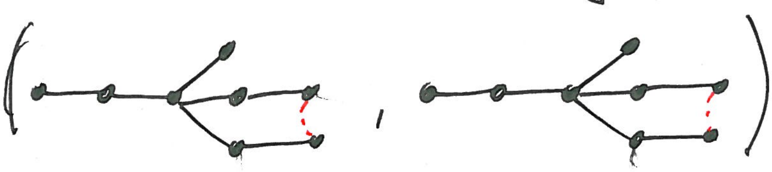


and its dual

- the 3311 GHJ planar algebra with index $3+\sqrt{3}$

and its dual

- and ~~the~~ Izumi's 2221 planar algebra with index $\frac{5+\sqrt{21}}{2}$



and its complex conjugate.

Our general approach is:

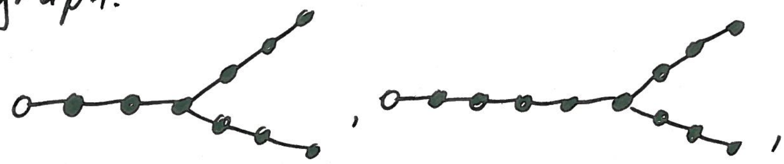
1. Enumerate possible principal graphs
2. Eliminate as many as possible using combinatorial or number theoretic constraints
3. Directly classify the planar algebras realizing each of the remaining graphs.

Enumerating principal graphs

Solely using combinatorial constraints, we cannot expect to get down to finitely many principal graphs up to some index limit L.

We need to introduce infinite families to organize the enumeration.

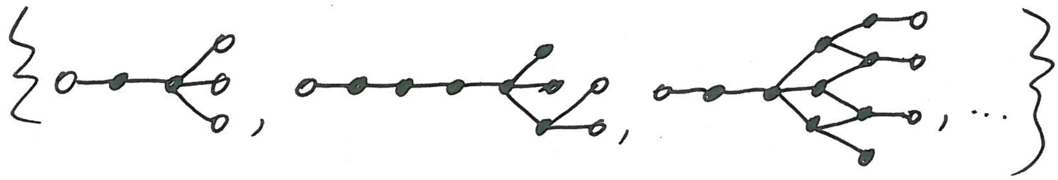
A vine represents all graphs obtained by 'translating' the given principal graph.

Example: $\{$  $, \ldots \}$

(we always translate by even amounts, to preserve parity.)

(We always work with a principal graph pair, with dual data, but I'll often only draw one of the graphs for brevity.)

A *weed* represents all graphs obtained by translating or 'extending' the given principal graph.

Example 

Clearly the weed  represents all irreducible principal graphs.

Now, a 'classification statement' (Γ_0, L, V, W) is the theorem

"Every subfactor principal graph ~~is~~ represented by the weed Γ_0 with index ~~below~~ in $(4, L)$ is in fact represented by one of the vines in V or one of the weeds in W."

We can begin with the trivial classification statement

$$(\bullet\!\!-\!\!\bullet, L, \{\}, \{\bullet\!\!-\!\!\bullet\})$$

but we'd like to reach a classification statement in which all the vines and weeds are 'manageable'

As it will turn out, all vines are 'manageable', in the sense that ~~we can~~ only finitely many of the represented graphs have cyclotomic index, and we can effectively determine these.

What do we do with an 'unmanageable' weed?

