In these lectures, I’m going to tell you about the classification of small index subfactors.

I’ll start out by summarizing the current situation
- explaining the regimes in which we have complete classifications, and
- describing the examples we see there.

As I do that, I’ll explain the constructions used in realizing these examples.

I want to emphasize that
- relatively few constructions are needed to realize almost everything we’ve seen,
- but that there are few exceptional examples, apparently isolated from the rest of the mathematical universe.

Finally, probably in the 3rd lecture, I’ll explain the mechanism of classification theorems, and tell you about some recent developments that have allowed us to look further out.
Even though these lectures are primarily about small index subfactors, it's important to realize this is a narrow view of the subject.

Perhaps someday we'll stand atop the mountain, and see the whole landscape of subfactors clearly. But just now we've barely left the trailhead. Understanding small index subfactors looked like a reasonable route into the foothills, but it's certainly not the only way.

To make matters worse, our trail is starting to become indistinct, and the undergrowth thicker. We know there are cliffs ahead we're not equipped to climb — but at least, through the trees, we're getting glimpses of the mountain.

I'll be trying, when possible, to mention the wider view.
We've been drawing a map!
Theorem (Index <4) (Oe, Jones, Izumi, Kawahigashi, et al.)

The principal graph must be a Dynkin diagram, with * on the longest arm.

The graphs $D_{odd}$ & $E_7$ don't extend to fusion rules.

- $A_n$ is realized by the quotient of $T_2$ at a root of unity.
- $D_{2n}$ can be obtained from $A_{4n-3}$ by 'de-equivariantization'.
- $E_6$ and $E_8$ arise as module categories over $A_{11}$ & $A_{29}$, as the module objects for a certain commutative algebra object.
Theorem (Index exactly 4) (Papa, et al.)

The principal graph must be an affine Dynkin diagram.

All are realized as \( P_n = \text{End}_G((C^2)^n) \)

for some \( G \subset SU(2) \). There are also relatives twisted by cohomological data.

<table>
<thead>
<tr>
<th>( A_{2n} )</th>
<th>( A_{2m-1} )</th>
<th>( D_n )</th>
<th>( E_6 )</th>
<th>( E_7 )</th>
<th>( E_8 )</th>
<th>( A_\infty )</th>
<th>( D_\infty )</th>
<th>( A_\infty )</th>
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<tbody>
<tr>
<td>binary</td>
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<td>sub(2)</td>
<td>binary</td>
<td>dihedral</td>
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</table>

0   n   n-2   1   1   1   1   1   1
Theorem (Index $\sqrt{5}$ in (4,5))

An extremal subfactor with index in (4,5) is either
non-amenable with standard invariant $A_\infty$,
or it is one of 10 cases, with planar algebra

$H, EH, AH, 3311, 2221$
or its dual or its complex conjugate
Theorem (Index exactly 5)

There are 7 subfactor planar algebras, all group-subgroups or their duals.

1. $\mathbb{Z}/5\mathbb{Z}$
2. $\mathbb{Z}/2\mathbb{Z}$ c $D_5$
3. $\mathbb{Z}/4\mathbb{Z}$ c $\mathbb{Z}/5\mathbb{Z} \times \text{Aut}(\mathbb{Z}/5\mathbb{Z})$
4. $A_4$ c $A_5$
5. $S_4$ c $S_5$

(The first 3 are self-dual, the last 2 are not)

* some loose ends!

In the interval \((5,3+\sqrt{5}]\), there are 10 subfactor planar algebras.

At index \((\sqrt{5}+1+\sqrt{5})^2 \approx 5.04892\)

two quantum group subfactors, coming from

\[ SU(2)_4 \subset C \subset C^3 \]

At index \(3+\sqrt{5} \approx 5.23\)

the Fuss-Catalan algebra \(A_3 \ast A_4\)

along with 3 quotients \(A_3 \ast A_4 = \ast \), \(\hat{\ast} \), \(\hat{\hat{\ast}} \)

(all 1-supertransitive)

and \(3^{\mathbb{Z}/2} \times \mathbb{Z}/2\) and \(3^{\mathbb{Z}/4}\)

along with "4442" (the \(\mathbb{Z}/32\) fixed points of \(3^{\mathbb{Z}/2} \times \mathbb{Z}/2\))

and "2D2" (which contains \(3^{\mathbb{Z}/4}\) as a \(\mathbb{Z}/2\) fixed pt)

