Khovanov homology as a 4-category

The goals of this talk are

1. to explain that Khovanov homology provides a beautiful example of a 4-category, and

2. to sketch our definition of "disklike n-category" which allows us to make the statement above.

What is Khovanov homology?

- a 'categorification' of the Jones polynomial:

  $Kh(L)$ is a doubly graded vector space, and
  $$J(L) = \sum_{i,j} (-1)^i q^j \dim Kh^{i,j}(L).$$

- a categorical link invariant, i.e. a functor

$$\begin{array}{ccc} \{\text{links in } B^3\} & \longrightarrow & \left\{\begin{array}{c}\text{doubly graded}\\ \text{vector spaces}\end{array}\right\} \\ \circlearrowright & & \circlearrowright \\ \left\{\begin{array}{c}\text{cobordisms}\\ \text{in } B^3\times I\end{array}\right\} & \longrightarrow & \{\text{linear maps}\} \end{array}$$

The Jones polynomial is intimately related to a certain 3-category, $\mathrm{Rep}\, U_q \mathfrak{sl}_2$.
(recall braided $\otimes$-categories are 2-tuply monoidal 3-categories)

If Khovanov homology is a 'categorification', is there a 4-category hiding nearby? <u>Yes.</u>

We'll give a 'tautological' construction of this category (but it's the real deal; following the same recipe we could recover $\mathrm{Rep}\, U_q \mathfrak{sl}_2$ from J).

What is the data required for a disklike 4-category?

- for each $0 \leq k \leq 4$, a functor $\mathcal{C}_k$

$$\begin{array}{ccc} \{k\text{-balls}\} & \longrightarrow & \{\text{sets}\} \\ \circlearrowleft & & \circlearrowleft \\ \{\text{homeomorphisms}\} & \longrightarrow & \{\text{bijections}\} \end{array}$$

- restriction maps; if $Y \subset \partial X$ is a sub-ball, natural transformations
$$\partial : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(Y)$$
- gluing maps: if $X^k = X_1 \cup_Y X_2$ is a gluing of balls,
$$\mathcal{C}_k(X_1) \times_\partial \mathcal{C}_k(X_2) \to \mathcal{C}_k(X)$$
- identities: for each projection $\pi : X^k \to Y^{k-1}$
$$\mathcal{C}_{k-1}(Y) \to \mathcal{C}_k(X)$$

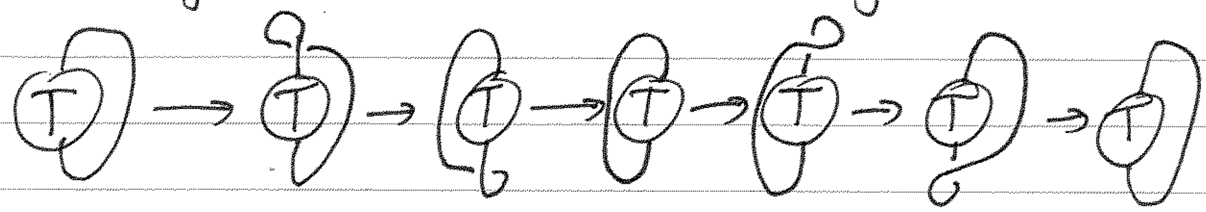
Briefly, what axioms do these obey?

- At <u>dimension n</u>, isotopic homeomorphisms must act identically.
- Gluing is strictly associative!
- homeomorphisms, restrictions, gluings and identities all interact nicely.

There's a subtle technical hurdle, however!
We need $\mathrm{Diff}(S^3)$ to act on $Kh(L \subset S^3)$.

The existing results on functoriality are all in B^3, not S

Essentially, we need one new thing:

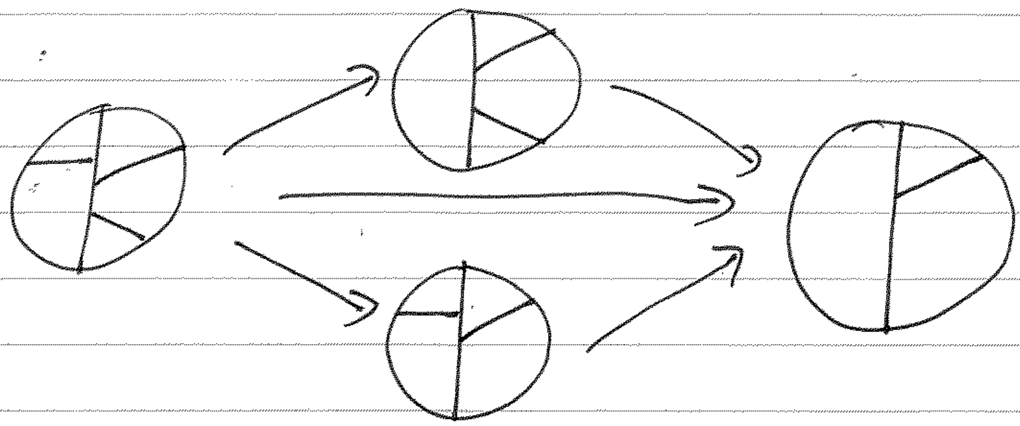


is the identity. (This is isotopic to the identity in S^3, but not in B^3.

We can prove this mod 2, but sadly not yet over the integers.

What's the payoff?
$\Rightarrow$ Invariants of 4-manifolds.

Define the 'poset of ball-decompositions' $D(M)$



A disklike 4-category gives a functor

$$\mathcal{C}_M : D(M) \longrightarrow \mathrm{Set}$$

- A ball decomposition is sent to the fibred product of sets
- a coarsening is sent to the gluing map.

The TQFT skein space is then

$$\underset{\rightarrow}{\mathcal{C}}(M) = \operatorname*{colim}_{D(M)} \mathcal{C}_M$$

and this lets us define vector space valued invariants of PL 4-manifolds from Khovanov homology.