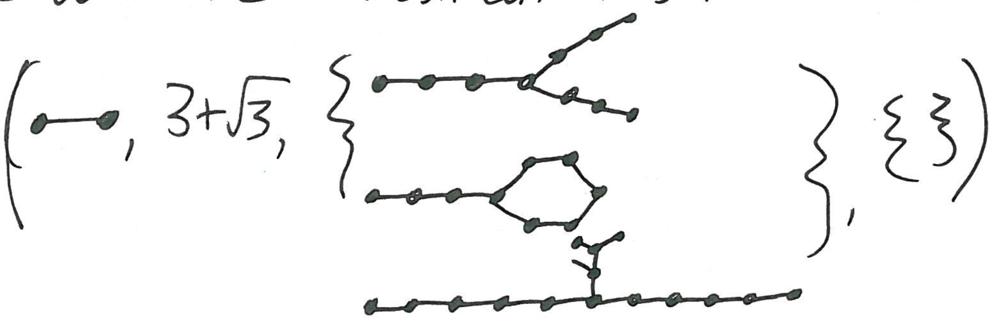
<u>Meta-theorem</u>: $(\Gamma_0, L, V, W \cup \{B\}) \Rightarrow (\Gamma_0, L, V \cup \{B\}, W \cup W_B)$

where W_B denotes all the depth 1 extensions of B staying below the index limit L, and further satisfying associativity.

<u>Lemma</u> W_B is always a finite set, as adding too many edges increase the index above L.

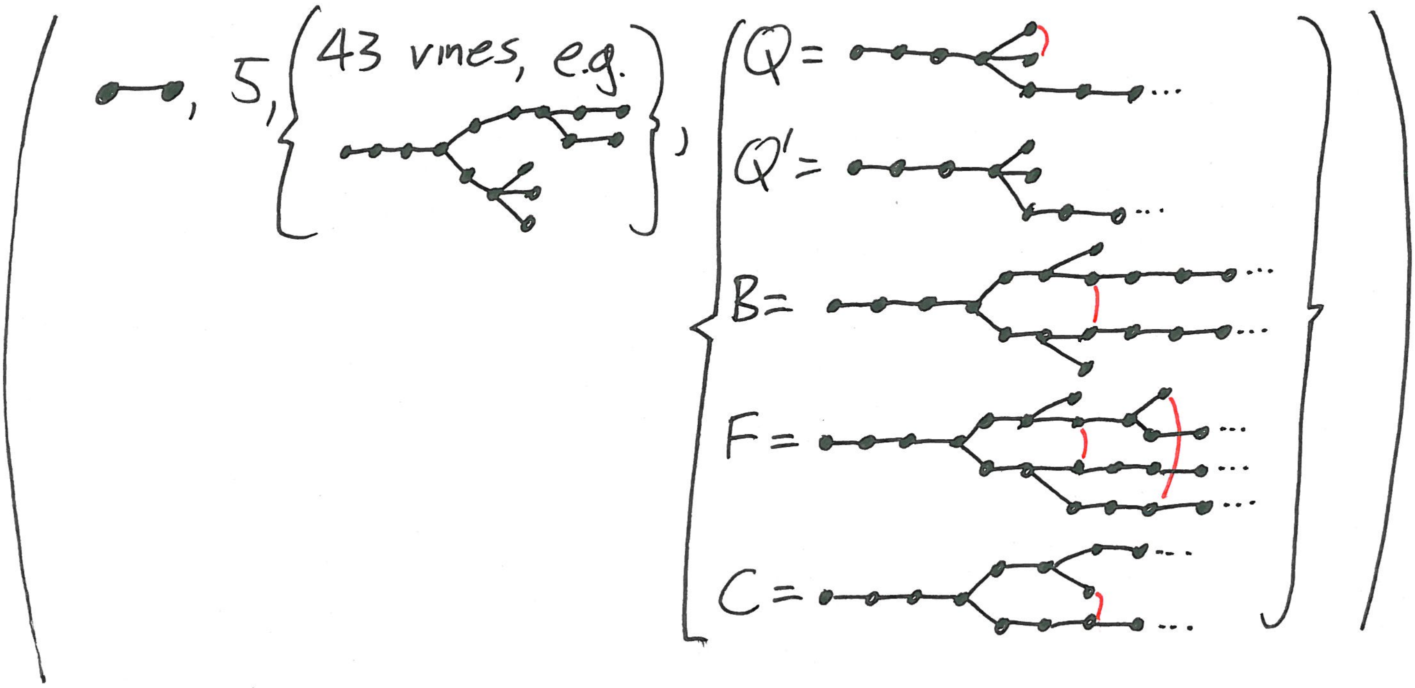
The challenge now is to skillfully apply the meta-theorem.
Each time we process a weed B in this way, we learn more, because each graph in W_B has been determined out to one greater depth, and hence might become more manageable.
We need to be careful the set of weeds does not grow too big, or to use a very large computer!

