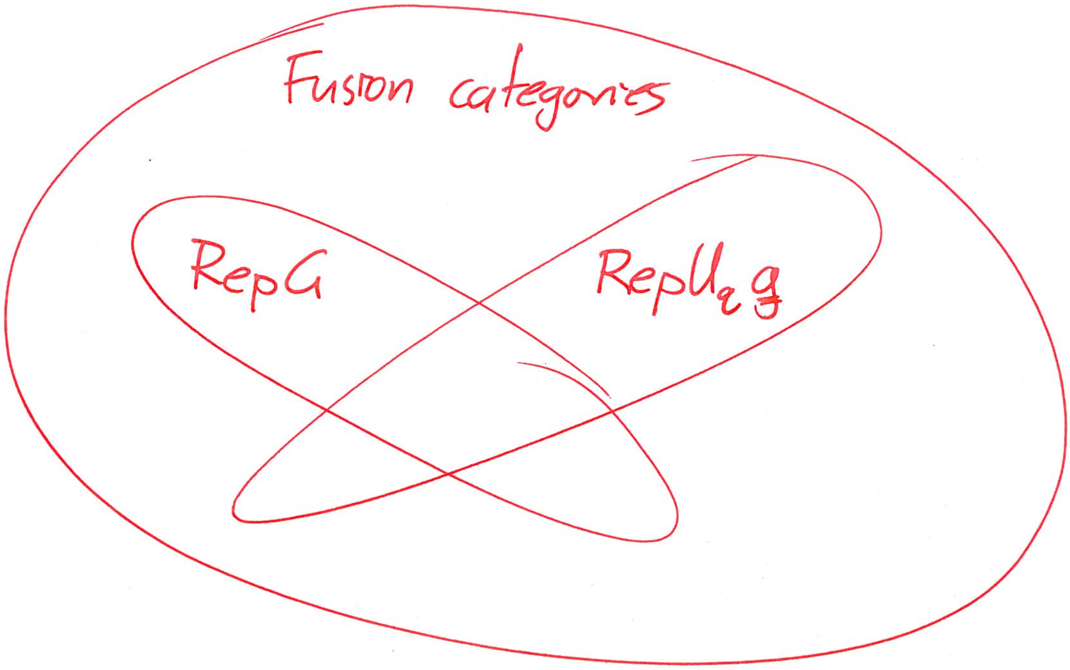
Fusion categories

A fusion category is a common generalization



of
- the category of representations of a finite group
- the (semisimplified) category of representations of a quantum group $U_q\mathfrak{g}$ at a root of unity.

Definition

A fusion category is a semisimple $\otimes$-category, with finitely many simple objects.

(Here we take $\otimes$-category to mean

- monoidal,
- linear, (Hom spaces are vector spaces)
- rigid (objects have well-behaved duals).)

Please just think about representations of finite groups to work out what these mean!

Every object in a fusion category has a dimension

$$\dim(X) = $$  $$\in k$$

the composition of the pairing and copairing maps

$$\cup^{X} : \mathbb{1} \to X^{*} \otimes X$$

$$\cap^{X} : X^{*} \otimes X \to \mathbb{1}$$

(Often, this is the asymptotic growth rate of $\dim \mathrm{Hom}(\mathbb{1} \to X^{\otimes n})$.)

This dimension is not necessarily an integer!
(Although it is an algebraic integer and in fact a cyclotomic integer.)

Just as a C^*-algebra is a 'noncommutative' Hausdorff space,

(or a von Neumann algebra is a noncommutative measure space)

a fusion category is a 'noncommutative' finite group.

A $\otimes$-category is symmetric if there are natural isomorphisms $X \otimes Y \to Y \otimes X$.

(giving a representation of S_n on each n-fold tensor product $X_1 \otimes \cdots \otimes X_n$)

<u>Theorem</u> (Deligne)

A symmetric fusion category with all objects having integer dimensions

(or for every object some Schur functor vanishes,

or for every object X $\dim \mathrm{Hom}(\mathbb{1} \to X^{\otimes n})$ is subexponential)

must be the representation category of some finite (super) group.

Why study fusion categories?

- We may learn something about finite groups!
- A fusion category gives a (2+1)-d local field theory:

$$M^3 \longmapsto Z(M^3) \in k, \text{ a number}$$
$$\Sigma^2 \longmapsto Z(\Sigma), \text{ a vector space}$$
$$\vdots$$

(in fact, there's a bijective correspondence.)

Come to more colloquium talk to hear more about this!

