The quantum exceptional trefoil

• What is a Lie algebra?
• What is Deligne's (quantum) exceptional series?
• What do we know about it?
• How might we prove it really exists?
Within each $A_n$, $B_n$, $C_n$, or $D_n$ series, there is a uniform labelling of irreps 
(E.g., for $A_n$, by partitions)
although at each $n$, some vanish.
(For $A_n$, those with $\geq n$ rows)
The dimension of each irrep is a rational function of the parameter $n$.
Tensor powers of the adjoint $g_n^{\otimes k}$ have uniform decompositions (although at each $n$ some irreps vanish).
The exceptional groups $G_2, F_4, E_6, E_7,$ and $E_8$ also exhibit this "uniform" behaviour.

As the coarsest example:

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\dim(\text{Inv}(g^\otimes k))$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$k=0$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>1</td>
</tr>
<tr>
<td>$F_4$</td>
<td>1</td>
</tr>
<tr>
<td>$E_6$</td>
<td>1</td>
</tr>
<tr>
<td>$E_7$</td>
<td>1</td>
</tr>
<tr>
<td>$E_8$</td>
<td>1</td>
</tr>
</tbody>
</table>

What might constitute an explanation of such phenomena?
What is a Lie algebra?

For today, a "(metric) Lie algebra" is a function

\[ F : \{ \text{vertex oriented cubic graphs} \} \rightarrow k \]

satisfying:

0) \[ F(G_1 \; G_2) = F(G_1) F(G_2) \]

1) \[ \begin{array}{c}
\begin{array}{c}
\text{antisymmetry}
\end{array}
\end{array} \]

2) \[ \begin{array}{c}
\begin{array}{c}
\text{IHX, or the}
\text{Jacobi identity}
\end{array}
\end{array} \]
Every honest metric Lie algebra provides an example!

- Pick a planar embedding of the graph
- Scan from bottom to top, interpreting

\[
\wedge \quad \text{as the Lie bracket } \ g \circ g \rightarrow g
\]

\[
\times \quad \text{as the switch map } \ g \circ g \rightarrow g \circ g
\]

\[
\wedge \quad \text{as the bilinear pairing } \ g \circ g \rightarrow k
\]

and \[ \vee \] as its dual

- We obtain a map \( k \rightarrow k \), i.e., an element of \( k \).
In the other direction, such a function $F$ determines a metric Lie algebra object in some symmetric pivotal category $C$.

We define
\[
\text{Obj}(\mathcal{E}) = \mathbb{N}
\]
\[
\text{Hom}_\mathcal{E}(n \rightarrow m) = k \left\{(\text{vertex oriented) cubic graphs with } n \text{ inputs and } m \text{ outputs}\right\} / R
\]
where the ideal $R$ is the 'negligible' morphisms
\[
R = \{ x \mid \forall y \in \text{Hom}(m \rightarrow n), \quad F(x \cdot y) = 0^3 \}.
\]
We then take $C$ to be the idempotent completion (aka the 'Karoubi envelope') of $\mathcal{E}$ — adjoin objects for each idempotent in $\text{Hom}(n, n)$. 
Does this recover the representation category, if we start with an honest Lie algebra.
Not quite!

1. We only recover representations appearing in a tensor power of the adjoint representation (highest weights in the root lattice, or equivalently representations of the adjoint form).

2. Even then, we might not realize all morphisms as graphs built out of the Lie bracket. In this case some irreps ‘glom’ together (equivariantly).

This happens in type $A$, type $D_n$ for $g_{\leq k}$ with $k \geq n$, and type $E_6$, but not otherwise.
In fact, any "Lie algebra" with dimensions 1,0,1,1,5 must satisfy relations (in addition to multiplicativity, antisymmetry, and IHX)

0) \( C = d \)

1) \( \phi = \text{zero} \)

2) \( \phi = 6 \) 

(since we can rescale the trivalent vertex, this is an arbitrary but convenient normalization)

3) \( \Delta = 3 \) 

(there must be some relation \( \Delta = \text{triv} \) since \( \dim \text{lnv} (q^0) = 1 \); the compute \( \Delta \) using this relation or using IHX)