If we're very lucky, W_B might be empty, and eventually we have only weeds.
This is the case below index $3+\sqrt{3}$, and Haagerup's initial result there was the classification statement



Up to 5, however, we have to chose a judicious place to stop. (In particular, by developing new tools to rule out persistent weeds.)

Thus we reach the classification statement



What to do about vines?

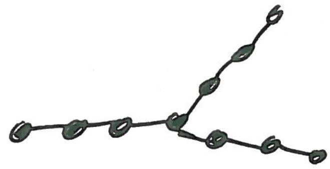
Coste and Gannon, building on work of De Boer and Goeree, showed the entries of the S-matrix of a modular category (and hence the dimensions of simple objects) are cyclotomic integers.

(Sketch: the Galois group acts faithfully on S either by permuting rows or columns; since these operations commute, the Galois group is abelian, hence the entries are cyclotomic.)

Etingof-Nikshych-Ostrik showed this implies the dimensions of simple objects in any fusion category (not necessarily modular or even braided) are also cyclotomic.

(In fact $\mathbb{Q}(\{\dim X\}_{X\in\mathcal{C}}) = \mathbb{Q}(\{\dim Y\}_{Y\in Z(\mathcal{C})})$.)

Richard Ng has recently given a constructive proof, writing the dimensions explicitly as sums of roots of unity.

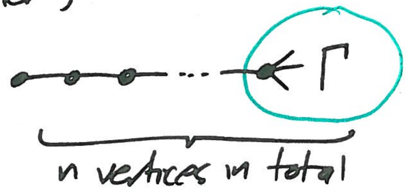
Asaeda-Yasuda used this obstruction to show the vine  only admitted two translations, by 0 and by 4.

Other vines required other ad-hoc arguments.

In fact, all dimension functions (and hence multiplicity free eigenvalues of the principal graph) must be cyclotomic.

Eventually, we found a hammer that deals with all vines.

<u>Theorem</u> (Calegari-Morrison-Snyder)

In a family of graphs $\Gamma_n =$ 

either 1) $\Gamma_n = A_n$ or D_n, or

2) there is an effective constant N, and $\forall n \geq N$, the adjacency matrix of Γ_n has a multiplicity free eigenvalue λ so $\mathbb{Q}(\lambda^2)$ is not abelian.

(In fact, we can show eventually the largest eigenvalue λ has $\mathbb{Q}(\lambda^2)$ not abelian but N may now be very large.)

Penneys-Tener in 'Subfactors with index less than 5; part IV' showed how to compute these constants, and checked all the cases below N as well.

This reduces the 43 vines from part I to 28 individual cases, and some other number theoretic results in that paper get us down to just 4 cases: H, EH, AH and 2221.

What about the weeds?

We need to learn about <u>connections</u>.

First, given a pair of principal graphs and dual data, we can define a <u>two-sided graph planar algebra</u>

- two string types, called 'dexter' and 'sinister'

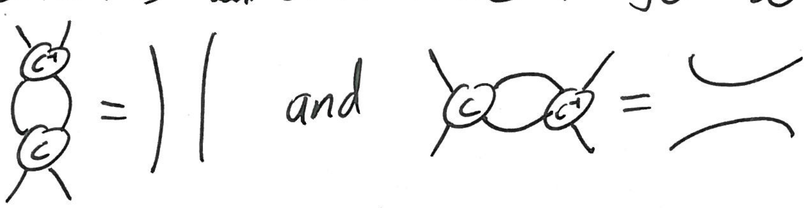
- P_w, for w a word in $\{d,s\}$.

$$= \left\{ \text{loops on } \begin{tikzcd} M\text{-}N \arrow[r, dash, "-\otimes X"] \arrow[d, dash, "X^* \otimes -"'] & M\text{-}M \arrow[d, dash, "X^* \otimes -"] \\ N\text{-}N \arrow[r, dash, "-\otimes X"] & N\text{-}M \end{tikzcd} \quad \begin{array}{l} \text{following horizontal edges at 'dexter' steps} \\ \text{vertical edges at 'sinister' steps} \end{array} \right\}$$

- the same action of planar tangles as for the usual graph planar algebras.

