The x-colour theorems

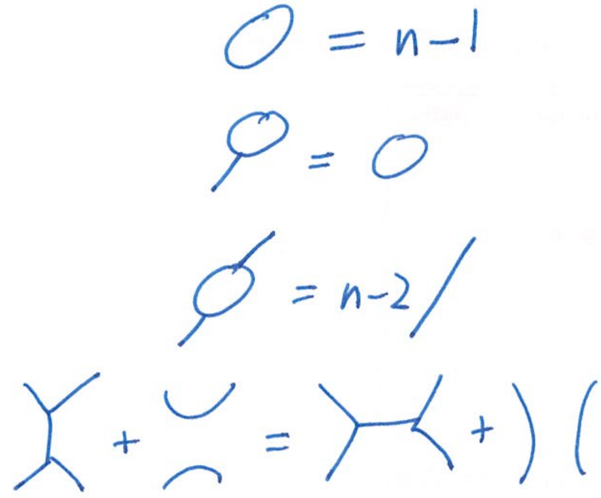
- The chromatic number $\chi_G(n)$, giving the number of n-colourings of a graph G, is polynomial in n.
- We can evaluate at any real x. The statement
  "$\chi_G(x) > 0$ for all 1-connected planar graphs G"
  is the <u>x-colour theorem</u>.
- It is famously true for $x = 4$. Much more easily, it is true for all real $x \geq 5$ (and false for $x < 4$).
  It remains open for $4 < x < 5$. (but is believed).

This talk will give a 'quantum' proof of the x-colour theorem for $x > 5$, and a plan, so far unsuccessful, to prove it for some $5 - \varepsilon$.

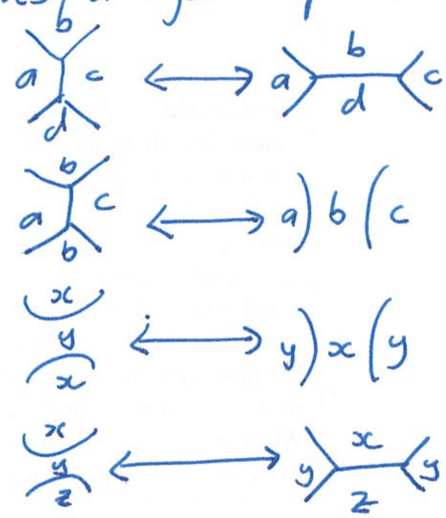
I'll also connect these conjectures to representation theory, and see they are special cases of positivity conjectures which are not obviously about colouring problems.

The chromatic number is determined by a few local relations

(note we're colouring faces here!)



here's a bijective proof:

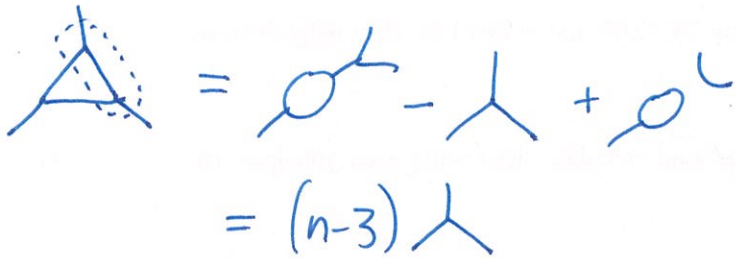


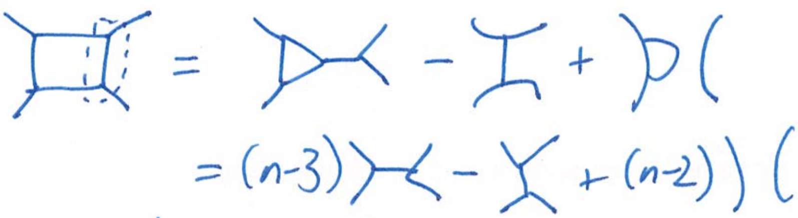
In fact, we can evaluate any planar trivalent graph using just these.

Using I=H, we can reduce the size of a chosen face (modulo terms with fewer vertices) until it has at most two edges, then reduce the number of vertices using one of the other relations.

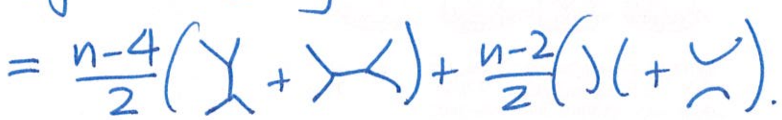
Moreover, this shows that chromatic numbers are polynomial (with integer coefficients!) in n.

Let's look at consequences of these relations.