The only worrisome loose end is

\[ \begin{array}{c}
\begin{array}{c}
\text{a} \\
\hline
\text{b}
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\text{a} \\
\hline
\text{b}
\end{array}
\end{array} \]

Suggestions welcome!
Constructions

- Quantum groups at roots of unity $A_n$
  - quantum subgroups (orbifolds, e.g. $D_{2n}$ and exceptional $E_6(E_8)$)
  (also conformal inclusions)
- Finite groups $- R^g < R^h$
  - twists by cohomological data
- Composites $-$ quotients of free products
  (at index 4, the $\hat{D}_n$'s are quotients of $\hat{D}_4 = A_3 \times A_3$)
  (at index $3 + \sqrt{5}$, there are only finitely many (3)
  quotients of $A_3 \times A_q$ Liu arXiv:
  Izumi-M-Penneys
  or arXiv:
- Near groups $-$ fusion categories with an
  abelian group subcategory with
  one another orbit.
  $(3^a, \text{ with Haagerup } = 3^{2/3}, 2^{a1})$
- Bimodule $-$ categories for algebra objects
  somewhere in the maximal atlas
  (e.g. $A_{11} C^* E_6$, let $A = \text{End}_{A_{11}}(1_{E_6}, 1_{E_6})$, then take $A_{11}-\text{mod}$, $A_{11}$-mod)
  (more complicated (Izumi-Grossman-Snyder):
  $A_1$ can be obtained from $3^{Z_2 \times Z_2}$)

- EH, constructed 'with our bare hands', as a subalgebra
  of the graph planar algebra.
From planar algebras to $\otimes$-categories and back again

A shaded planar algebra gives a pair of $\otimes$-categories.
(If our planar algebra came from a subfactor, $A \subset B$, these are just $A\text{-mod-}A$ and $B\text{-mod-}B$.)

Given $P_0$, $\hat{C}_\pm(P_0)$ is the $\otimes$-category

$$\text{Obj} = \mathbb{N}$$

$$\text{Hom}(n \to m) = \begin{array}{c}
\begin{array}{c}
\text{boxes with } 2n+2m \\
\text{boundary points}
\end{array}
\end{array} \quad P_{n+m, \pm}$$

(composition \begin{array}{c}
\begin{array}{c}
\text{tensor product}
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\text{tensor product}
\end{array}
\end{array}
)

We can idempotent complete to obtain $C_\pm(P_0)$.
The simple objects are the even depth vertices on the principal graph $P_\pm$.

There are also bimodule categories between these: $C_+ \otimes C_-$ and $C_+ \otimes C_+$.
Here $C_+ \otimes C_+ \otimes C_+ = C_- \otimes C_-$.
(Again, if we obtain our planar algebra from a subfactor $A \subset B$, these are just $A\text{-mod-}B$ and $B\text{-mod-}A$.)

In $\hat{C}_\pm(P_0)$,

$$\text{Obj} = 2\mathbb{N}+1$$

$$\text{Hom}(2n+1 \to 2m+1) = P_{n+m+1, \pm} \quad \begin{array}{c}
\begin{array}{c}
\text{boxed with } 2n+1 \\
\text{boundary points}
\end{array}
\end{array}
$$

These are Morita equivalences: $C_- \otimes C_- \cong C_+$, etc.
The maximal atlas

An algebra in a \(\mathcal{C}\)-category is
  
  * an object \(A\), along with
  * a map \(\Delta: A \otimes A \rightarrow A\)

such that \(\Delta = \Delta\).

A module over an algebra \(A\) in a \(\mathcal{C}\)-category is
  
  * an object \(M\), along with
  * a map \(\varepsilon: A \otimes M \rightarrow M\)

such that

\[
\varepsilon \circ \Delta = \varepsilon
\]

The collection of all \(A\)-modules forms a category "\(A\)-mod".

