In these lectures, I'm going to tell you about the classification of small ~~index~~ subfactors.

I'll start out by summarizing the current situation
- explaining the regimes in which we have complete classifications, and
- describing ~~what~~ the examples we see there.

As I do that, I'll explain the constructions used in realizing these examples.
I want to emphasize that
- relatively few constructions are needed to realize almost everything we've seen,
- but that there are few exceptional examples, apparently isolated from the rest of the mathematical universe.

Finally, probably in the 3rd lecture, I'll explain the mechanism of classification theorems, and tell you about some recent developments that have allowed us to look further out.

Even though these lectures are primarily about <u>small index</u> subfactors, it's important to realize this is a narrow view of the subject.

Perhaps someday we'll stand atop the mountain, and see the whole landscape of subfactors clearly.

But just now we're barely left the trailhead.

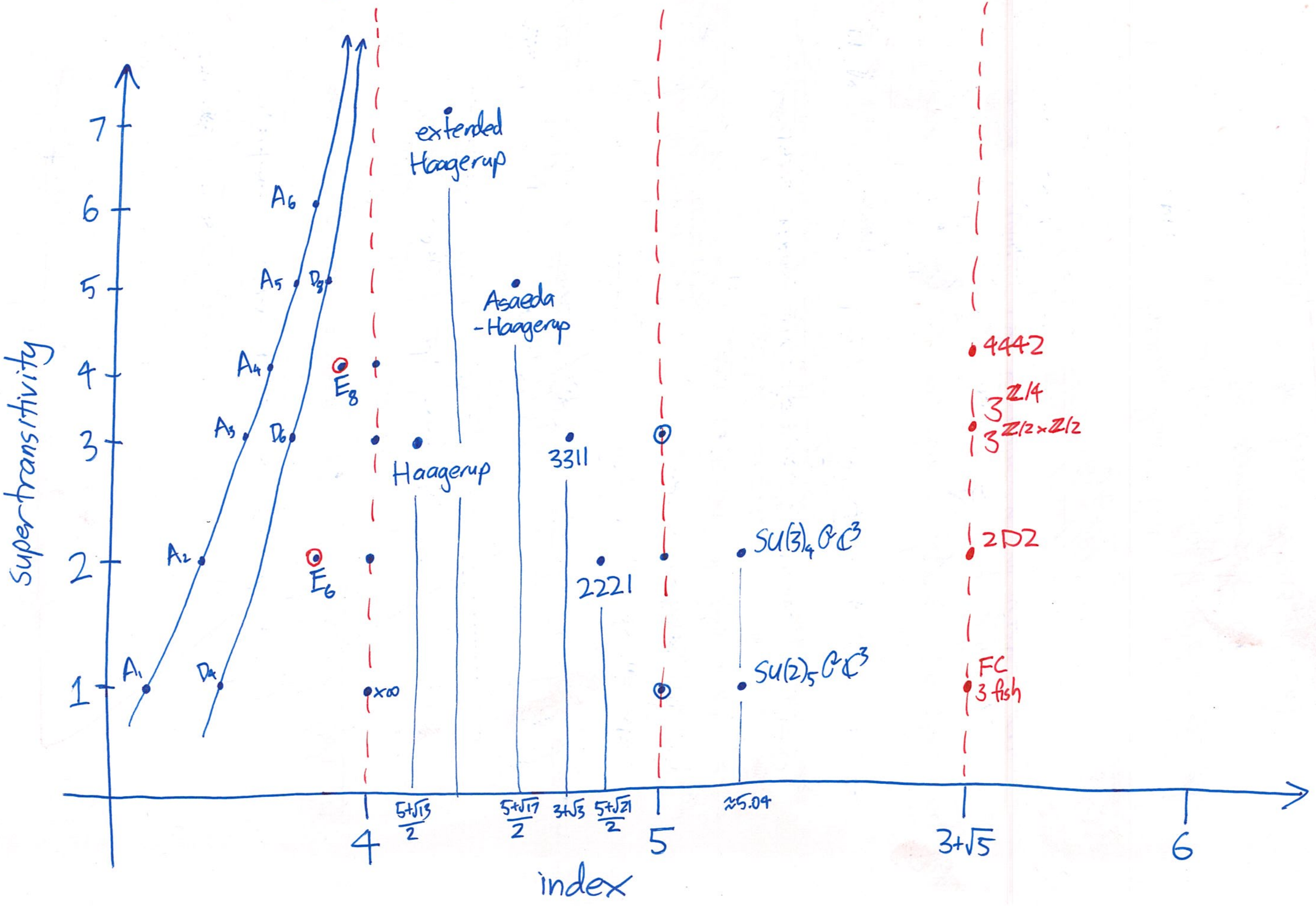
Understanding small index subfactors looked like a reasonable route into the foothills, but it's certainly not the only way.

To make matters worse, our trail is starting to become indistinct, and the undergrowth thicker!

We know there are cliffs ahead we're not equipped to climb — but at least, through the trees, we're getting glimpses of the mountain.

I'll be trying, when possible, to mention the wider view.

We're been drawing a map!



Theorem (Index <4) (Ocneanu, Jones, Izumi, Kawahigashi, et. al)

The principal graph must be a Dynkin diagram, with ★ on the longest arm.

The graphs D_{odd} & E_7 don't extend to fusion rules.

- A_n is realized by [redacted] the quotient of TL at a root of unity.
- D_{2n} can be obtained from A_{4n-3} by 'deequivariantization'.
- E_6 and E_8 arise as module categories over A_{11} & A_{29}, as the module objects for a certain commutative algebra object.

Theorem (Index exactly 4) (Popa, et al.)

The principal graph must be an affine Dynkin diagram.

All are realized as $P_n = \mathrm{End}_G((\mathbb{C}^2)^{\otimes n})$

for some $G \subset SU(2)$. There are also relatives twisted by cohomological data.