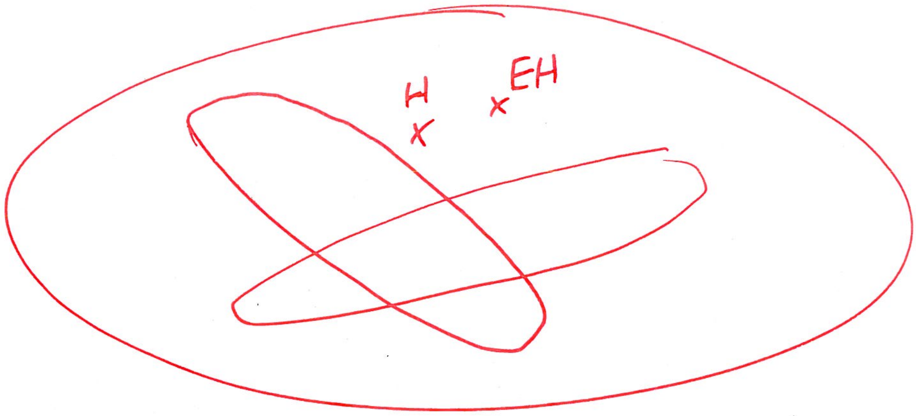
- A II_1 subfactor $N \subset M$

  (a pair of von Neumann algebras, with trivial centre, and a trace with $tr(1)=1$)

  gives a (not necessarily finite) fusion category, as the category of N-N bimodules generated by ${}_N M_N$.

This is a strong (with some hypotheses, complete) invariant of the subfactor.

Today I want to exhibit some points



outside the 'classical' regime, and discuss just how 'exotic' these are.

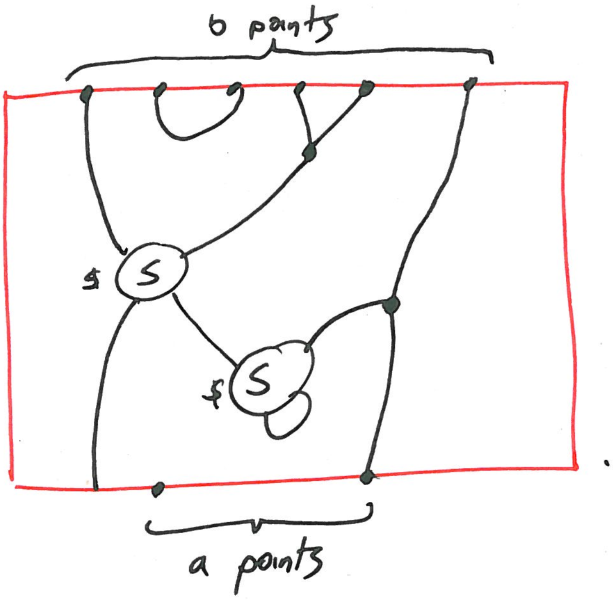
These fusion categories were initially discovered via systematic combinatorial searches.

(in fact looking for small index subfactors, cf. arXiv:1304.6141).

We'll consider a $\otimes$-category $\mathcal{C}$ determined by an integer parameter ℓ, $n = 2\ell + 2$

$\mathcal{C}$ is $\otimes$-generated by an self-dual object X, and morphisms $t: X \otimes X \to X$ and $S: \mathbb{1} \to X^{\otimes n}$.

That is, every object is a summand of $X^{\otimes k}$ for some k, and every morphism in $\mathrm{Hom}(X^{\otimes a} \to X^{\otimes b})$ is built out of t and S by 'planar' operations, i.e. (linear combination of) diagrams:



To interpret such a diagram, scan from bottom to top, interpreting

$\cup$ as the duality copairing $\mathbb{1} \to X \otimes X$

$\cap$ as the duality pairing $X \otimes X \to \mathbb{1}$

 as the map $t: X \otimes X \to X$

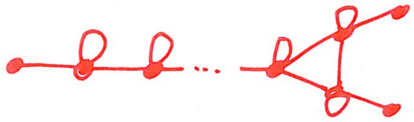
 as the map $S: \mathbb{1} \to X^{\otimes 2n}$

Finally, these morphisms satisfy 'local relations':

(first, some numbers:

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$$

$[3]$ is the largest eigenvalue of the adjacency matrix of



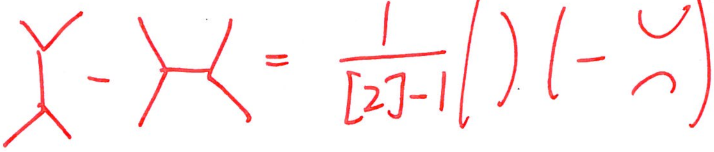
$$= 1 - \zeta^2 - \zeta^5 - \zeta^6 - \zeta^7 - \zeta^8 - \zeta^{11}$$