A map of \(A\)-modules satisfies

\[
\varepsilon_M = \varepsilon
\]
As we work in a unitary fusion category, we also require:
\[
\bigotimes = \mathbb{1} \quad \text{and} \quad \mathbb{1}^* = \mathbb{1} = \mathbb{1}^* (\mathbb{1} = \bigotimes)
\]

**Fact** A-mod-A, the category of A-A bimodules, is a unitary tensor category.

Observe that if we started with a planar algebra and build \( \hat{\mathcal{E}} \) as before, \([1] \) has a canonical algebra structure given by \( \bigotimes \in \mathcal{P}_{3,+} \).

\( \mathcal{E} \) Given an inclusion of algebras \( A \subset B \) in \( \mathcal{E} \), we can extract a shaded planar algebra. Write \( X := _A B_B \). Write \( \bigotimes^n V = \bigotimes_{i=1}^n V \otimes V^* \) for \( n \) factors.

Define
\[
P_{n,+} = \text{Hom}_{A-A} (A_A \rightarrow \bigotimes^n_A X_A)
\]
\[
P_{n,-} = \text{Hom}_{B-B} (B_B \rightarrow \bigotimes^n_B X_B^*)
\]

**Claim** Starting with \( (A=[0]) \subset (B=[1]) \) in \( \hat{\mathcal{E}} \), we recover the planar algebra we built \( \hat{\mathcal{E}} \) from.
How do we find algebras?

(1) $V^* \otimes V$ is always an algebra, with

Then the construction above simplifies to for $1 \leq V^* \otimes V$

$$P_{n,+} = \text{Hom}(1 \to \hat{\otimes}^n V)$$

$$P_{n,-} = \text{Hom}(1 \to \hat{\otimes}^n V^*)$$

This is sometimes called 'shading on unshaded planar algebra'.

E.g. Start with the Fibonacci category

$$\text{Obj} = \{1, x, x^2 \} \quad x^2 = 1 \otimes x.$$  

The principal graph for the shading is

$$1 \quad x \quad x \quad x \quad 1.$$
Quantum groups at roots of unity

Associated to any $U_q(g)$, with $q$ a root of unity (of sufficiently large order), there is an associated semisimple braided $\otimes$-category $\text{Rep}U_q(g)$. The simple objects correspond to simple objects in $\text{Rep}(g)$ which are below a certain 'wall' (depending on $q$) in the Weyl chamber.

Picking a simple object $V$, we obtain a planar algebra

\[
P_{n,+} = \text{End}_{U_q(g)}(V \otimes V^* \otimes V \otimes \ldots) / \text{factor}
\]

\[
P_{n,-} = \text{End}_{U_q(g)}(V^* \otimes V \otimes V^* \otimes \ldots)
\]

If $q$ is a root of unity closest to 1, this planar algebra is positive definite (see Wenzl), and in fact a subfactor planar algebra.

Example $U_q(SU_2)$ at a $14$th root of unity has $6$ simple objects. If we take the standard representation $V = C_{e^2}$, we obtain the $A_5$ subfactor.

Example If we take the $3d$ representation instead, you can compute the principal graph of the resulting subfactor using the $\otimes$-product rule in $A_5$.

Exercise its
Quantum subgroups

Consider $C$, a braided $\otimes$-category, and $A$, a commutative algebra.

(since we have a map $\lambda : A \otimes A \to A$, satisfying

\[ \lambda = \lambda \quad \text{and} \quad \lambda \otimes \lambda = \lambda, \]

and we can normalize it as $\lambda = 1$)

(More generally, one can work in a $\otimes$-category with an algebra that lifts to centre: We'll need this later. This allows us to capture equivariantization as a special case.)

The category of $A$-modules

(objects $M$ equipped with a map $\lambda : A \otimes M \to M$

satisfying $\lambda \otimes \lambda = \lambda$)

naturally forms a $\otimes$-category:

\[
\begin{array}{c}
A \\
M
\end{array} \quad = \quad \begin{array}{c}
A \\
M
\end{array} \quad = \quad \begin{array}{c}
A \\
M \end{array}
\] (Exercise check associativity)

Example The $A_n$ planar algebra is braided

(it's the planar algebra associated to $U_{\mathbb{C}^2} C_C C_\infty^2$, $q^{24} = 1$)

and $A = V_0 + V_8 + V_{10}$

is a commutative algebra.