$A_{2n}^{(1)}$	$A_{2n-1}^{(1)}$	$D_n^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$	A_∞	D_∞	$A_\infty^{(1)}$
	binary cyclic	binary dihedral	binary tetra	binary octa	binary icosa	$SU(2)$	infinite binary dihedral	infinite binary cyclic
0	n	n-2	1	1	1	1	1	1

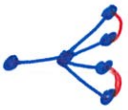
Theorem (Index [illegible] in $(4,5)$)

An extremal subfactor with index in $(4,5)$ is either non-amenable with standard invariant A_∞, or it is one of 10 cases, with planar algebra

$$\underbrace{H,\ EH,\ AH,\ 3311}_{\text{or its dual}},\ \underbrace{2221}_{\text{or its complex conjugate}}.$$

Theorem (Index exactly 5)

There are 7 subfactor planar algebras, all group-subgroups or their duals.

$1 \subset \mathbb{Z}/5\mathbb{Z}$  "

$\mathbb{Z}/2\mathbb{Z} \subset D_{10}$  "

$\mathbb{Z}/4\mathbb{Z} \subset \mathbb{Z}/5\mathbb{Z} \rtimes \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z})$  "

$A_4 \subset A_5$  

$S_4 \subset S_5$  

(the first 3 are self-dual, the last 2 are not)

Theorem* (Afzaly-M-Penneys, [redacted] Liu arXiv:1308.5691
also Izumi-M-Penney arXiv:1308.5723
M-Penneys arXiv:1208.3637
and ...forthcoming...)

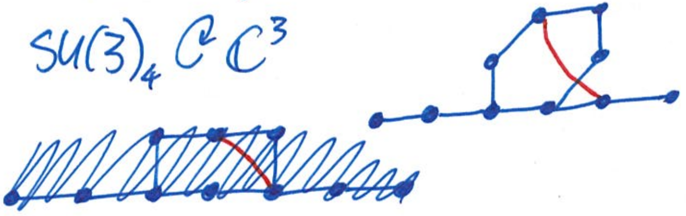
* some loose ends!

In the interval $(5, 3+\sqrt{5}]$, there are 10 subfactor planar algebras

At index $(\zeta_{14}^{2}+1+\zeta_{14}^{-2})^2 \approx 5.04892$

two quantum group subfactors, coming from

$SU(2)_5 \subset \mathbb{C}^3$ and $SU(3)_4 \subset \mathbb{C}^3$



At index $3+\sqrt{5} \approx 5.23$

the Fuss-Catalan algebra $A_3 * A_4$

along with 3 quotients $A_3 \otimes A_4 =$ 

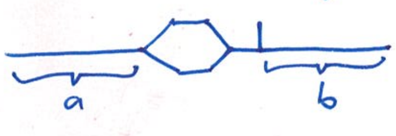
(all 1-supertransitive)

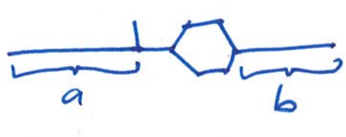
and $3^{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$ and $3^{\mathbb{Z}/4\mathbb{Z}}$

along with "4442" (the $\mathbb{Z}/3\mathbb{Z}$ fixed points of $3^{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$)

and "2D2" (which contains $3^{\mathbb{Z}/4\mathbb{Z}}$ as a $\mathbb{Z}/2\mathbb{Z}$ fixed pt)

The [redacted] only worrisome loose end is





Suggestions welcome!

Constructions

- Quantum groups at roots of unity A_n
  - quantum subgroups (orbifolds, e.g. D_{2n} and exceptionals E_6 & E_8)
    (also conformal inclusions)
- Finite groups — $R^G \subset R^H$
  - twists by cohomological data
- Composites — quotients of free products
  (at index 4, the $\hat{D}_n$'s are quotients of $\hat{D}_\infty = A_3 * A_3$)
  (at index $3+\sqrt{5}$, there are only finitely many (3)
  quotients of $A_3 * A_4$ Liu arXiv:
  Izumi-M-Penneys
  arXiv:
- near groups — fusion categories with an abelian group subcategory with one other orbit.
  (3^G, with Haagerup $= 3^{\mathbb{Z}/3}$, $2^G 1$)

[redacted]

- bimodule [redacted] categories for algebra objects
  ~~somewhere in the maximal atlas~~
  (e.g. $A_{11} \curvearrowright E_6$, let $A = \underline{\mathrm{End}}_{A_{11}}(1_{E_6}, 1_{E_6})$, then take $\begin{matrix} 1\text{-mod-}1, & 1\text{-mod-}A \\ A\text{-mod-}1, & A\text{-mod-}A \end{matrix}$
  (more complicated (Izumi-Grossman-Snyder):
  AH can be obtained from $3^{\mathbb{Z}_4 \times \mathbb{Z}_2}$)

- EH, constructed 'with our bare hands', as a subalgebra of the graph planar algebra.

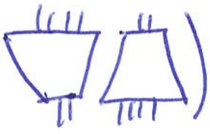
From planar algebras to ⊗-categories and back again

A shaded planar algebra gives a pair of ⊗-categories.
(If our planar algebra came from a subfactor, $A \subset B$, these are just A-mod-A and B-mod-B.)

Given $P_\bullet$, $\hat{\mathcal{C}}_\pm(P_\bullet)$ is the ⊗category

$\mathrm{Obj} = \mathbb{N}$

$\mathrm{Hom}(n \to m) =$ [redacted] $P_{n+m,\pm}$ ↙ boxes with $2n+2m$ boundary points.

(composition  tensor product )

We can idempotent complete to obtain $\mathcal{C}_\pm(P_\bullet)$.
The simple objects are the even depth vertices on the principal graph $\Gamma_\pm$.