A connection is merely an element $c \in P_{dsds}$.

A connection is bi-invertible if $\exists c^{-1}$ so



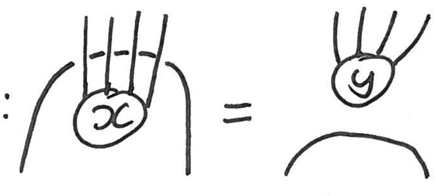
A connection is bi-unitary if $c^{-1} = c^*$.

There is a <u>gauge group</u> $\mathcal{G}^d = \prod_{\substack{\text{bimodules} \\ P,Q}} U(\dim\langle P \otimes X, Q\rangle)$, $\mathcal{G} = \mathcal{G}^d \times \mathcal{G}^s$.

which acts of P, preserving bi-unitary connections.

Given a connection we can define the planar algebra of flat elements:

$$F_n = \left\{ x \in P_{\underbrace{dd\cdots d}_{n \text{ times}}} \;\middle|\; \exists y \in P_{ss\cdots s} : \right.$$

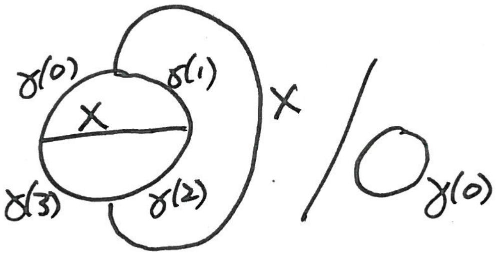


$$\left. \right\}$$

Gauge equivalent connections give equivalent planar algebras.

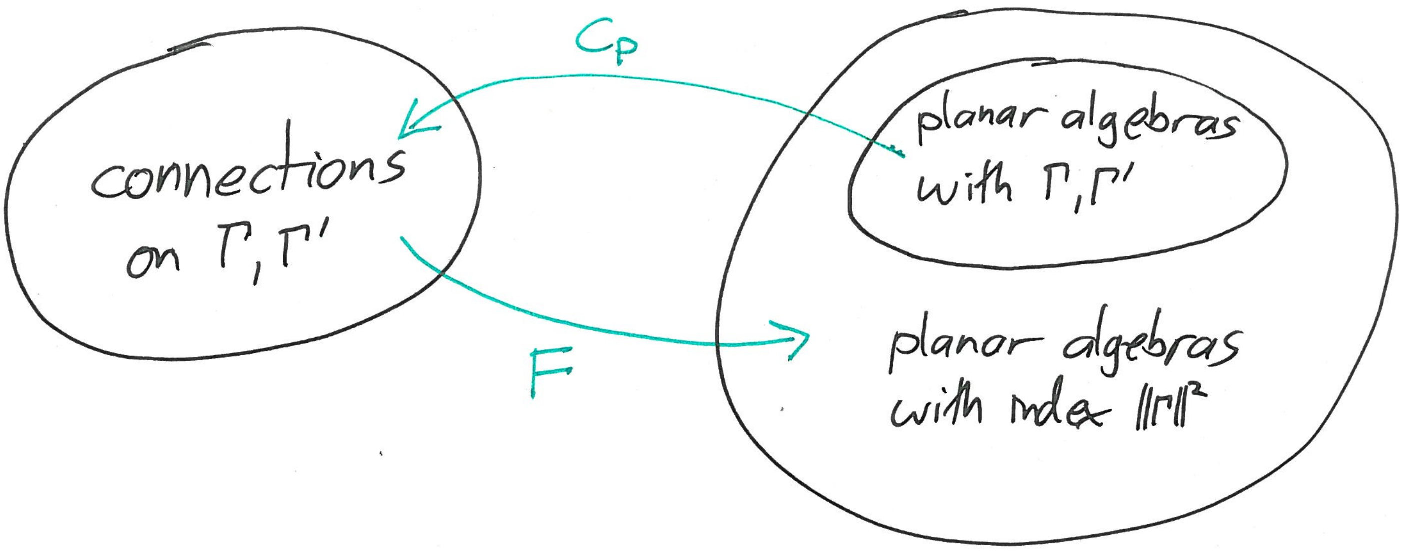
Conversely, every planar algebra P with principal graphs (Γ, Γ') gives a connection $C_P \in GPA(\Gamma,\Gamma')_{dsds}$