The category of $A$-modules is the $E_6$ planar algebra.
(De)equivariantization

If \( G \), a finite group, acts on a planar algebra \( P_0 \),
the fixed point subalgebra \( P^G_0 \) is called the equivariantization.

If \( P_0 \) contains an abelian group of invertible objects,
it is sometimes possible to add morphisms making all
these invertible objects isomorphic to \( 1 \).

Generally, if there is a \( \text{Rep}G \) subcategory lifting to the centre,
we can take \( C[GJ\text{-mod}] \).

This is called de-equivariantization.

When it's possible, there an action of \( G \) on the resulting
planar algebra, and fixed points recovers the original \( P_0 \).

Example In \( A_{4n-3} \), we can add
an isomorphism \( S:1 \to Sg \), obtaining

\[
D_{2n} = \begin{array}{c}
\cdots \\
\bullet \\
\cdots 
\end{array}
\]

(There's a \( Z/2Z \) action negating \( S \).)

How do we find algebras? It is always and algebra ‘standing on unbalanced’.

(2) One way is using internal Hom. For $M$ a $C$-module category, this is characterized by $M_1, M_2 \in M$,
\[ \text{Hom}(M_1, M_2) \in C \]
characterized by
\[ \text{Hom}_C(C, \text{Hom}(M_1, M_2)) \cong \text{Hom}_M(C \otimes M_1, M_2). \]

Internal endomorphisms is always an algebra object.

Example $E_6$ is a module category over $A_{11}$,
\[ \text{End}(1_{E_6}) =: A \]
is an algebra object, in $A_{11}$
and the planar algebra for $1 \subset A$ is the $3311$ planar algebra.

Question What are all the algebra objects in your favourite $C$-category?

Example Izumi-Grossman-Snyder have constructed a $3^{Z/4 \times Z/2}$ subfactor, and shown:
The $Z/2 \subset Z/4$ subcategory lifts to the centre, and we can de-equivalanantize by it, obtaining a subfactor with principal graph
\[
\begin{array}{c}
\circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array}
\]

There is an inclusion of algebras
\[ 1 + X \subset 1 + 2X + qX + q^2X + q^3X, \]
and the corresponding PA is $A_{14}$. 

Composite planar algebras

Given planar algebras \( P \) & \( Q \), we can form

\[(P \otimes Q)_n = P \otimes Q \]

with planar tangles acting as \( T(p \otimes q) = T(p) \otimes T(q) \).

(You should think of \( P \otimes Q \) diagrams as being a \( P \) diagram 'superimposed' over a \( Q \) diagram.)

We can also form the free product \( P * Q \).

\[(P * Q)_n = \bigoplus_{n\text{-paintings}} \left( \bigotimes_{P\text{-regions}} P_{r_1} \right) \otimes \left( \bigotimes_{Q\text{-regions}} Q_{l_1} \right) / \text{an empty } P\text{-region} \\
\text{an empty } Q\text{-region}
\]

An \( n \)-painting is division of a disc with \( 4n \) points on the boundary, labelled \( PQPQP \ldots QQP \), into "\( P\)-regions" and "\( Q\)-regions", with boundary points in appropriate regions.

Some 2-paintings:

\[\]

Some typical elements of \((TL \ast TL)_2\):

\[\]

\[\]

\[=\]
Observe that \((P \ast Q) \subset (P \otimes Q)\) as the non-crossing diagrams.

Alternatively, we can obtain \(P \otimes Q\) from \(P \ast Q\) by adding a generator: \(\times\) satisfying appropriate relations.

**Definition** A composite of \(P\) and \(Q\) is a planar algebra \(F\) with \((P \ast Q) \subset F\).

Every composite has a special element \(\otimes \in F_2\), which is a bi-projection:

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{bi_projection.png}
\end{array}
\]

Conversely, given any planar algebra \(F\) with a bi-projection \(\phi\), we can construct \(P\) and \(Q\) and \((P \ast Q) \subset F\), recovering the bi-projection.

This theory has been extensively developed by Dietmar Bisch and Vaughan Jones.

**Question** Should we expect lots of composites besides the tensor product?

\(\mathbb{Z}/2 \ast \mathbb{Z}/3 = \text{SL}(2,\mathbb{Z})\), which has infinitely many finite quotients.

