Fusion categories, modular tensor categories, and subfactors.

My goal today is to tell you about some 'exotic' fusion categories, and their associated modular tensor categories and subfactors.

We know very little about the classifications of these objects.

We've attempted classifications in various 'small' regimes.

$\Rightarrow$ Do most examples fall into families?

Or is it just a mess?

We haven't really looked for enough to have a good guess at the answer.

The intriguing summary so far is:—

- Every known fusion category is related (by taking subcategories, tensor products, graded extensions, (de)equivariantizations, or Morita equivalences) to one of the following:

  1. Rep G, for G a finite group (or $\mathrm{Vec}^\omega G$)
  2. Rep $U_q\mathfrak{g}$, with q a root of unity
  3. a 'quadratic category', with a group of invertible objects, and one other orbit of objects under this group
  4. the extended Haagerup subfactors.

First, however, lets recall what

- fusion categories
- modular tensor categories, and
- subfactors

are, and the relationships between them.

A fusion category is a semisimple tensor category, with finitely many simple objects.

(It's not necessarily symmetric or braided, although some of the first examples, e.g. RepG and Rep$U_q\mathfrak{g}$, happen to be. The category VecG, for G noncommutative is neither.)

(Here 'tensor' is doing a lot of work for us — it means monoidal, and also rigid, so objects and morphisms have well behaved duals. The definition doesn't require the category is pivotal, but it's a very plausible conjecture that this is an automatic consequence.)

A modular tensor category $\mathcal{C}$ is a fusion category which is braided (and spherical), and whose S-matrix

$$S = \Bigg($$  $$\Bigg)_{i,j \in \operatorname{Irr}\mathcal{C}}$$

is invertible.

The representations of a quantum group at a root of unity is often modular,

but Rep G is not, since it is symmetric, so

 $=$  $= \dim(i)\dim(j)$,

and the S-matrix is rank one.

Although this definition formally says an MTC is a special type of fusion category, really you should think of them as lying at different levels

(fusion categories are 2-categories, MTCs are 3-categories).

The centre construction builds an MTC out of a fusion category.

$$Z(\mathcal{C}) := \{ (X, \beta) \mid X \in \mathrm{Obj}(\mathcal{C}),\ \beta : X \otimes - \to - \otimes X \}$$

Two fusion categories $\mathcal{C}$ and $\mathbb{D}$ are <u>Morita</u> equivalent if there is a $\mathcal{C}$-$\mathbb{D}$ bimodule category $\mathcal{M}$ so $\mathcal{M} \underset{\mathbb{D}}{\otimes} \mathcal{M}^* \cong \mathcal{C}$ and

$$\mathcal{M}^* \underset{\mathcal{C}}{\otimes} \mathcal{M} \cong \mathbb{D}.$$

In fact, $\mathcal{C}$ and $\mathbb{D}$ are Morita equivalent if and only if $Z(\mathcal{C}) \cong Z(\mathbb{D})$ as MTCs.

Finally, a (finite depth) subfactor "$\cong$"
an algebra object A in a fusion category $\mathcal{C}$
(ie an object A and an associative map $A \otimes A \to A$).

We then find that the A-A bimodule objects form a second fusion category $\mathbb{D}$, and the 1-A bimodule objects form a $\mathcal{C}$-$\mathbb{D}$ bimodule category giving a Morita equivalence between $\mathcal{C}$ and $\mathbb{D}$.

(Before Vaughan has a heart attack, this situation is always realized by a pair of II_1 factors $N \subset M$, where $\mathcal{C} = {}_N\mathrm{mod}_N$, $\mathbb{D} = {}_M\mathrm{mod}_M$, and $A = M$ as an N-N bimodule.
In fact, any such subfactor gives the situation above, although the categories don't necessarily have finitely many simples.)

Said another way, a finite depth subfactor is a Morita equivalence $(\mathcal{C}, \mathcal{M}, \mathbb{D})$ along with a choice of generating object in $\mathcal{M}$.

Many people have been trying to classify 'small' fusion categories for various senses of 'small':

- small rank (number of simple objects)
  - this is hard; we know the answers for rank ≤ 3, even rank $= 4$ is incomplete
  - with the additional of modular, or integral (every object has an integer dimension) or weakly integral, (dimensions in $\sqrt{\mathbb{N}}$), there's further progress
- small morphism spaces
  - eg if $\otimes$-generated by an object X and $\dim \mathrm{Hom}(\mathbb{1} \longrightarrow X^{\otimes k})$ is bounded by the sequence $1, 0, 1, 1, 4, 11, 40$, we know all possibilities (see Emily's talk?)
  - related classifications by Tuba-Wenzl and Bisch-Jones-Liu.
- small global dimension

$$\dim \mathcal{C} = \sum_{X \in \mathrm{Irr}(\mathcal{C})} \dim(X)^2$$

  - embarrassingly little is known!
- small index subfactors (here the index is $\dim_{\mathcal{C}} A$)
  - the remainder of this talk.