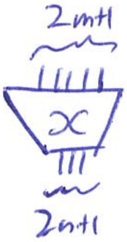
There are also bimodule categories between these: $\mathcal{C}_{+-}(P)$ & $\mathcal{C}_{-+}(P$

Here $\mathcal{C}_+ \curvearrowright \mathcal{C}_{+-} \curvearrowleft \mathcal{C}_- \curvearrowright \mathcal{C}_{-+} \curvearrowleft \mathcal{C}_+$.

(Again, if we obtain our planar algebra from a subfactor $A \subset B$, these are just A-mod-B and B-mod-A.)

In [redacted] $\mathcal{C}_{\pm\mp}(P)$,

$\mathrm{Obj} = 2\mathbb{N}+1$

$\mathrm{Hom}(2n+1 \to 2m+1) = P_{n+m+1,+} \ni$ 

These are Morita equivalences: $\mathcal{C}_{+-} \otimes_{\mathcal{C}_-} \mathcal{C}_{-+} \simeq \mathcal{C}_+$, etc.

The maximal atlas

~~Given a subfactor $N \subset M$,~~

An *algebra* in a $\otimes$-category is

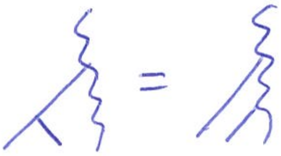
- an object A, along with
- a map  $: A \otimes A \to A$

such that 

A *module* over an algebra A in a $\otimes$-category is

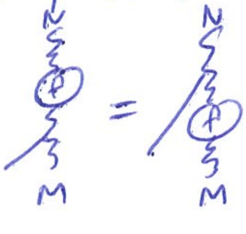
- an object M, along with
- a map  $: A \otimes M \to M$

such that



The collection of all A-modules forms a category "A-mod".

A map of A-modules satisfies



As we work in a unitary fusion category (in particular, rigid), we also require:

Fact A-mod-A, the category of A-A bimodules, is a unitary tensor category.

Observe that if we started with a planar algebra and build $\hat{\mathcal{C}}_+$ as before, [1] has a canonical algebra structure given by  $\in P_{3,+}$.

Given an inclusion of algebras $A \subset B$ in $\mathcal{C}$ we can extract a shaded planar algebra.

Write $X := {}_A B_B$. Write $\widehat{\bigotimes}^n_{C\,D} V = \underbrace{V \otimes_D V^* \otimes_C \cdots \otimes V^\circ}_{n \text{ factors}}$.

Define $P_{n,+} = \mathrm{Hom}_{A-A}\left({}_A A_A \longrightarrow \widehat{\bigotimes}^{2n}_{A} X_{B}\right)$

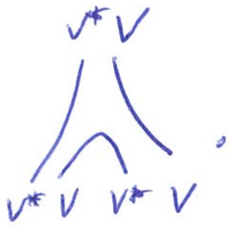
$P_{n,-} = \mathrm{Hom}_{B-B}\left({}_B B_B \longrightarrow \widehat{\bigotimes}^{2n}_{B} X^*_{A}\right)$

Claim Starting with $(A = [0]) \subset (B = [1])$ in $\hat{\mathcal{C}}_+$, we recover the planar algebra we built $\hat{\mathcal{C}}_+$ from.

~~$(V^*\otimes V)\otimes_{V^*\otimes V}(V^*\otimes V)\otimes_{1}(V^*\otimes V)\otimes_{V^*\otimes V}(V^*\otimes V)\otimes\cdots$~~

How do we find algebras?

① $V^*\otimes V$ is always an algebra, with



Then the construction above ~~simplifies to~~ for $1\subset V^*\otimes V$ simplifies to

$$P_{n,+} = \mathrm{Hom}_{\mathcal{C}}(1\longrightarrow \hat{\otimes}^{2n} V)$$

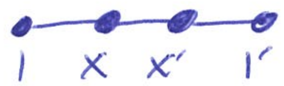
$$P_{n,-} = \mathrm{Hom}_{\mathcal{C}}(1\longrightarrow \hat{\otimes}^{2n} V^*).$$

This is sometimes called 'shading an unshaded planar algebra'.

E.g. Start with the Fibonacci category

$\mathrm{Obj}=\{1, X\}$, $X^2\cong 1\oplus X$.

The principal graph for the 'shading' is



~~② The~~

Quantum groups at roots of unity

Associated to any $U_q(\mathfrak{g})$, with q a root of unity (of sufficiently large order) there is an associated semisimple braided $\otimes$-category $\mathrm{Rep}\, U_q(\mathfrak{g})$.

The simple objects correspond to simple objects in $\mathrm{Rep}(\mathfrak{g})$ which are below a certain 'wall' (depending on q) in the Weyl cham

Picking a simple object V, we obtain a planar algebra

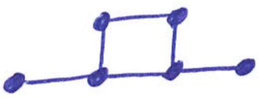
$$P_{n,+} = \mathrm{End}_{U_q(\mathfrak{g})}\big(\underbrace{V \otimes V^* \otimes V \otimes \cdots}_{n \text{ factors}}\big)$$

$$P_{n,-} = \mathrm{End}_{U_q(\mathfrak{g})}\big(V^* \otimes V \otimes V^* \otimes \cdots\big)$$

If q is a root of unity closest to 1, this planar algebra is positive definite (see Wenzl), and in fact a subfactor planar algebra.

Example $U_q(\mathfrak{sl}_2)$ at a 14th root of unity has 6 simple objects. If we take the standard representation $V = \mathbb{C}_q^2$, we obtain the A_6 subfactor .

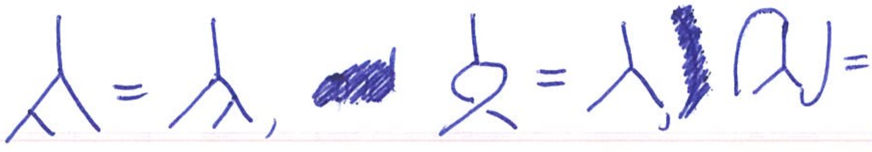
Example If we take the 3rd representation instead, you can compute the principal graph of the resulting subfactor using the $\otimes$-product rule in A_6.