4) \[ \begin{array}{c}
\hline
\hline
\hline
\end{array} \]

(there must be some relation amongst these 6 diagrams since \( \dim \text{lnv} (q^0) = 5 \); easy skein theory determines the coefficients)
Deligne’s conjectures:

1. These relations uniquely determine a function on cubic graphs.

2. Over \( \mathbb{Q}(d) \), the associated category is semisimple and nonzero.

3. At the following values of \( d \), there are additional negligible morphisms, and the quotients are representation categories of the exceptional groups:

<table>
<thead>
<tr>
<th>( d )</th>
<th>( \text{Rep}_g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>( A_1 )</td>
</tr>
<tr>
<td>8</td>
<td>( A_2 : 2 )</td>
</tr>
<tr>
<td>14</td>
<td>( G_2 )</td>
</tr>
<tr>
<td>52</td>
<td>( F_4 )</td>
</tr>
<tr>
<td>78</td>
<td>( E_6 : 2 )</td>
</tr>
<tr>
<td>133</td>
<td>( E_7 )</td>
</tr>
<tr>
<td>248</td>
<td>( E_8 )</td>
</tr>
</tbody>
</table>
We have a 'quantum' version of Deligne's conjectures:

(Conjecture (M-Snyder-D.Thurston))

- There is a unique invariant of knotted cubic graphs with dimensions 1, 0, 1, 1, 5, 16

- It is determined by
  1) \( \gamma = v^{12} \) and \( \delta = -v^6 \gamma \)
  2) \( \varphi = \text{zero} \)
  3) \( v^{-3} \frac{v^{-7} - v^{-1}}{v + v^{-1}} \alpha \left( \frac{v^{13} - v^{-8}}{v^{1} - v^{-1}} \right) = 0 \)

where \( \alpha = -\frac{(v - v^{-1})(v^{1} - v^{-3})}{v^{1} - v^{-1}} \).

- Specialising at \( d = \dim_{\mathbb{C}} g \), \( v^{12} = q^{\alpha_{1} + 2\alpha_{2}} \)
  recovers the (adjoint) representation category of \( U_{g} \) for \( g = A_{1}, A_{2}, 2, G_{2}, F_{4}, E_{6}, 2, E_{7}, E_{8} \).

(We won't be too surprised if this isn't quite right — perhaps you need to specify a few more dimensions, or impose some extra relations...
How might we prove such a conjecture? (where does it even come from?)

Just knowing we have an invariant of knotted graphs with specified dimensions, we can derive many relations between small graphs with boundary (and guess at more).

If we can find enough such relations, we may be able to show that they suffice to evaluate any knotted graph by successive simplifications.

This would prove uniqueness.

Proving existence seems really hard!

- If we have enough relations to simplify, presumably there are many choices along the way. Perhaps we could show the answer is independent of the choices (e.g. by a collection of diamond lemmas).

- Perhaps we can guess a nice basis for each Hom space, and find enough relations to rewrite everything into that basis.

- ??
How do we derive relations in a braided category generated by a trivalent vertex with \( \dim \text{Inv}(X^{\otimes k}) = 1, 0, 1, 1, 5, 16, \ldots \) ?

We start with relations
\[
\mathcal{O} = d, \quad \mathcal{Q} = \text{zero}, \quad \mathcal{Q} = \mathcal{B}, \quad \mathcal{A} = t \mathcal{A}
\]

and
\[
X = \alpha_1 \circ \circ \circ + \alpha_2 \circ \circ \circ + \alpha_3 \circ \circ \circ + \alpha_4 \circ \circ \circ + \alpha_5 \circ \circ \circ.
\]

We must have
\[
\gamma = \gamma_{12} \circ \circ \circ \quad \text{for some \( \gamma \), and}
\]

\[
\gamma = \gamma_0 \circ \circ \circ \quad \text{(since \( \gamma = \gamma_0 \)).}
\]