Bisch-Haagerup subfactors give infinitely many composite subfactors planar algebras of index 6.

\(A = \mathbb{A}(\mathbb{Z}/3), B = \mathbb{A}(\mathbb{Z}/2)\) and \(A \cong \mathbb{A}(\mathbb{Z}/6) \otimes \mathbb{A}(\mathbb{Z}/3)\).
Example Consider the fusion categories

\[ A_2 = \begin{pmatrix} \infty & \infty \\ \infty & \infty \end{pmatrix}, \quad \sigma = 1 \]

\[ T_2 = \begin{pmatrix} \infty \\ \infty \end{pmatrix}, \quad \sigma = \frac{1 + \sqrt{5}}{2}, \quad \chi = \left( 1 - \frac{1 + \sqrt{5}}{2} \right) \]

What composites are possible?

An relatively easy argument shows that any composite would contain an element \( U \) :\( (ab) \rightarrow 1 \)

- \( UU^* = 1 \)
- \( U \cup^a = U^* \cap^a \)
- \( \sigma = 2, \quad U^* = U \cap^2 \)

These are \( \sigma \) instances of "jellyfish" relations, and suffice to evaluate any closed diagram. This shows that a composite of \( A_2 \) & \( T_2 \) is uniquely determined by the parameters \( n \) and \( \sigma \).
Since the successful classification of subfactors with index at most 5 (leaving out, as always, the non-amenable subfactors with trivial standard invariant), there have been four significant advances which make it plausible to push further.

1. Liu's results on the presence of intermediate subfactors (arXiv:1308.5656)


3. Penney's results on triple point obstructions (arXiv:1307.5890)

4. Afzal's new isomorph-free principal graph enumerator.
Recall a principal graph $\Gamma$ consists of a pair $(\Gamma_0, \Gamma_1)$ of locally finite, connected, pointed graphs, and a involution preserving depth (distance from the base point), and taking even depth vertices on $\Gamma_i$ to $\Gamma_i$, and taking odd depth vertices on $\Gamma_i$ to $\Gamma_{1-i}$, such that $x(\overline{x(v)}) = \overline{x(x(w))}$.

where $x(w)$ denotes the multiset of neighbours of $w$.

This is called the associativity condition.

(Recall a subfactor $A \subset B$ gives a principal graph —
$\Gamma_0$ has vertices the simple $A$-$A$ and simple $A$-$B$ bimodules $\Gamma_1$— $B$-$B$ $B$-$A$ —,
there are $\dim \text{Hom}(V \otimes B, W)$ edges from $V$ to $W$.
The involution is the dual, and the associativity condition says $(B \otimes V) \otimes B \cong B \otimes (V \otimes B)$.

We write $P(L, d)$ for the set of principal graphs with index $\leq L$ (index is the square of the largest eigenvalue of the adjacency matrix) and depth $\leq d$.}
We say the associativity condition holds between vertices \( v \) and \( w \) if the multiplicities of \( w \) in 
\[ x(v) \] and in \[ x(w) \] agree.

**Definition** A partial principal graph is a pair \((P, n)\) where \( P \) is a principal graph and \( n = \text{depth} P \) or \( \text{depth} P + 1 \), except that the associativity condition need not hold between vertices \( v \) and \( w \) when \((\text{depth} v, \text{depth} w) = (n-1, n-1), (n, n) \) or \((n-2, n)\).

If \((P, n)\) is a partial principal graph, and \( P' \) is obtained from \( P \) by deleting a self-dual vertex at depth \( n \) or by deleting a pair of dual vertices at depth \( n \), then \((P', n)\) is also a partial principal graph.

If \((P, n)\) is a partial principal graph and \( n = \text{depth} P + 1 \), then \((P, n-1)\) is also a partial principal graph. These operations are called reductions.
Lemma There are only finitely many partial principal graphs
with index $\leq L$ that reduce in one step
to a given $(P,n)$.

Lemma Every partial principal graph can be recursively
reduced to $(\emptyset,0)$.

An expansion is the inverse of a reduction.

Lemma Only finitely many vertex-addition expansions can
be applied to $(P,n)$ without increasing the
index above $L$.