The classification of small index subfactors

(For this talk, we're only interested in the finite depth case.)

We begin with a fundamental invariant, the <u>principal graph</u>.

with vertices the simple 1-1, 1-A, A-1, or A-A bimodules in $\mathcal{C}$.

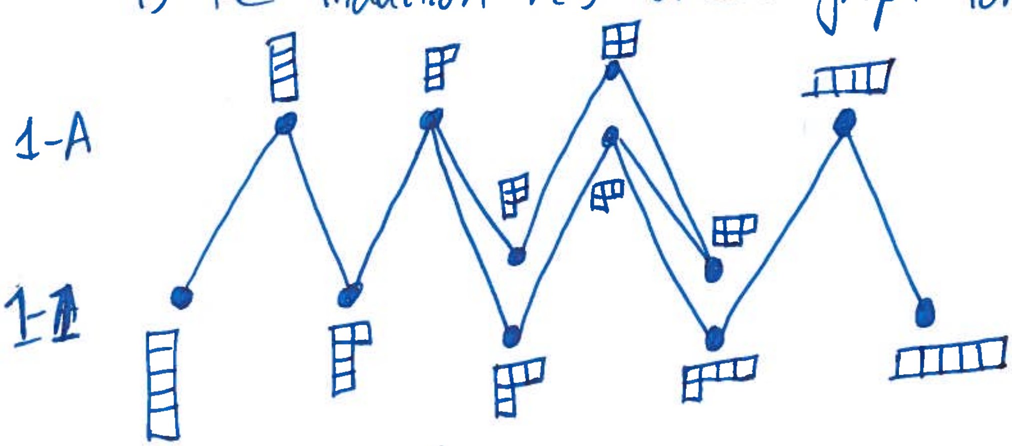
and edges from X to Y according to

$$\dim \mathrm{Hom}(X \otimes A^? \to Y)$$

where we think of A as a 1-A bimodule, and $A^?$ denotes A or A^* (an A-1 bimodule) as appropriate.

<u>Example</u> • $\mathcal{C} = \mathrm{Rep}\, S_5$, $A = k[S_5/S_4]$ with convolution.

The category of 1-A bimodules is $\mathrm{Rep}\, S_4$, and one component of the principal graph is the induction-restriction graph for $S_4 \subset S_5$:



the length of this initial chain

Two basic facts underlie the use of the principal graphs:

① the graph norm $\|\Gamma\|$ (the largest eigenvalue of the adjacency matrix) is the square root of the index:

$$\dim A = \|\Gamma\|^2$$

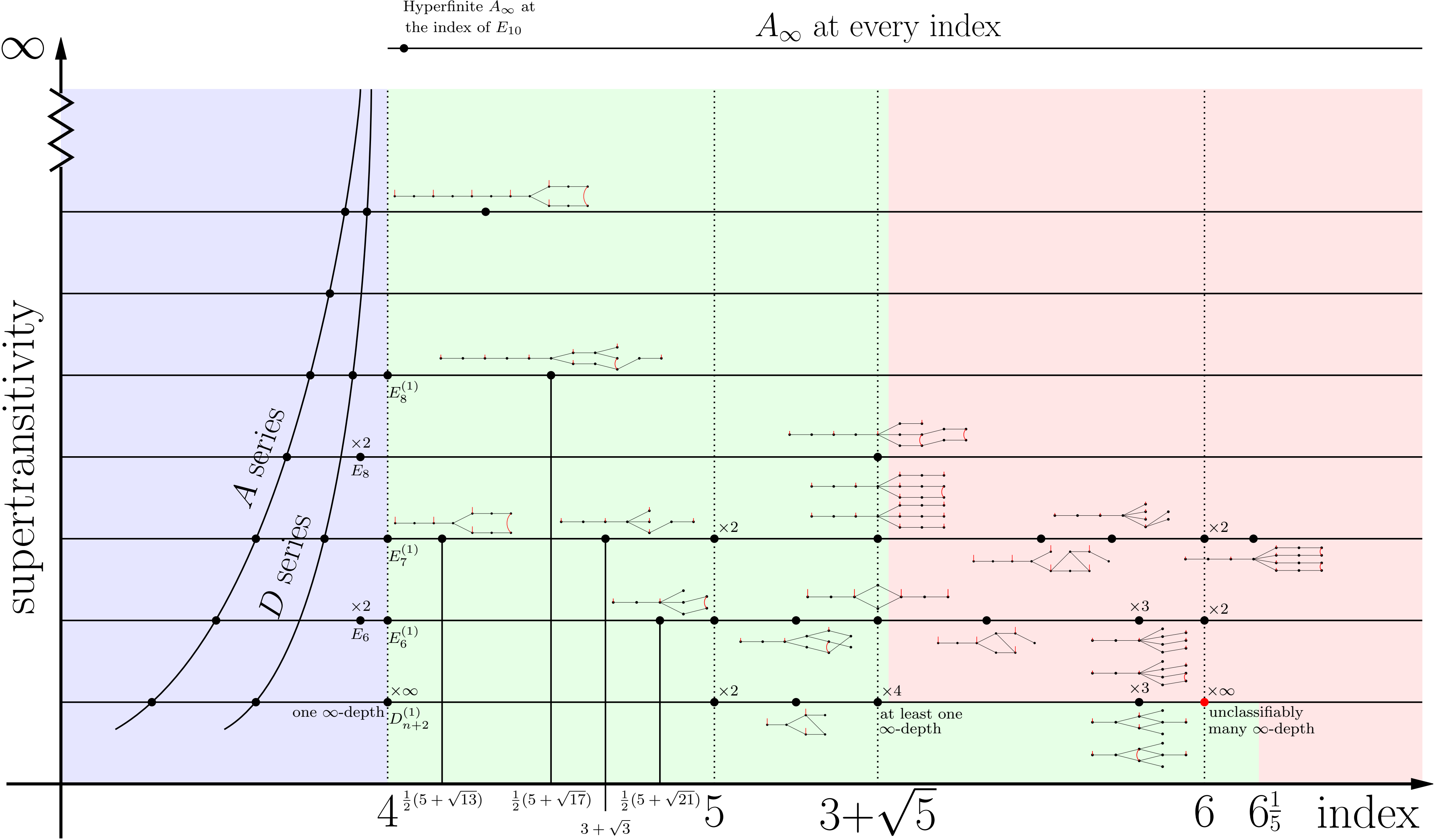
and if $\Gamma \subsetneq \Gamma'$, then $\|\Gamma\| < \|\Gamma'\|$

② there are at most finitely many subfactors with a given principal graph ("Ocneanu compactness" + ε)

Classifications are achieved by:

- enumerating all principal graphs up to the target index, aided by 'obstructions' which eliminate families of principal graphs (typically applicable when we only know the graph up to some radius from the trivial object)

- once we're (hopefully) eliminated all but finitely many graphs, we attempt ~~the realization~~ the 'categorification' problem.



Hyperfinite A_∞ at the index of E_{10}
A_∞ at every index
∞
supertransitivity
A series
D series
$\times 2$
E_8
$E_8^{(1)}$
$E_7^{(1)}$
$\times 2$
E_6
$E_6^{(1)}$
$\times\infty$
one ∞-depth
$D_{n+2}^{(1)}$
$\times 2$
$\times 2$
$\times 4$
at least one ∞-depth
$\times 2$
$\times 3$
$\times 2$
$\times 3$
$\times\infty$
unclassifiably many ∞-depth
4
$\frac{1}{2}(5+\sqrt{13})$
$\frac{1}{2}(5+\sqrt{17})$
$3+\sqrt{3}$
$\frac{1}{2}(5+\sqrt{21})$
5
$3+\sqrt{5}$
6
$6\frac{1}{5}$
index

Some observations: —

- high supertransitivity
  (roughly, having an object X for which
  $\mathrm{Inv}(X^{\otimes k})$ is as small as possible for high k)

  is rare
  - the extended Haagerup subfactor (supertransitivity 7) is the record (besides the A and D series)
  - Liu's work on $A_3 * A_4$ at index $3+\sqrt{5}$ shows a related phenomenon

- the quadratic categories account for a lot of previously mysterious stuff (Izumi, Evans-Gannon, Pinhas-Snyder)
  - Haagerup, 2221, 3333 (and (de)equivariantizations)
  - particularly Asaeda-Haagerup is (very non-obviously) Morita equivalent to a quadratic category

- the extended Haagerup category is currently the only known 'exotic' category
  - it can't be defined over a cyclotomic field (unlike Rep G or Rep $U_q\mathfrak{g}$, but like some other quadratic
  - we have an 'upper bound' on its Morita equivalence class, and nothing looks more familiar