(for $l=0$)

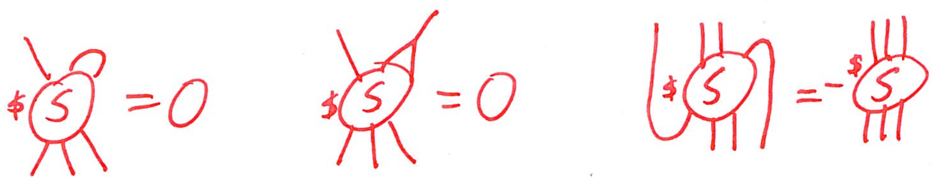
$$= 2 + \zeta^2 + \zeta^3 + \zeta^{10} + \zeta^{11} \quad (\text{for } l=1) \qquad \zeta = \exp(2\pi i/13)$$

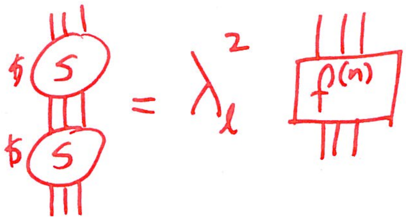
$$\lambda_l^2 = -\frac{1}{[2]^2}\frac{[2n+2]}{[2n]}\Big)$$

$$\bigcirc = [2]$$

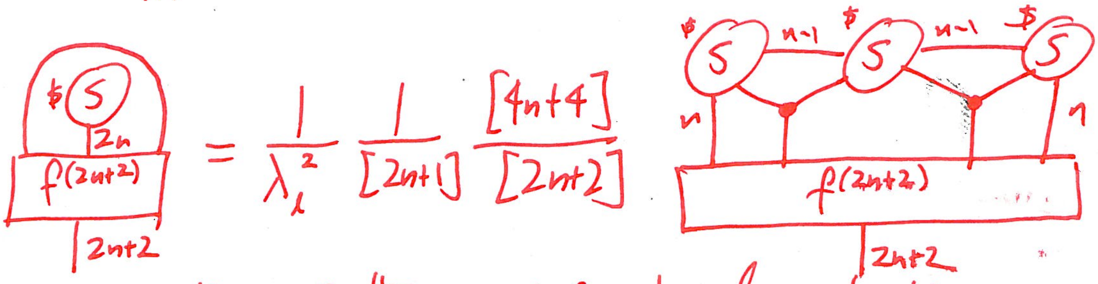


(Q)





and



where $f^{(j)}$ is the j-th "Jones-Wenzl idempotent", a specified linear combination of trivalent diagrams.

What a crazy definition!

<u>Theorem</u> (Haagerup, Bigelow–Morrison–Peters–Snyder) arXiv:0909.4099

For $\ell=0$ and $\ell=1$, this construction really gives fusion categories! ("Haagerup" and "extended Haagerup")

In particular,

- they don't collapse to zero
- $\mathbb{1}$ is simple (equivalently, these relations let you evaluate all closed diagrams)
- they are semisimple (in fact, unitary)
- and have finitely many simple objects.

<u>Theorem</u> (Asaeda–Yasuda, Calegari–Morrison–Snyder) arXiv:1004.0665

For $\ell \geq 2$, these constructions collapse: all morphism spaces are just the zero vector space.

<u>Proof</u> by number theory!

Theorem (many: see arXiv:1304.6141 and references therein)

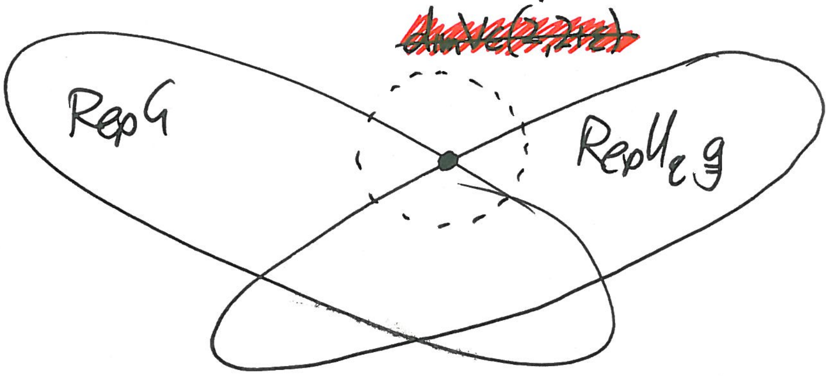
There are only three *unitary* fusion categories with a "Q-system" (Q) containing an object with dimension in $(2, \sqrt{3+\sqrt{3}})$.