Exercise it's 

Quantum subgroups

Consider $\mathcal{C}$, a braided $\otimes$-category, and A, a commutative alge

(So we have a map $\lambda : A \otimes A \to A$, satisfying



and we can normalize it as

=1)

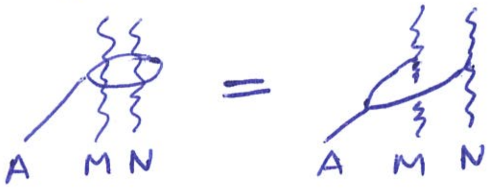
(More generally, one can work [redacted] a $\otimes$-category with an algebra that lifts to centre. ~~We'll need this later.~~) This allows us to capture deequivariantization as a special case.

The category of A-modules

(objects M equipped with a map

$: A \otimes M \to M$ satisfying

naturally forms a $\otimes$-category:



(Exercise check associativity)

Example The A_{11} planar algebra is braided

(it's the planar algebra associated to $U_q\underline{\mathfrak{sl}}_2 \circlearrowleft$ [redacted] $\mathbb{C}^2_q$, $q^{24}=1$)

and $A = V_0 + V_8 + V_{10}$

is a commutative algebra.

The category of A-modules is the E_6 planar algebra.

(De)equivariantization

If G, a finite group, acts on a planar algebra $P_\bullet$, the fixed point subalgebra $P^G_\bullet$ is called the equivariantization.

If $P_\bullet$ contains an abelian group of invertible objects, it is sometimes possible to add morphisms making all these invertible objects isomorphic to $\mathbb{1}$.

(Generally, if there is a Rep G subcategory lifting to the centre, we can take $\mathbb{C}[G]$-mod.)

This is called de-equivariantization.
When it's possible, there an action of G on the resulting planar algebra, and fixed points recovers the original $P_\bullet$.

Example In A_{4n-3} 

we can add an isomorphism $S: 1 \xrightarrow{\cong} g$, obtaining

$D_{2n} =$ 

(There's a $\mathbb{Z}/2\mathbb{Z}$ action negating S.)
See arXiv:0808.0764 for details.

~~How do we find algebras?~~ ~~$\mathbb{1}$ is always an algebra 'shading on unshaded'.~~

② One way is using <u>internal Hom</u>. For $\mathcal{M}$ a $\mathcal{C}$-module category,
~~This is characterized~~ $M_1, M_2 \in \mathcal{M}$,

$$\underline{\mathrm{Hom}}(M_1, M_2) \in \mathcal{C}$$

characterized by

$$\mathrm{Hom}_{\mathcal{C}}(C, \underline{\mathrm{Hom}}(M_1, M_2)) \cong \mathrm{Hom}_{\mathcal{M}}(\mathcal{C} \otimes M_1, M_2).$$

'Internal endomorphisms is always an algebra object.

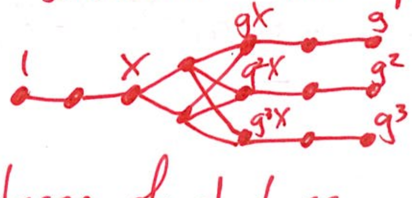
<u>Example</u> E_6 is a module category over A_{11},

$$\underline{\mathrm{End}}(1_{E_6}) =: A$$

is an algebra object, in A_{11}
and the planar algebra for $1 \subset A$ is the 3311 planar algebra.

<u>Question</u> What are all the algebra objects in your favourite $\otimes$-category?

<u>Example</u> Izumi-Grossman-Snyder have constructed a $3^{\mathbb{Z}/4 \times \mathbb{Z}/2}$ subfactor, and shown:
The $\mathbb{Z}/2 \subset \mathbb{Z}/4$ subcategory lifts to the centre, and we can deequivariantize by it, obtaining a subfactor with principal graph



There is an inclusion of algebras
$1 + X \subset 1 + 2X + gX + g^2X + g^3X$, and the corresponding PA is AH.

Composite planar algebras

Given planar algebras $P_\bullet$ & $Q_\bullet$, we can form

~~[illegible]~~ $(P\otimes Q)_\bullet = P_\bullet \otimes Q_\bullet$

with planar tangles acting as $T(p\otimes q) = T(p)\otimes T(q)$.

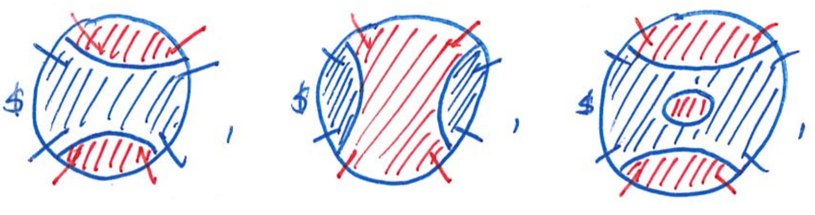
(You should think of $P\otimes Q$ diagrams as being a P diagram 'superimposed' over a Q diagram.)

We can also form the free product $P * Q$.

$$(P*Q)_n = \bigoplus_{n\text{-paintings}} \left(\bigotimes_{\substack{P\text{-regions}\\ r}} P_{|r|}\right) \otimes \left(\bigotimes_{\substack{Q\text{-regions}\\ r}} Q_{|r|}\right) \Bigg/ \begin{array}{c}\text{an empty } P\text{-region}\\ \shortparallel \\ \text{an empty } Q\text{-region}\end{array}$$

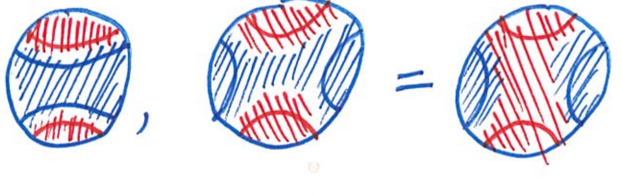
An n-painting is division of a disc with $4n$ points on the boundary, labelled $PQQPP\cdots QQP$, into "P-regions" and "Q-regions", with boundary points in appropriate regions.

