Khovanov homology and 4-manifolds

Today I'll define an invariant of 4-manifolds

$\underrightarrow{Kh}(W^4)$

along with a relative version $\underrightarrow{Kh}(W^4, L \subset \partial W)$ with

$\underrightarrow{Kh}(B^4, L \subset S^3) = Kh(L)$.

This 4-category probably has some connection with the representation theory of categorified $U_q \mathfrak{sl}_2$, but the point of today's talk is that we can construct the 4-category directly from Khovanov homology as a knot invariant.

Why another 4-manifold invariant?

Khovanov homology depends on smooth structure, and there are ~~many~~ difficult questions remaining about smooth 4-manifolds, in particular SPC4.

Previously, we discovered a link L such that $s(L) \neq 0$ would imply a particular 4-manifold (the Rasmussen s-invariant) was a counterexample.

Unfortunately $s(L) = 0$, and moreover Akbulut showed this 4-manifold was just smooth S^4, and later Kronheimer-Mrowka showed our whole approach was doomed — the s-invariant can't detect counterexamples.

But there's much more information in Khovanov homology!

Khovanov homology and 4-manifolds

The Jones polynomial is an invariant of links

$$J\Big($$  $$\Big) = A^7 + A^3 + A^{-1} - A^{-9}$$

which is easy to compute from the following rules

$$J\Big($$  $$\Big) = A\ )\,(+ A^{-1}$$ 

$$J(O) = -A^2 - A^{-2}.$$

Moreover, it's easy to show it's an invariant directly from these rules.

This is only the tip of an iceberg! The Jones polynomial gives us a <u>3-category</u>.

In the setting of 'disklike n-categories', for every k-ball with $k \leq 3$ we need to describe the collection of k-morphisms 'of that shape'.

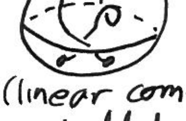
We have

$$J(\bullet) = \{\bullet\}$$

$$J(\sim) = \{\sim\}$$

$$J(O) = \Big\{$$  $$\Big\}$$

finite subsets

$$J\Big($$  $$\Big) = \Big\{$$  $$\Big\} \Big/$$ the Jones relations

(linear combos of) embedded tangles

Why does this deserve to be called a 3-category?

$\Rightarrow$ given a k-morphism α 'of shape X', and
a k-morphism β 'of shape Y'

(here X and Y are k-balls)

if we glue X and Y together along a $(k-1)$-ball S
to obtain $X \underset{S}{\cup} Y$, another k-ball,

we naturally obtain a k-morphism $\alpha \underset{S}{\cup} \beta$ of shape $X \underset{S}{\cup} Y$.

(moreover, $\mathrm{Diff}(X^k)$ acts on k-morphisms of shape X.
If $f, g \in \mathrm{Diff}(X^n)$ are isotopic rel ∂, they act the same way on n-morphisms.)

You may well have seen this 3-category in another guise, as the braided monoidal category of representations of $U_q \underline{\mathfrak{sl}}_2$.

(Roughly,  $\longmapsto \left(\mathbb{C}_q^2\right)^{\otimes k}$

k points

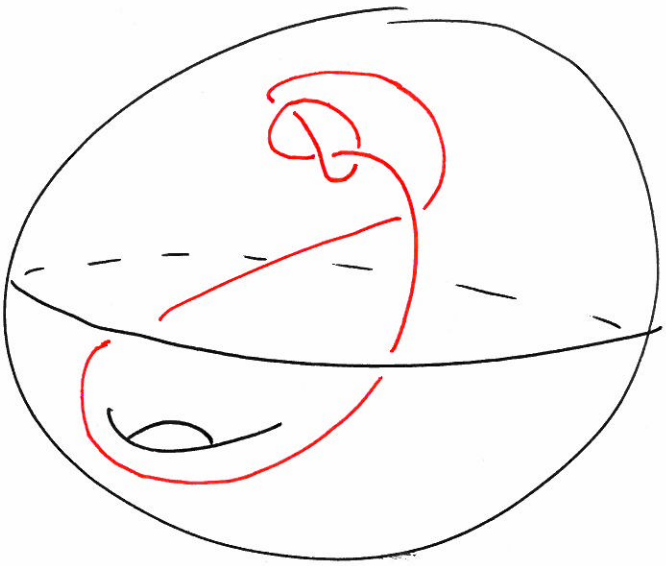
 $\longmapsto \beta_{\mathbb{C}_q^2}$, the braiding.