Corollary $\mathcal{P}(L,d)$ is finite, and may be enumerated by
starting at $(d,0)$ and recursively expanding.
An expansion is the inverse of a reduction.

Lemma Only finitely many vertex-addition expansions can be applied to $(P, n)$ without increasing the index above $L$.

Corollary $P(L, d)$ is finite, and may be enumerated by starting at $(0, 0)$ and recursively expanding.

The challenge, however, is to efficiently enumerate $Q(L, d)$.

There are three essential problems:

1. Two expansions of the same graph may be equivalent, e.g. \[ \begin{array}{c}
\text{equivalent} \\
\end{array} \]

2. Two inequivalent expansions may nevertheless give isomorphic results, e.g. 

\[ \begin{array}{c}
\text{isomorphic} \\
\end{array} \]
expansions of two different parents may give the same result,

\[ \text{e.g. } \quad \bullet - \bullet \rightarrow \bullet - \bullet \]

and

\[ \text{and } \quad \bullet - \bullet \rightarrow \bullet - \bullet \]

The solution is ‘canonical generation’ [McKay “Isomorph-free exhaustive
geneneration”, ’98].

Amongst all the reductions of an object, we choose our
favourite one, or rather a favourite orbit under
the automorphism group.

Now, when enumerating, we apply each expansion,
but only accept it if the inverse reduction lies
in our favorite orbit of reductions.

Example We prefer removing dual-pairs over removing
self-dual vertices.

Thus we don’t accept

\[ \bullet - \bullet \rightarrow \bullet - \bullet \]

because the favorite reduction is

\[ \bullet - \bullet \rightarrow \bullet - \bullet \]

When our preferences are ‘transparent’ we can often
save work by not even building certain expansions.

Other times it is hard to choose favorites — and there
is some cost in doing so, typically handled by
nauty’s fast graph canonical labelling algorithms.
Now we know how to enumerate $P(L,d)$ efficiently. Write $P(L) = \bigcup_{d \geq 1} P(L,d)$.

Unfortunately whenever $L \geq 4$ this is certainly an infinite set, because it contains
\[ \ldots, \ldots, \ldots, \ldots, \ldots \]

Write $T$ for the operation of increasing the supertransitivity by 2. Observe if $\Gamma$ is associative, $T(\Gamma)$ is too, and $\|T(\Gamma)\| > \|\Gamma\|$.

Write $P^{(k)}(L)$ for the exactly $k$-supertransitive
\[ = \{ \Gamma \in P(L) \mid \Gamma \text{ is exactly } k \text{-supertransitive} \} \]

Then $P^{(k+2)}(L) \subset T(P^{(k)}(L))$

so we can write
\[ P(L) = P^{(1)}(L) \cup \bigcup_{k \geq 0} T^k P^{(0)}(L) \cup T^k P^{(2)}(L) \]

or better, \[ \ldots \] because typically the $1$-supertransitive case is so different.

These are still unlikely to be finite sets!

\[ \text{We want to identify some "obstructions" } f: P^{(k)}(L) \rightarrow \{ \text{true, false} \} \]
so that all principal graphs come if $\Gamma$ comes from a subfactor $f(\Gamma) = \text{true},$ and $f(\Gamma) = \text{false} \Rightarrow T(\Gamma) = \text{false}$ if $\Gamma^*$ is an expansion of $\Gamma$.
The two important obstructions are

1. Ocneanu's triple point obstruction:

```
  ---\-
  |   |
  |   |
```

are forbidden, but

```
  --\-
  |   |
  |   |
```

is allowed.

(and its generalizations, \(\text{Jones arXiv:}\)

\(\text{Snyder arXiv:}\)

\(\text{Penneys arXiv:}\)

2. Popa / Bigelow-Penneys 'stability obstruction'.
Stability

We say a principal graph is stable at depth $n$ if each vertex at depth $n$ is connected to at most one vertex at depth $n+1$, and each vertex at depth $n+1$ is connected to exactly one vertex at depth $n$.

Examples:
- $\bullet - \bullet$ is not stable at depth 1
- $\bullet - \bullet - \bullet$ is not stable at depth 2
- $\bullet - \bullet - \bullet - \bullet$ is stable at depth 3.