Some 2-paintings:



etc.

Some typical elements of $(TL*TL)_2$:



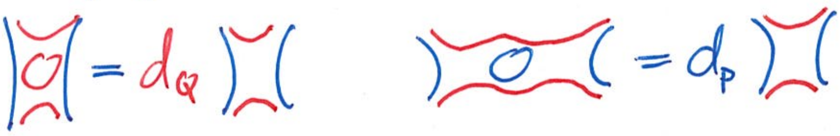
Observe that $(P * Q)_\bullet \subset (P \otimes Q)_\bullet$

as the non-crossing diagrams.

Alternatively, we can obtain $P \otimes Q$ from $P * Q$ by adding a generator:  satisfying appropriate relations.

Definition A <u>composite</u> of $P_\bullet$ and $Q_\bullet$ is a planar algebra $F_\bullet$ with $(P * Q)_\bullet \subset F_\bullet$

Every composite has a special element  $\in F_2$, which is a biprojection:



Conversely, given any planar algebra $F_\bullet$ with a biprojection p we can construct $P_\bullet$ and $Q_\bullet$ and $(P * Q)_\bullet \subset F_\bullet$ recovering the biprojection.

This theory has been extensively developed by Dietmar Bisch and Vaughan Jones.

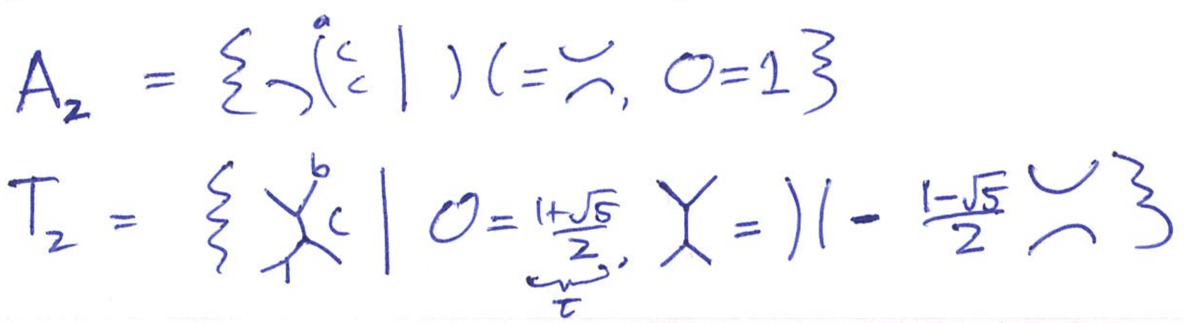
Question Should we expect lots of composites besides the tensor product?

$\mathbb{Z}/2 * \mathbb{Z}/3 = SL(2, \mathbb{Z})$, which has infinitely many finite quotients.

The ~~group-subgroup~~ Bisch-Haagerup subfactors give infinitely many composite subfactor planar algebras at index 6.

$A = \mathbb{C}[\mathbb{Z}/2]$, $B = \mathbb{C}[\mathbb{Z}/3]$ $\quad A \subset \mathbb{C}[G] \subset \mathbb{C}[G] \otimes_B \mathbb{C}[G]$.

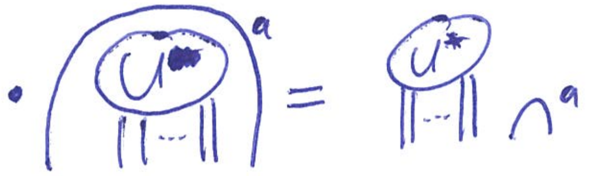
Example Consider the fusion categories

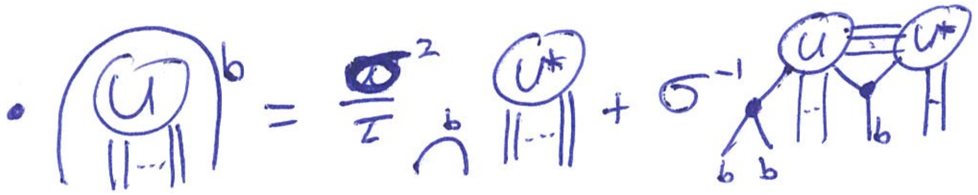


What composites are possible?

A relatively easy argument shows that any composite would contain an element U $:(ab)^n \to 1$

- $UU^* = 1$





$\sigma^{2n} = 1$

These are instances of "jellyfish" relations, and suffice to evaluate any closed diagram. This shows that a composite of A_2 & T_2 is uniquely determined by the parameters n and σ.

Since the successful classification of subfactors with index at most 5 (leaving out, as always, the non-amenable subfactors with trivial standard invariant) there have been four significant advances which make it plausible to push further.

① Liu's results on the presence of intermediate subfactors (arXiv:1308.5656)

② [redacted] Bigelow-Penneys' (arXiv:1208.1564) (extending Popa) on principal graph stability.

③ Penneys results on triple point obstructions (arXiv:1307.5890)

④ Afzaly's new isomorph-free principal graph enumerator.

Recall a principal graph Γ consists of a pair (Γ_0, Γ_1) of locally finite, connected, pointed bipartite graphs, and a involution — preserving depth (distance from the base point), and taking even depth vertices on Γ_i to Γ_i, and taking odd depth vertices on Γ_i to Γ_{1-i},

such that $x(\overline{x(\bar{v})}) = \overline{x(\overline{x(v)})}$ ⊛

where $x(w)$ denotes the multiset of neighbours of w.

⊛ is called the associativity condition.

(Recall a subfactor $A \subset B$ gives a principal graph —

Γ_0 has vertices the simple A-A and simple A-B bimodules

Γ_1 ——————— B-B ——— B-A ———,

there are $\dim \mathrm{Hom}(V \otimes B, W)$ edges from V to W.