This is an equivalence of categories!)

What can we do with such a 3-category?
$\Rightarrow$ build 3-manifold invariants

If M^3 is a 3-manifold
$J(M)$ is the 'skein module' of
embedded links in M, modulo the <u>local</u> Jones relations.

(We also have $J(M, X \subset_{\text{finite}} \partial M)$, using tangles ending on X.)



① If $f: M \hookrightarrow N$, we get a map of skein modules
We can think of this as giving a linear map for a cobordism $W: M \to N$, at least for W a mapping cylinder.

② For <u>some</u> 3-categories (satisfying a 'finiteness' condition) we get linear maps for arbitrary cobordism
$$W^4: M^3 \longrightarrow N^3.$$
For the Jones polynomial, this holds precisely when A is a root of unity.

3 For the Jones polynomials, $J(M)$ 'barely' depends on the interior of M, and reduces to a vector space valued invariant of <u>2-manifolds</u>.

Similarly, we have a dimension reduction for cobordisms.

These 2- and 3-manifold invariants are usually known as Reshetikhin-Turaev invariants.

(This dimension reduction happens ~~becaeause~~ the braided monoidal category is modular.)

What is Khovanov homology?

$Kh(L)$ is a doubly graded vector space
(L a link in B^3)

$Kh(\underbrace{\Sigma : L_1 \to L_2}_{\text{a link cobordism in } B^3\times I})$ is a (grading preserving) linear map, which only depends on Σ up to isotopy rel ∂.

It has an explicit, direct construction — more complicated than the Jones polynomial, but still 'elementary'. We just need to know about links, link cobordisms, and chain complexes.

Just as the Jones polynomial 'comes from' the representation theory of $U_q\mathfrak{sl}_2$, (because the 3-category 'is' $\mathrm{Rep}\,U_q\mathfrak{sl}_2$) one hopes that the 4-category we're about to define is in some sense 'the representation theory' of Khovanov and Lauda's categorification of $U_q\mathfrak{sl}_2$.

The details are not yet worked out.

Today, let's ignore that (very interesting!) representation theoretic story, and

construct a 4-category 'with our bare hands'.

As before the Khovanov 4-category 'starts off boring': ⑥

Kh(•) = { • }

Kh(~) = { ~ }

Kh() = {  }

Kh() = {  }

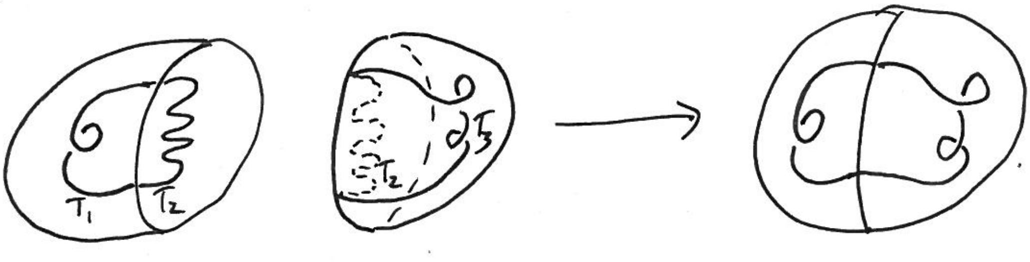
embedded ~~[illegible]~~ tangles with given boundary

Kh() = Kh(L). (anticlimactic?)

How do we glue morphisms together?

For k-balls with $k \leq 3$ this is clear.

Consider now two 4-balls



We need a map $Kh(T_1 \# T_2) \otimes Kh(T_2 \# T_3) \to Kh(T_1 \# T_3)$.

There is a natural link cobordism $T_1 \# T_2 \cup T_2 \# T_3 \to T_1 \# T_3$,

so we hope that the associated linear map gives us what we want.

Does this work?

<u>Theorem</u> In characteristic 2, this defines a disklike 4-category.

The essential difficulty is defining Khovanov homology for links (and link cobordisms) in S^3, rather than B^3.

The problem is that while

- links (generically) avoid the north pole,
- link cobordisms avoid the north pole,
- isotopies between link cobordisms do not.