Theorem (Popa, c.f. also Bigelow-Penneys arXiv:1208.1564)
If $(\Gamma, \Gamma')$ is exactly $k$-supertransitive and the principal graph of a subfactor, with index $>4$, and stable at depth $n$, for some $n \geq k$, then it is stable at all depths $n' > n$.

(Also, (Bigelow-Penneys) if just $\Gamma$ is stable at depths $n, n+1$, then both $\Gamma$ and $\Gamma'$ are stable at $n' \geq n+1$.)

Example
- $\bullet - \bullet - \bullet - \bullet$ is not the principal graph of a subfactor.
To prove the stability theorem, we need the notion of trains. For a subset $W \subseteq P_n$, define

$$\text{trains}_k(W) = \text{span}\left\{ \sum_{i=1}^{k} w_i \right\} \quad w_i \in W, \quad z \in T_n$$

Write $i : P_n \rightarrow P_{n+1}$ for the inclusion $x \mapsto x^k$.

**Lemma 1** $\text{trains}_{n+k}(P_{\leq n}) = \langle i^k(P_n), TL_{n+k} \rangle$.

We say a planar algebra $P$ is stable of depth $n$ if

$$P_{n+1} = P_n \oplus P_{n+1}$$

**Lemma 2** $P$ is stable of depth $n$, $n+1$, ..., $k-1$ if and only if

$$\text{trains}_{n+k}(P_{\leq n}) = P_{n+k}$$

**Lemma 3** $\Gamma(P)$ is stable of depth $n$ if and only if $P$ is stable of depth $n$.
Proof of the stability theorem

Take $Q_0 \subseteq P_n$ to be the planar subalgebra generated by $P_n$.

Claim $Q_0$ is stable at all depths $n' \geq n$.

Proof By Lemma 3, $P_n$ is stable at depth $n$.

By Lemma 2, $P_{n+1} = \text{trans}_{n+1}(P_n)$.

In particular $\forall x \in \text{trans}_{n+1}(P_n)$, $\forall x \in P_n$.

Thus we have jellyfish relations for all the generators of $Q_0$, and so $Q_{n+k} = \text{trans}_{n+k}(P_n)$ for all $k$.

Applying Lemma 2 again gives the result.

Now, $\Gamma(Q_0)$ is stable at all depths $n' \geq n$ (Lemma 3), and $\Gamma(P_n)$ and $\Gamma(Q_0)$ agree are identical up to depth $n$.

A stable graph cannot have infinite tails, so $Q_0$ is finite depth, and hence $P_n$ is also.

A purely graph theoretic argument now gives $\Gamma(P_n) = \Gamma(Q_0)$. (See Theorem 3.10 of the paper.)

(and moreover, $P_n = Q_0$)
Proof of Lemma 1

(Let's write composition horizontally.)

An element of \( \langle c_k(P_n), T_{L\text{mtk}} \rangle \) is already in train form.

Given a train in \( \text{trains}_{\text{mtk}}(P_n) \), we may assume

1) no box from \( P_{\Delta n} \) has a cap

2) no two adjacent boxes are connected by at least half their strands

(as otherwise we multiply them, reducing the number of boxes)

and as a consequence

3) no two non-adjacent boxes are connected at all.

(otherwise, a corner argument as in the jellyfish algorithm contradicts 2).

Thus our train looks like

![Diagram of train]

It's now easy to see that dropping a vertical line from the midpoint of an m-box crosses at most k strings, so cutting along these lines we can write the train as a word in \( \langle c_k(P_n), T_{L\text{mtk}} \rangle \).
Proof of Lemma 2

\textbf{k=1 case:}\quad P_{n+1} = i(P) + i(P) e_{n} i(P) \implies P_{n+1} = \text{trains}(P_{n})

Given any train in \text{trains}_{n+1}(P_{n}), by Lemma 1 we can write it as

\[ W_{0} e_{n} W_{1} e_{n} \cdots Z_{k-1} W_{k} e_{n} i(P), \quad Z_{i} \in \text{T}_{n+1} \]

We may assume each \( Z_{i} \) is just \( e_{n} \);

all other T generators can be absorbed into those traces are in \( i(P_{n}) \).