Then computing pairings of \( X \) with other diagrams (e.g. \( \langle X, \circ \circ \circ \rangle = \circ \circ \circ = \circ \circ \circ = t \mathcal{O} = t \mathcal{B} \mathcal{D} \)) we can find the \( \alpha_i \) as rational functions in \( d, b, t, v, \) and \( \Box \).
We can compute

\[ O = \langle \mathcal{O}^{-1}, \mathcal{X} \rangle \quad \text{for} \quad \mathcal{X} \in \{ \mathcal{O}, \mathcal{X}, \mathcal{X}, \mathcal{X} \} \]

and learn the values of

\[ \square, \star, \text{and} \quad \star \]

as rational functions in \( d, b, t, \) and \( v. \)

Next

\[ O = \langle -|K^-|, \mathcal{X} \rangle \quad \text{for} \quad \mathcal{X} \in \{ \mathcal{S}, \mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X} \} \]

and rotations

and discover

\[ t = b \cdot \frac{\sqrt{10} \cdot \sqrt{v^2 + (d-2) + v^{-8} + v^{10}}}{(v^2 + v^{-2})(v^8 + d + v^{-8})} \quad (\text{notice} \ v \to 1 \ \text{gives the classical} \ t = b/2.) \]

and the values of several more polyhedra.

A change of variables and choosing a normalization of \( \mathcal{X} \) and we've discovered the 'quantum IHX' relation of the conjecture.
Next, the 6 diagrams
\[ \{, \circ, \times, \triangle, \square, \Box \} \]
must have a relation between them, which we can detect as the kernel of the matrix of inner products.
(Which has entries rational functions in $v$ and $w$, using the polyhedral relations discovered so far.)

Now we can simplify adjacent squares as linear combinations of ‘simpler’ diagrams.
Onwards and onwards, we see

\begin{align*}
\tau & \quad + \text{ rotations,} \\
\exists & \quad + \text{ rotations,} \\
\varphi & \quad + \text{ rotations, and}
\end{align*}

must form a basis for \( \text{inv}(X^{\otimes 5}) \).

Thus there's some relation simplifying

\begin{align*}
\text{as a linear combination of these.}
\end{align*}

We still can't simplify all polyhedra, e.g.

\begin{align*}
\text{But } \dim \text{inv}(X^{\otimes 6}) = 80 \Rightarrow \quad \text{as } + \text{ lower order terms}
\end{align*}

\begin{align*}
= & \quad + \text{ lower order terms}
\end{align*}

and thus lets us use the branding to evaluate

the dodecahedron.
We're still far from having "enough" relations, but there are hopefully plenty more to discover.

We already know enough about the quantum exceptional series to compute some knot invariants.

These are a good sanity check, and also amusing.

Here's the quantum exceptional trefoil:

$$E_{\theta}(\mathbb{R}) = -\frac{d}{z^{16}(1-dz+z^{2})} \times$$

$$\left(-1-2z^{2} - z^{4} - z^{5} + z^{6} - z^{7} + 2z^{8} + 2z^{10} - z^{12} + 4z^{13} - z^{14} + 2z^{15} - z^{17} + z^{18} - 3z^{19} + z^{20} - z^{21} - z^{22} + z^{23} - 6z^{24} - 4z^{26} - 2z^{27} + z^{28} - 4z^{29} + 4z^{30} - 2z^{31} + 3z^{32} + z^{33} + 3z^{35} + 2z^{37}\right)$$

$$+ d (-2z^{6} + 2z^{7} - 3z^{8} + 2z^{9} + 6z^{12} - 2z^{13} + 5z^{14} + 6z^{17} - 5z^{18} + 6z^{19} - 5z^{20} - 2z^{21} - 6z^{23} + 3z^{24} - 4z^{25} + 3z^{26} + 2z^{28} - z^{30} - z^{36})$$

$$+ d^{2} (-2z^{12} + z^{13} - 2z^{14} + z^{15} - 2z^{17} + 2z^{18} - 2z^{19} + 2z^{20} + 2z^{23} - z^{24} + z^{25} - z^{26})$$

Plugging in $d=14$, $z = q^{\frac{11}{14}+2\rho} = q^{6}$ gives the quantum $G_{2}$ adjoint invariant.

Presumably plugging in $d=248$, $z = q^{15}$ gives the $E_{8}$ invariant!