That is, we may have two cobordisms

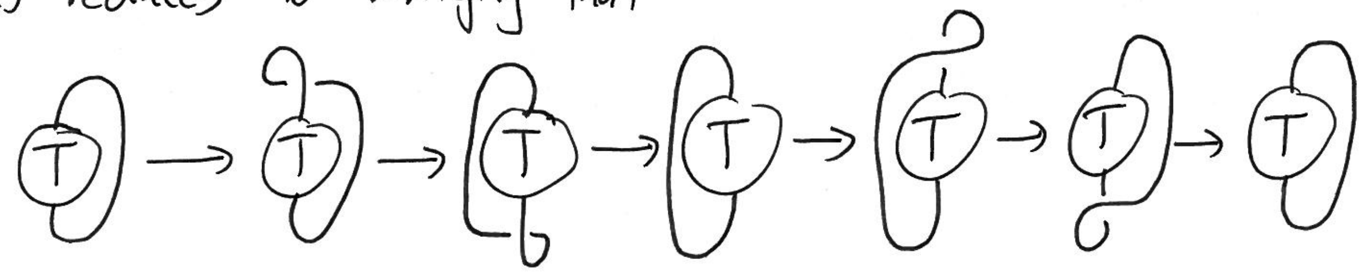
$$\Sigma_a, \Sigma_b : L_1 \to L_2$$

which are not isotopic in B^3, but are in S^3.

We need to show that these induce the same linear maps

$$Kh(K_a), Kh(K_b) : Kh(L_1) \to Kh(L_2).$$

This reduces to verifying that

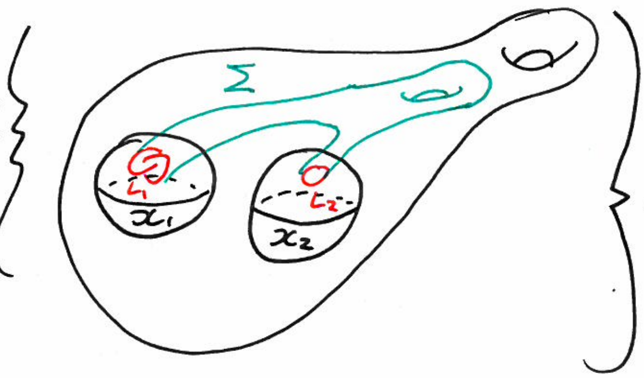


is the identity map.

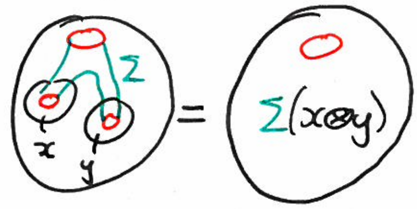
In characteristic 2, the maps associated to link cobordisms are significantly easier to describe. (In particular, $\simeq$ ( ) ($\to$), and the map $\to$ has components induced by cobordisms. The map $\to$ is 'diagonal' w.r.t. resolutions of T, and given by the obvious cobordisms.)

This gives us an invariant of 4-manifolds, $\underset{\rightarrow}{Kh}(W^4)$.

$\underset{\rightarrow}{Kh}(W^4) = \mathbb{Z}/2\mathbb{Z}$



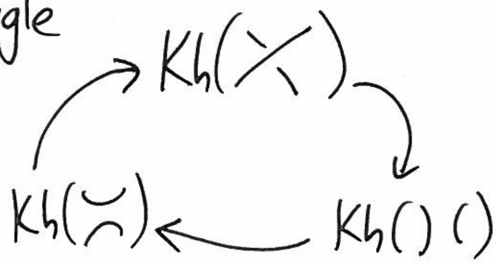
locally,



There's also a 'relative' version $\underset{\rightarrow}{Kh}(W^4, L \subset \partial W)$,

and $\underset{\rightarrow}{Kh}(B^4, L \subset S^3) = Kh(L)$.

Unfortunately the main computational tool for Khovanov homology, the exact triangle



<u>fails</u> for the 4-manifold version.

(This is an indication that the usual TQFT construction is the wrong thing to do with a 'homotopy-theoretic' 4-category like Kh. This is exactly what we invented blob homology for!)

Some questions

- Is this invariant actually computable?
- Will it tell us anything about smooth 4-manifolds?
- Is there a 'dimension reduction' phenomenom as for the Jones polynomial?

  (i.e. perhaps this vector space only depends weakly on the interior, and is 'really' an invariant of $(\partial W, L)$.)