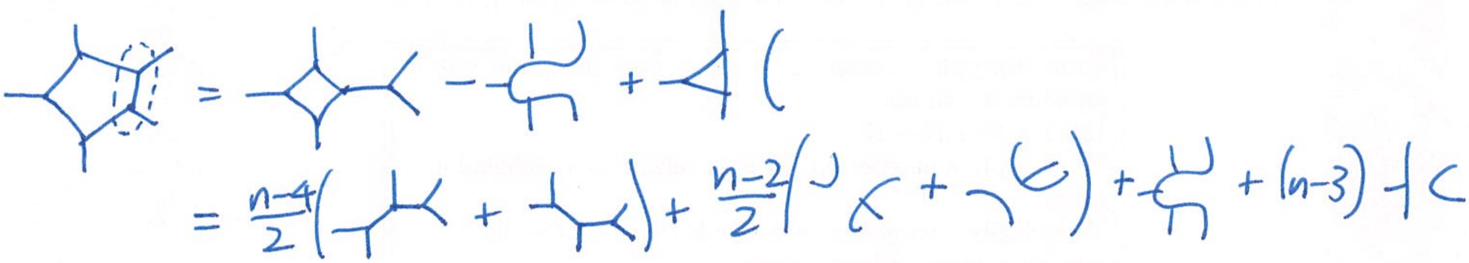


and symmetrizing



~~This already tells us something interesting~~

and one more:



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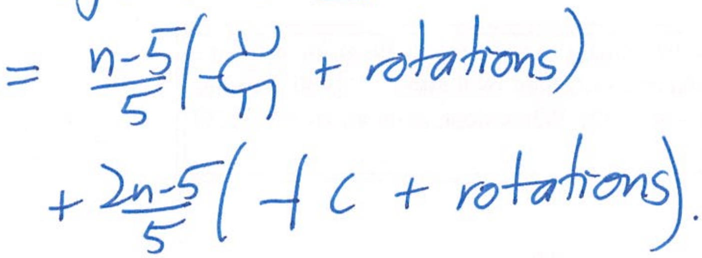


This lets us prove the $x>5$ colour theorem:

- Every planar trivalent graph has a face with at most 5 sides.
- All the coefficients in the reduction formulas above are positive for $x>5$.

(• If the left hand side is 1-connected, at least one term on the right hand side is 1-connected.)

Note that this is a quantum proof, rather than an existential ~~enumerative~~ proof.

We show that every graph is a positive linear combination of simpler graphs.

An existential ~~enumerative~~ ~~graph~~ proof would show that for every graph without an n-colouring, there exists a simpler graph also without an n-colouring.

(With a little more work we can prove the $x=5$ colour theorem by the same technique — we just ~~need~~ to show that at least one of the $\exists C$ terms is 1-connected.)

At this point the talk bifurcates into two separate stories

1. an attempt to give a quantum proof of a $5-\varepsilon$ colour theorem
2. connections with representation theory, and some stronger conjectures.

A $5-\varepsilon$ colour theorem?

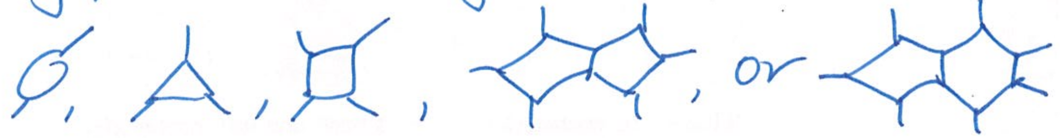
- There are many interesting statements analogous to "Every planar trivalent graph contains a pentagon or smaller face".

Can we use these?

We might hope to find a set S of subgraphs, and for each subgraph a <u>positive</u> reduction formula for $x = 5-\varepsilon$.

(Even better S_n for x_n with $x_n \to 4$ in finitely many steps!)

<u>Example</u> Every planar trivalent graph contains one of:



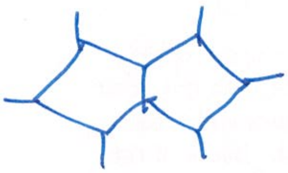
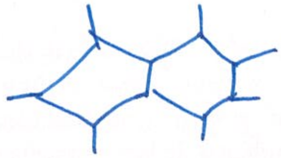
<u>Proof</u> Give each n-gon $\$6-n$.

By an Euler characteristic argument, there are $\$12$.

If there's no small face, have each pentagon distribute its $\$1$ amongst its neighbours.

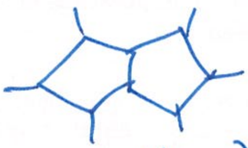
~~Sh~~ Show me the money. $\square$

But are there positive reduction formulas for

 and  ?

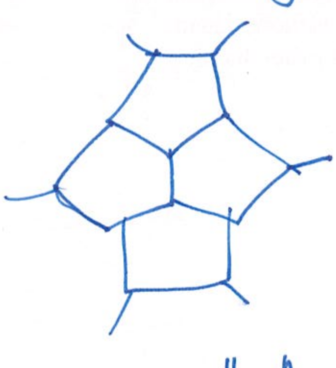
(We have to say exactly what we mean by a reduction formula — let's say for now the terms on the right hand side have strictly fewer vertices.)

Unfortunately, <u>no</u>.

(To see this, we write  and all simpler diagrams in a basis (planar partitions), and then check if  lies in the cone spanned by the simpler diagrams.)

Perhaps one should give up at this point!
But having written the programs to do these calculations, let's press on.