The involution is the dual, and the associativity condition says

$(B \otimes V) \otimes B \cong B \otimes (V \otimes B)$.)

We write $P(L,d)$ for the set of principal graphs with index $\leq L$ (index is the square of the largest eigenvalue of the adjacency matrix) and depth $\leq d$.

We say the associativity condition holds between vertices v & w if the multiplicities of w in $x(\overline{x(\bar{v})})$ and in $\overline{x(x(v))}$ agree.

__Definition__ A __partial principal graph__ is a pair (P, n) where P is a principal graph and $n = \operatorname{depth} P$ or $\operatorname{depth} P + 1$, __except__ that the associativity condition need not hold between vertices v and w when $(\operatorname{depth} v, \operatorname{depth} w) = (n-1, n-1)$, (n, n) or $(n-2, n)$.

If (P, n) is a partial principal graph, and P' is obtained from P by deleting a self-dual vertex at depth n ~~or~~ or by deleting a pair of dual vertices at depth n, then (P', n) is also a partial principal graph.

If (P, n) is a partial principal graph and $n = \operatorname{depth} P + 1$, then $(P, n-1)$ is also a partial principal graph.

These operations are called __reductions__.

Lemma There are only finitely many partial principal graphs with index $\leq L$ that reduce in one step to a given (P, n).

Lemma Every partial principal graph can be recursively reduced to $(\phi, 0)$.

An expansion is the inverse of a reduction.

Lemma Only finitely many vertex-addition expansions can be applied to (P, n) without increasing the index above L.

Corollary $P(L, d)$ is finite, and may be enumerated by starting at $(\phi, 0)$ and recursively expanding.

An *expansion* is the inverse of a reduction.

[redacted]

~~**Lemma** Only finitely many vertex-addition expansions can be applied to (P,n) without increasing the index above L.~~

~~**Corollary** $P(L,d)$ is finite, and may be enumerated by starting at $(\emptyset,0)$ and recursively expanding.~~

The challenge, however, is to *efficiently* enumerate $\mathcal{P}(L,d)$.

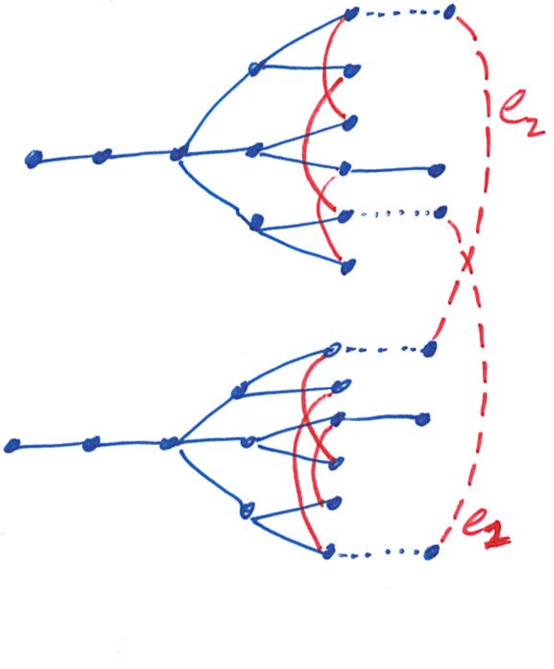
There are three essential problems—

① two expansions of ~~the~~ a graph may be ~~[illegible]~~ equivalent

e.g.

② two inequivalent expansions may nevertheless give isomorphic results

e.g.



③ expansions of two different parents may give the same result

e.g.

and

The solution is 'canonical generation' [McKay "Isomorph-free exhaustive generation", '98].

Amongst all the reductions of an object, we choose our favourite one, or rather a favourite orbit under the automorphism group.

Now, when enumerating, we apply each expansion, but only accept it if the inverse reduction lies in our favorite orbit of reductions.

Example We prefer removing dual-pairs over removing self-dual vertices.

Thus we don't accept

because the favorite reduction is

When our preferences are 'transparent' we can often save work by not even building certain expansions.

Other times it is hard to choose favorites — and there is some cost in doing so, typically handled by nauty's fast graph canonical labelling algorithms.

Now we know how to enumerate $\mathcal{P}(L,d)$ efficiently.

Write $\mathcal{P}(L) = \bigcup_{d \geq 1} \mathcal{P}(L,d)$.

Unfortunately whenever $L \geq 4$ this is certainly an infinite set, because it contains

Write T
for the operation of increasing the supertransitivity by 2.

(observe if Γ is associative, $T(\Gamma)$ is too, and $\|T(\Gamma)\| > \|\Gamma\|$)

Write $\mathcal{P}^{(k)}(L)$ ~~for the exactly k-supertr~~
$= \{\Gamma \in \mathcal{P}(L) \mid \Gamma \text{ is exactly } k\text{-supertransitive}\}$.

Then $\mathcal{P}^{(k+2)}(L) \subset T(\mathcal{P}^{(k)}(L))$

so we can write

$$\mathcal{P}(L) = \mathcal{P}^{(1)}(L) \cup \bigcup_{k \geq 0} T^k \mathcal{P}^{(2)}(L) \cup T^k \mathcal{P}^{(3)}(L)$$

$$= \bigcup_{k \geq 0} T^k \mathcal{P}^{(1)} L \cup T^k \mathcal{P}^{(2)}(L)$$

or better, ——, because typically the 1-supertransitive case is so different.

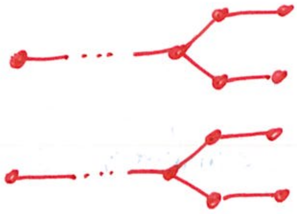
These are still unlikely to be finite sets!