via $C_P(\gamma) =$

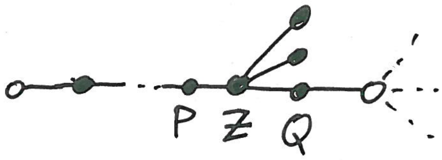


(here the edges are labelled by idempotents/vertices of Γ, Γ', vertices are labelled by intertwiners/edges of Γ, Γ')

and the flat elements recover P.



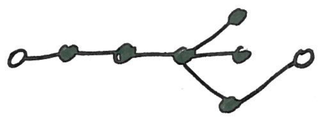
Theorem

A connection on 

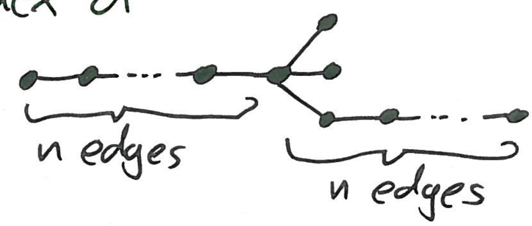
exists only if $\dim P = \dim Q$.

(Proof: unitarity of the 4×4 matrix of connection entries passing through Z and Z^*) arXiv:1109.3190

Corollaries:

* a subfactor represented by the weed 

  has index equal to the index of



* in fact has principal graph nn11
* in fact has principal graph 3311

  (otherwise the index isn't cyclotomic)

There's a circle worth ~~of~~ of bi-unitary connections on 3311. An analysis of the relationship between the connection and the rotational eigenvalues of the planar algebra generators picks out two points giving 3311 subfactors.

(For the others, the flat subalgebra is just Temperley-Lieb.)

We still need to eliminate the weeds B, C and F. This was done in 'Subfactors with index less than 5, part II', arXiv: 1007.2240, but there are now better arguments.

Say a principal graph is <u>stable</u> at depth n if each vertex at depth n is connected to at most one, distinct, vertex at depth $n+1$ (on both graphs).

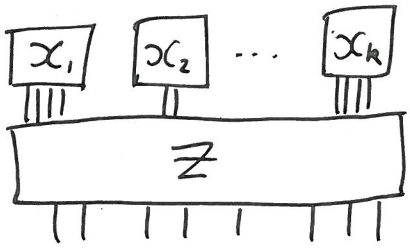
<u>Theorem</u> (Popa, c.f. Bigelow-Penneys)
Stable at depth $n \Rightarrow$ stable at depth $n+1$.

Since the weed B is stable at depth 6, it must remain stable, and these graphs are easy to rule out.

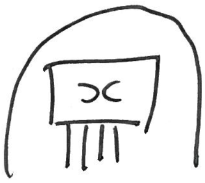
∎

Sketch of theorem

For a subset $W \subset P_\bullet$, define

trains$(W) = \Big\{ \sum$  $\Big| \; x_i \in W, \; Z \in TL \Big\}$

If P is stable at depth n, for any $x \in P_{n,\pm}$

 $\in$ trains$(P_{\leq n})$ $\qquad (*)$

Let Q be the subalgebra of P generated by $P_{\leq n}$.