We (eventually) find

 $\in \mathbb{R}_{>0}\{\text{simpler diagrams}\}$

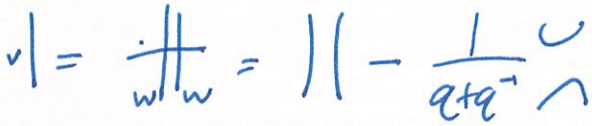
all the way down to $n > 4.517$.

Also a few other examples — but it's hard to complete this to a full set of subgraphs

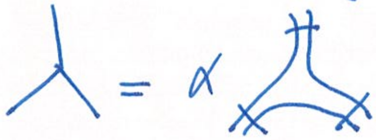
What does this have to do with representation theory?

The chromatic polynomial "is" the standard representation of $SO(3)_q$ at $n = q^2 + 2 + q^{-2}$.

The standard representation V is $\mathrm{Sym}^2_q W$, W the standard representation of $SU(2)_q$, or:



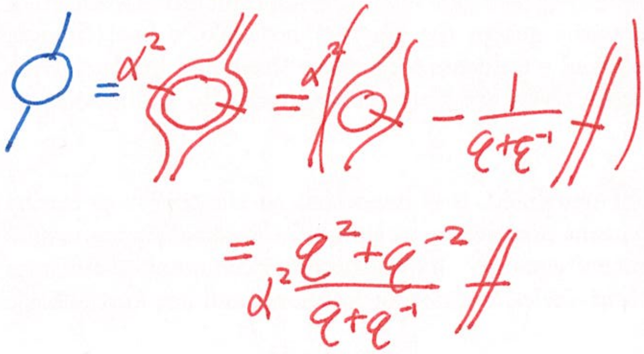
There is a (rotationally invariant) map $V \otimes V \to V$



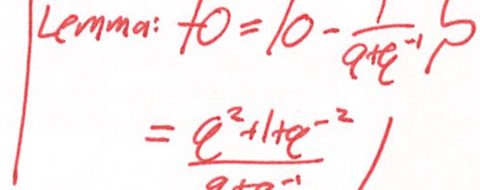
(for some undetermined scalar α)

We have:

$= q^2 + 1 + q^{-2} = n - 1$

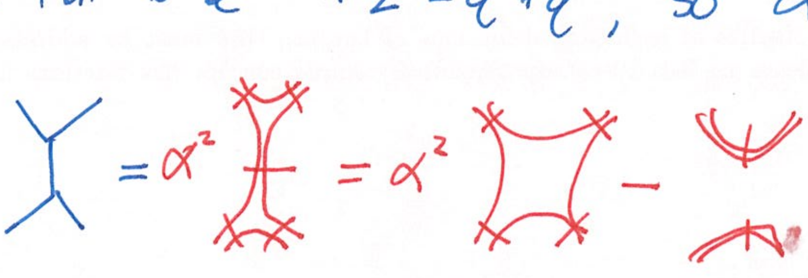


Lemma:



Wed like that to be $n-2 = q^2 + q^{-2}$, so $\alpha = \sqrt{q+q^{-1}}$.

Finally



giving

We thus have an interpretation of the 4-colour theorem as:

"the $SO(3)$ evaluation of any 1-connected planar trivalent graph is positive"

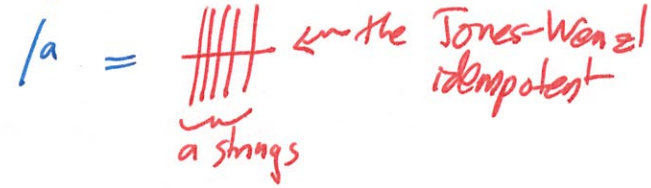
and the $x \geq 4$ colour theorem as

"the $SO(3)_q$ evaluation of any 1-connected planar trivalent graph is positive at any $q \geq 1$"

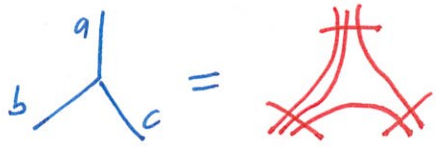
(Recall we have this for $x \geq 5$, corresponding to $q \geq \frac{1+\sqrt{5}}{2}$.)

Here's a generalization: label a planar trivalent graph with positive integers, ~~and~~ satisfying the triangle inequality at each vertex.

We can interpret this in the $SU(2)$ ~~$SO(3)$~~ representation theory via



and



<u>Conjecture</u> this is always positive (at $q=1$).

<u>Question</u> Is there a normalization of these trivalent vertices so this is always positive for $q \geq 1$?

<u>Question</u> Can we do this for other groups?

(E.g. no — for G_2's 7-dimensional representation
in the 'usual' normalization

 $= -126$  $= 3024$  $= 9450$

so this can't be fixed by a rescaling)

<u>Finally</u> a strengthening of the $x \geq 4$ colour theorem
(supported by some, but not enough, evidence)
is that $$\left.\frac{d^k}{dn^k} \chi_a(n)\right|_{n=4} \geq 0 \text{ for each } k.$$

This is a collection of linear relations on the coefficients
(rather than infinitely many evaluations of polynomials).

But how, given a 1-parameter family of tensor categories,
should we study the derivative?