We want to identify some "obstructions" $f : \mathcal{P}^{(k)}(L) \to \{\text{true}, \text{false}\}$ so that ~~all principal graphs com~~ if Γ comes from a subfactor, $f(\Gamma) = \text{true}$, and $f(\Gamma) = \text{false} \Rightarrow f(T(\Gamma)) = \text{false}$ | if Γ^* is an expansion of Γ $f(\Gamma^*) \Rightarrow f(\Gamma)$

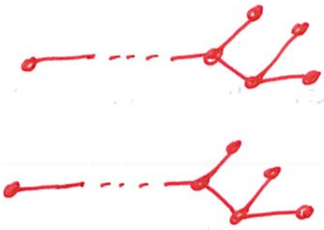
The two important obstructions are

① Ocneanu's triple point obstruction:

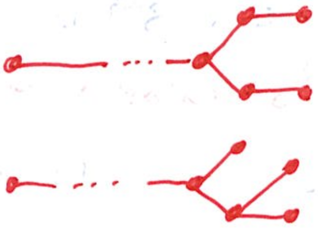
"



and



are forbidden, but



is allowed."

(and its generalizations, Jones arXiv:
Snyder arXiv:
Penneys arXiv:

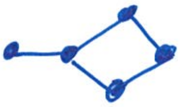
② Popa / Bigelow-Penneys 'stability obstruction':

Stability

We say a principal graph is <u>stable</u> at depth n if each vertex at depth n is connected to <u>at most one</u> vertex at depth $n+1$, and each vertex at depth $n+1$ is connected to exactly one vertex at depth n.

Examples:

is not stable at depth 1



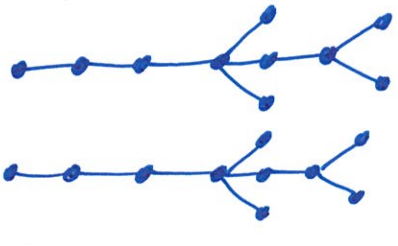
is not stable at depth 2

is stable at depth 3.

<u>Theorem</u> (Popa, c.f. also Bigelow-Penneys arXiv:1208.1564)
If (Γ, Γ') is exactly k-supertransitive and the principal graph of a subfactor, with index >4, and stable at depth n, for some $n > k$, then it is stable at all depths $n' \geq n$.

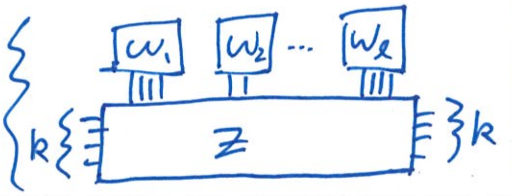
(Also, (Bigelow-Penneys) if just Γ is stable at depths $n, n+1$, then both Γ and Γ' are stable at $n' \geq n+1$.)

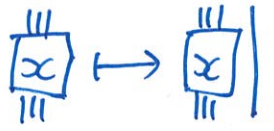
Example



is not the principal graph of a subfactor.

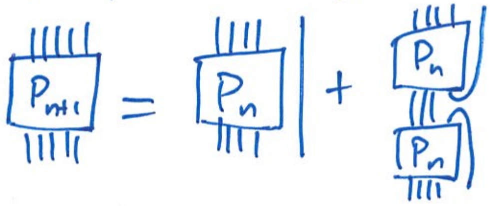
To prove the stability theorem, we need the notion of trains. For a subset $W \subset P_\bullet$, define

$$\mathrm{trains}_k(W) = \mathrm{span}\left\{ \right.$$  $$\left. \middle| \begin{array}{l} w_i \in W \\ Z \in TL \end{array} \right\}$$

Write $i : P_n \longrightarrow P_{n+1}$ for the inclusion .

<u>Lemma 1</u> $\mathrm{trains}_{n+k}(P_{\leq n}) = \langle i^k(P_n), TL_{n+k} \rangle$

the associative algebra generated by $i^k(P_n)$ and TL_{n+k}.

We say a planar algebra $P_\bullet$ is stable at depth n if  [redacted].

<u>Lemma 3</u> $\Gamma(P_\bullet)$ is stable at depth $n \iff P_\bullet$ is stable at depth n

<u>Lemma 2</u> $P_\bullet$ is stable at depths $n, n+1, \ldots, k-1$

$$\Updownarrow$$

$$\mathrm{trains}_{n+k}(P_{\leq n}) = P_{n+k}$$

Proof of the stability theorem

Take $Q_\bullet \subset P_\bullet$ to be the planar subalgebra generated by P_n.

Claim $Q_\bullet$ is stable at all depths $n' \geq n$.

Proof By Lemma 3, $P_\bullet$ is stable at depth n.

By Lemma 2, $P_{n+1} = \text{trains}_{n+1}(P_n)$.

In particular  $\in \text{trains}_{n+1}(P_n)$ $\forall x \in P_n$.

Thus we have jellyfish relations for all the generators of $Q_\bullet$, and so

$$Q_{n+k} = \text{trains}_{n+k}(P_n) \text{ for all } k.$$

Applying Lemma 2 again gives the result.

Now, $\Gamma(Q_\bullet)$ is stable at all depths $n' \geq n$ (Lemma 3), and $\Gamma(P_\bullet)$ and $\Gamma(Q_\bullet)$ are identical up to depth n.

A stable graph cannot have infinite tails, so $Q_\bullet$ is finite depth, and hence $P_\bullet$ is also.

A purely graph theoretic argument now gives
└ see Theorem 3.10 of the paper.

$$\Gamma(P_\bullet) = \Gamma(Q_\bullet).$$ (this is the only place we use index > 4)

(and moreover, $P_\bullet = Q_\bullet$) $\square$

Proof of Lemma 1

(Let's write composition horizontally.)

An element of $\langle i^k(P_n), TL_{n+k}\rangle$ is already in train form.