Repeatedly applying $(*)$, every element of Q is in trains$(P_{\leq n})$.

Therefore, $\Gamma(Q)$ agrees with $\Gamma(P)$ up to depth n, but is stable thereafter.

Note index P = index Q, so $\|\Gamma(P)\| = \|\Gamma(Q)\|$.

An analysis of Frobenius-Perron eigenvalues shows $\Gamma(P) = \Gamma(Q)$ (and indeed $P = Q$).

Theorem (Snyder) arXiv:

Consider an $n-1$ supertransitive subfactor with excess 1. (i.e both graphs begin with a triple point)

Suppose a vertex on Γ at depth n is univalent.

Let r be the ratio of dimensions of the vertices of Γ' at depth n.

Let λ be the rotational eigenvalue of the new n-box.

Then $r + \frac{1}{r} = \frac{\lambda + \lambda^{-1} + 2}{[n][n+2]} + 2$

Sketch

For the 3×3 matrix U of connection entries passing through the two branch points, there is a 'nice' gauge choice (using the TL-intertwiners).

In this gauge choice, we have a formula relating $\rho^{\frac{1}{2}}$ and U, acting on elements in $TL^{\perp}$.

Finally, the identity falls out of unitarity for U. $\square$

Applying this to a weed is somewhat tricky!
We don't know r, n or the index.

However, F&C are 'manageable' in that
we can write r, $[n]$ and $[n+2]$ in terms of n and q.

Using $-2 \leq \lambda + \lambda^{-1} \leq 2$, we rule out all but finitely
many n, and for the remaining cases
see λ is not a root of unity.

Lecture IV: constructing exotic examples

Today we'll tackle the problem of constructing subfactors 'with our bare hands'.

Given a principal graph (or the entire combinatorial fusion data) can we determine how many (if any!) subfactors realize it?

3 'general purpose' approaches:

Ⓐ Look for 6-j symbols satisfying the pentagon equations.

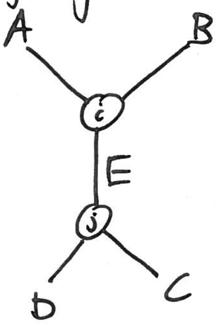
Gauge orbits of solutions $\overset{1:1}{\longleftrightarrow}$ planar algebras

Ⓑ Look for subalgebras of the graph planar algebra. Maybe we can identify planar equations satisfied by some element? Even if we can locate such elements, how do we characterize the subalgebra they generate? Show it has the right principal graph?

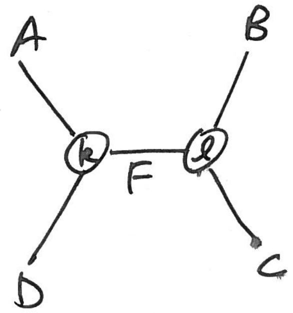
Ⓒ Find bi-unitary connections, and identify which give ~~the~~ flat algebras with the right principal graph.

Solving the pentagon equations is generally totally impractical.

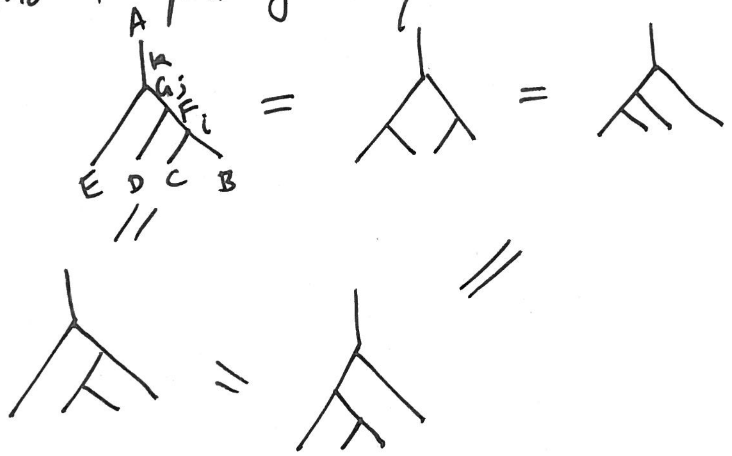
The 6j-symbols give the ~~equations~~ identities