(Equivalently, there are only three finite depth subfactors with index in $(4, 3+\sqrt{3})$.)

Two of these are realized by this construction.

Where do these categories sit?

$|\dim V - 2| < \varepsilon$



But are they 'classical'?

To see they are not, we introduce some number theory.

__Theorem__ (Brauer)
Rep G can be defined over $\mathbb{Q}(\zeta_e)$, for e the exponent of G.

__Theorem__
Rep $U_{\zeta_n}\mathfrak{g}$ can be defined over $\mathbb{Q}(\zeta_n)$
(by construction, using Weyl modules).

What field can we define $\mathcal{C}_0$ and $\mathcal{C}_1$ over?

Finally, let's try to understand which fields we can define these categories over.

In the relations given, every coefficient (at least for $l=0,1$) is in fact in $\mathbb{Q}(\zeta_{13})$.

Does this mean these fusion categories are defined over $\mathbb{Q}(\zeta_{13})$?

Not quite — it turns out they are not semisimple there!

(In order to be defined over $k \subset \mathbb{C}$, we need to be able to write down all the projections onto simple objects using only coefficients in k.)

(arXiv:1002.0168)

Theorem If the categories $\mathcal{C}_{l=0,1}$ are defined over a subfield $k \subset \mathbb{C}$, they are semisimple iff $k \ni \sqrt{\lambda_l}$, which is not cyclotomic.

Sketch

If $\mathcal{C}_k \otimes_k \mathbb{C} = \mathcal{C}_{\mathbb{C}}$ and $\mathcal{C}_k$ is semisimple, each idempotent P in $\mathcal{C}_{\mathbb{C}}$ has a 'pre-image' in $\mathcal{C}_k$.

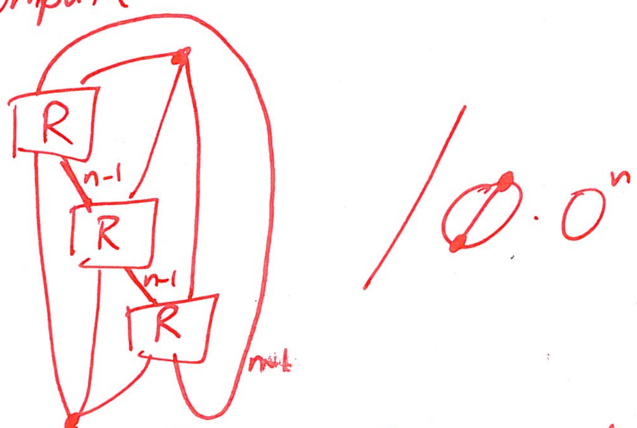
Now pick $\curlywedge : X \otimes X \to X$ in $\mathcal{C}_k$ and $\cup : \mathbb{1} \to X^{\otimes 2}$, $\cap : X^{\otimes 2} \to \mathbb{1}$.

Take $R = \frac{1}{2}(P^+_k - P^-_k)$

where $P^{\pm} = \frac{1}{2}\left(f^{(n)} \pm \frac{1}{\lambda_\ell} S \right)$, idempotents in $\mathcal{C}_{\mathbb{C}}$.

Now we compute

$M_3(R) =$



which is independent of our choices, and must be in k.

We can compute it in $\mathcal{C}_{\mathbb{C}}$:

$$M_3(R) = \lambda_\ell \frac{[2n+2]}{[n+2]} \frac{[2]^2}{([5]+1)[3]^{2\ell+1}}$$

which is not cyclotomic! $\square$

These fusion categories are thus 'exotic'.
But how exotic?

At this point this seems a hard question to answer.

For finite groups, we have some 'structure theory';
e.g. the notions of normal subgroups
and simple groups.

For general fusion categories, we so far don't
have analogous notions.

Recently ENO have introduced a homotopy theoretic
description of 'fusion categories graded by a finite group';
and using this we've seen that the $\ell=0$ example above
is perhaps 'closer' to finite groups than we
at first expected. The $\ell=1$ example is still
completely mysterious.

Borrowing Terry Gannon's analogy, fusion categories are
like the night sky; at first large swathes of it appear
completely blank, but when we develop a telescope able
to zoom in on one of these blank patches, we
see that it is full: a myriad of remote stars.