Given a train in $\mathrm{trains}_{n+k}(P_{\leq n})$, we may assume

1) no box from $P_{\leq n}$ has a cap

2) no two adjacent boxes are connected by at least half their strands

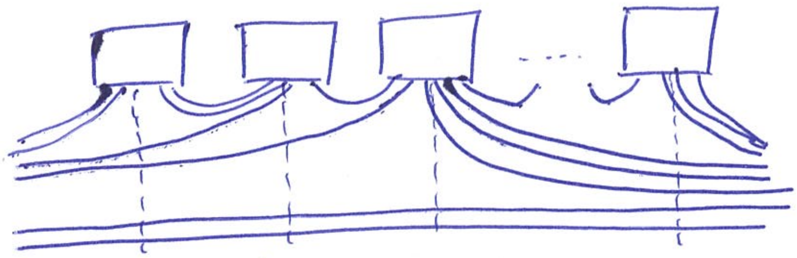
(as otherwise we multiply them, reducing the number of boxes)

and as a consequence

3) no two non-adjacent boxes are connected at all.

(otherwise, a corner argument as in the jellyfish algorithm contradicts 2).)

Thus our train looks like



It's now easy to see that dropping a vertical line from the midpoint of an m-box crosses at most k strings, so cutting along these lines we can write the train as a word in

$\langle i^k(P_n), TL_{n+k}\rangle$.

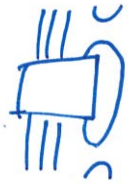
Proof of Lemma 2

k=1 case: $P_{n+1} = i(P) + i(P) e_n i(P) \Rightarrow P_{n+1} = \mathrm{trains}(P_{\le n})$.

Given any train in $\mathrm{trains}_{n+1}(P_{\le n})$, by Lemma 1 we can write it as

$$w_0 z_0 w_1 z_1 \cdots z_{k-1} w_k, \quad w_i \in i(P_n), \; z_i \in TL_{n+1}$$

We may assume each z_i is just e_n; all other TL generators ~~can be absorbed into~~ are in $i(P_n)$.

However $e_n w_i e_n =$  $= \mathrm{tr}_1(w_i) e_n$

so we can reduce to the case $k=1$, and we're done.

general k: Suppose $P_\bullet$ is stable at depths $n, n+1, \ldots, k-1$

Then $\mathrm{trains}_{n+j+1}(P_{n+j}) = P_{n+j+1}$ for $j = 0, \ldots, k-1$

Thus $P_{n+k} = \mathrm{trains}_{n+k}(P_{n+k-1})$

$= \mathrm{trains}_{n+k}(\mathrm{trains}_{n+k-1}(P_{n+k-2}))$

$= \ldots$

Conversely, if $\mathrm{trains}_{n+k}(P_n) = P_{n+k}$ then by taking partial traces

$$P_{n+j} = \mathrm{trains}_{n+j}(P_n) \text{ for } j = 0, \ldots, k$$

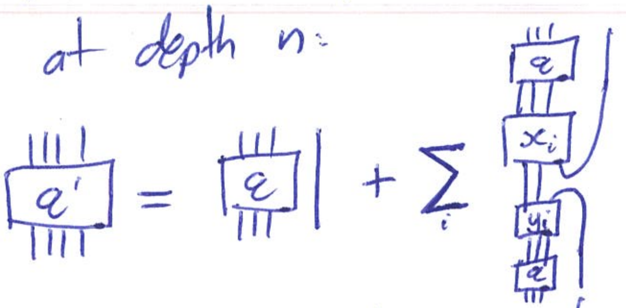
Then $P_{n+j+1} = \mathrm{trains}_{n+j+1}(P_n) = \mathrm{trains}_{n+j+1}(\mathrm{trains}_{n+j}(P_n))$

$= \mathrm{trains}_{n+j+1}(P_{n+j})$

Proof of Lemma 3

Suppose $P_\bullet$ is stable at depth n.
We want to show $\Gamma(P_\bullet)$ neither forks nor fuses at depth n.

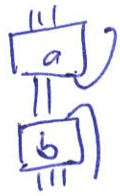
Wenzl's formula gives us an expression for all the vertices at depth $n+1$ connected to a given vertex q at depth n:



We want to show q' is simple (this shows no forks)

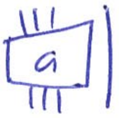
Suppose we have some $f : q' \to q'$.

If ~~[illegible]~~ $f =$



, $q' f q' = 0$ since q' is depth n

If $f =$



, then $q' f q' = \alpha\, q' q q' = \alpha q'$ since q is simple.

Thus there is just a 1-d space of maps $q' \to q'$ in $P_n + P_n e_n P_n$.

~~Suppose p & q are projections at depth n+1.~~

Suppose ~~pq=0. We want to prove~~
Hom$(p \to q) = 0$. We want to show Hom$(p' \to q') = 0$.

Certainly $p'($

$)q' = 0$ since Hom$(p \to q) = 0$.

For $f \in P_n e_n P_n$, $p' f q' = 0$ since p' & q' are depth $n+1$.

Thus there is no fusing.

Suppose now $\Gamma(P_\bullet)$ is stable at depth n.

Recall

Theorem (Calegari-M-Snyder) arXiv:

In a family of graphs $\Gamma_n =$ ,

there is an explicit integer $N(\Gamma_0)$ so for $n \geq N(\Gamma_0)$

the index of Γ_n is not cyclotomic.

(and hence Γ_n is not the principal graph of a subfactor).

In practice this integer is ≤ 300, and we can directly check that $\mathrm{index}(\Gamma_n)$ is not cyclotomic for $n < N(\Gamma_0)$ too.

(exceptions tend to be principal graphs of subfactors!)

Recently,

Theorem (Zoey Guo, a student of Calegari at Northwestern.)

Fix $n, k > 0$. Let $S_{n,k}$ be the set of finite graphs with

the degree of every vertex bounded by k, and

at most n vertices of degree > 2.

Then at most finitely many graphs in $S_{n,k}$ have cyclotomic index.

Unfortunately, this theorem isn't (yet?) effective.

It guarantees that at most finitely many graphs represented by a cylinder can be principal graphs of subfactors, but without providing an upper bound on the cylinder lengths.