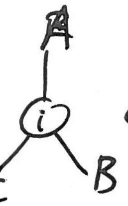


$$= \sum F^{AB\;Eij}_{DC\;Fkl}$$

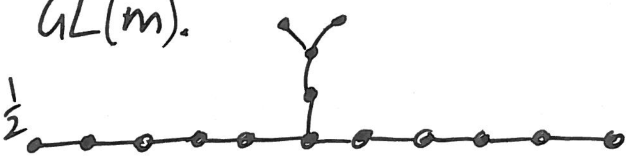


and the pentagon equations are



For each trivalent vertex space  $\in \mathrm{Hom}(1 \to A \otimes B \otimes C)$

we have a gauge group $GL(m)$.

E.g. for the even part of Asaeda-Haagerup 

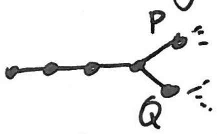
there are 465976 equations in 22208 unknowns

and a 356 dimensional gauge group

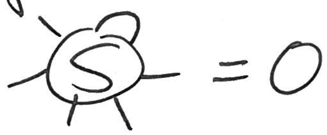
The tools of applied algebraic geometry are hopelessly inadequate.

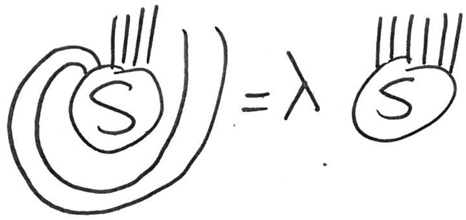
To search in the GPA, we need some equations.

For Haagerup, Asaeda-Haagerup or extended Haagerup, (or generally magic numbers $10^{n-1}1\ldots$)

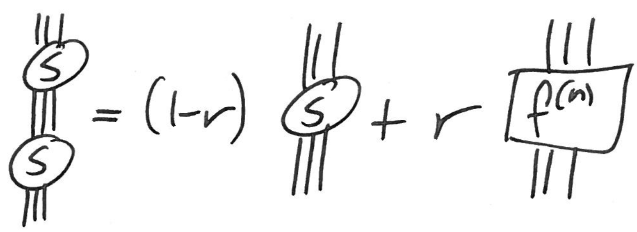


the planar algebra must contain an element $S \in P_n$ satisfying:

① 

② 

and

③ 

where $r = \frac{\dim P}{\dim Q} > 1$

(since S and $f^{(n)}$ must form a 2d algebra with idempotents with traces $\dim P$ and $\dim Q$)

Using Jones/Snyder's triple point obstruction, we usually know λ.

Thus the first step is to solve the linear equations ①&② in the graph planar algebra.

This is easy and cuts down to a small vector space (e.g. dim = 4, 16, 21 for H, AH, EH.)

We then need to solve ③. This is already very hard for EH and especially AH.

(The best strategy is to solve by hand for about half the variables, the use numerical methods to isolate the discrete solutions, then guess and check algebraic solutions)

At this point we have a candidate subalgebra

$$\mathcal{G}(S) \subset GPA(\Gamma)$$

(and indeed, if such a planar algebra exists it must be $\mathcal{G}(S)$ for one of our solutions S.)

This subalgebra is certainly unitary and spherical (inherited from the GPA)

but it is not obvious that $\mathcal{G}(S)_0 = \mathbb{C}$, or $\Gamma(\mathcal{G}(S)) = \Gamma$, or even $\mathcal{G}(S) \subsetneq GPA(\Gamma)$!

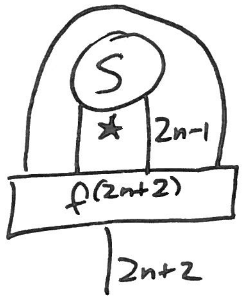
To show $\mathcal{G}(S)_0 = \mathbb{C}$, we need to identify relations satisfied by S that suffice to evaluate closed diagrams built from S.