However \( e_{n} W_{i} e_{n} = \begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array} = \text{tr}_{i}(W_{i}) e_{n} \)

so we can reduce to the case \( k=1 \), and we're done.

general \( k \):

\textbf{Assume}\quad P_{n} \text{ is stable at depths } n, n+1, \ldots, k-1

Then \text{trains}_{n+j+1}(P_{n+j}) = P_{n+j+1} \quad \text{for } j = 0, \ldots, k-1

Thus \quad P_{n+k} = \text{trains}_{n+k}(P_{n+k-1})

\[ = \text{trains}_{n+k}(\text{trains}_{n+k-1}(P_{n+k-2})) \]

Conversely, if \text{trains}_{n+k}(P_{n}) = P_{n+k}

then by taking partial traces

\[ P_{n+j} = \text{trains}_{n+j}(P_{n}) \text{ for } j = 0, \ldots, k \]

Then \quad P_{n+j+1} = \text{trains}_{n+j+1}(P_{n}) = \text{trains}_{n+j}(\text{trains}_{n+j}(P_{n}))

\[ = \text{trains}_{n+j}(P_{n+j}) \]
Proof of Lemma 3

Suppose \( P_0 \) is stable at depth \( n \).
We want to show \( \Gamma(P_0) \) neither forks nor fuses at depth \( n \).

Wenzl's formula gives us an expression for all the vertices at depth \( n+1 \) connected to a given vertex \( \epsilon' \) at depth \( n \):

\[
\mathcal{Q}' = \mathcal{Q} + \sum \text{terms}
\]

We want to show \( \mathcal{Q}' \) is simple (this shows no forks)

Suppose we have some \( f: \mathcal{Q}' \to \mathcal{Q}' \).

If \( f = \begin{pmatrix} a & \epsilon' \end{pmatrix} \), \( \mathcal{Q}' \mathcal{Q}' = 0 \) since \( \mathcal{Q}' \) is depth \( n \).

If \( f = \begin{pmatrix} a \end{pmatrix} \), then \( \mathcal{Q}' \mathcal{Q}' = \alpha \mathcal{Q} \) \( \mathcal{Q}' \mathcal{Q}' = \alpha \mathcal{Q}' \)
since \( \mathcal{Q}' \) is simple.

Thus there is just a 1-d space of maps \( \mathcal{Q}' \to \mathcal{Q}' \) in \( P_n \cup P_n e_n P_n \).

Suppose \( p \) & \( q \) are projections at depth \( n+1 \).

Suppose \( p \mathcal{Q} = 0 \). We want to prove \( \text{Hom}(p \to \mathcal{Q}) = 0 \).
We want to show \( \text{Hom}(p \to \mathcal{Q}) = 0 \).

Certainly \( p'((\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})) \mathcal{Q}' = 0 \) since \( \text{Hom}(p \to \mathcal{Q}) = 0 \).
For \( f : P_n \to P_n \), \( p' \mathcal{Q}' = 0 \) since \( p' \) & \( \mathcal{Q}' \) are depth \( n+1 \).

Thus there is no fusing.
Suppose now \( T(P_0) \) is stable at depth \( n \).
Recall

Theorem (Calegari–M. Snyder) arXiv:

In a family of graphs $\Gamma_n$, there is an explicit integer $N(\Gamma)$ so for $n \geq N(\Gamma)$ the index of $\Gamma_n$ is not cyclotomic.

(and hence $\Gamma_n$ is not the principal graph of a subfactor.)

In practice this integer is $B \leq 300$, and we can directly check that $\text{index}(\Gamma_n)$ is not cyclotomic for $n < N(\Gamma)$ too.

(exceptions tend to be principal graphs of subfactors!)

Recently,

Theorem (Zoey Guo, a student of Calegari at Northwestern.)

Fix $n, k > 0$. Let $S_{n,k}$ be the set of finite graphs with

the degree of every vertex bounded by $k$, and

at most $n$ vertices of degree $\geq 2$.

Then at most finitely many graphs in $S_{n,k}$ have cyclotomic index.

Unfortunately, this theorem isn’t (yet?) effective.

It guarantees that at most finitely many graphs represented by a cylinder can be principal graphs of subfactors, but without providing an upper bound on the cylinder lengths.