(Often, this also lets us show $\dim(\mathcal{G}(S)_k) = \dim \mathcal{TL}_k$ for $k < n$, and indeed lets us identify $\Gamma(\mathcal{G}(S))$.)

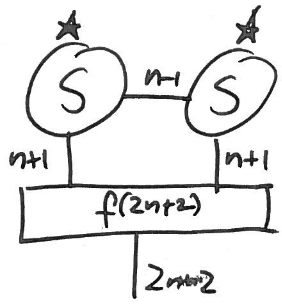
Our favourite approach to this is 'jellyfish relations' (5)
pioneered in the construction of extended Haagerup.
(arXiv:0909.4099)

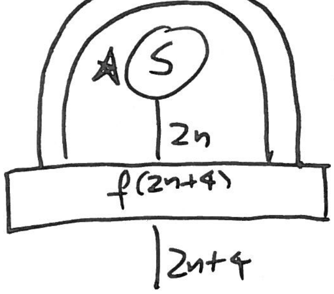
Various generalizations are proving very fruitful,
and we're still discovering new consequences.

Theorem the 'candidate generators' for H $(n=4)$ and EH $(n=8)$
satisfy relations

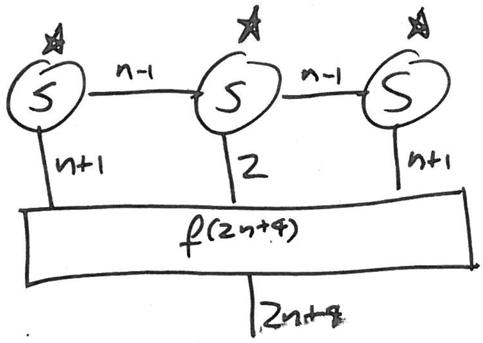


$$= i\,\frac{\sqrt{[n][n+2]}}{[n+1]}$$





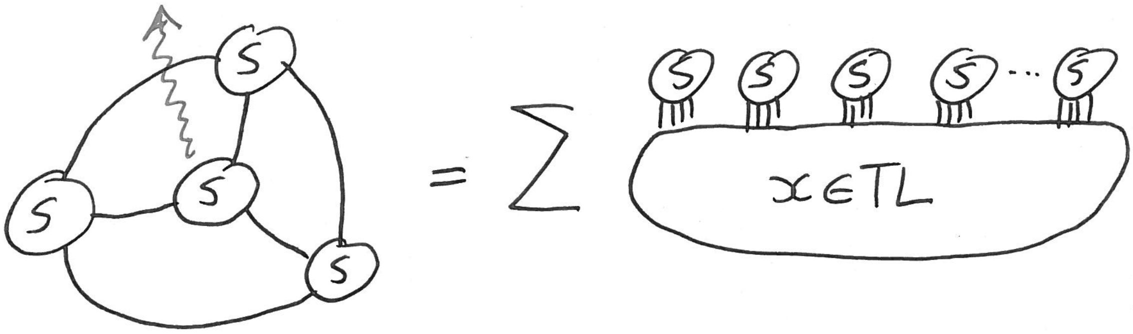
$$= \frac{[2][2n+4]}{[n+1][n+2]}$$



Theorem these relations suffice to evaluate all closed diagrams.

Proof

Float all the jellyfish to the surface:



Next, a simple argument about diagonals in polygons shows there must be a pair of S's connected by at least n strands.

The relation $S^2 = (1-r)S + r f^{(n)}$ lets us rewrite the diagram with fewer S's, all still 'at the surface'.

Eventually we're left with just Temperley-Lieb diagrams, which are easy to evaluate $\square$.

This evaluation algorithm is remarkable in that it increases the 'complexity' at intermediate steps.

With little additional work we can completely determine $\Gamma(G(S))$.