Small examples

Fusion categories (finitely semisimple tensor categories) give a nice description of 'finite quantum symmetries'.

The classical case is $\mathcal{C} = \operatorname{Rep} G$, for G a finite group.

There are physical systems (honest, lab bench physical systems!) whose symmetries form a fusion category.

To interpret this, recall the idea that particles should correspond to the irreducible representations of the symmetry group.

We want to see a physical system in which the 'point-like excitations' are indexed by the objects in our category of symmetries.

Claim (Walker?)

Given $\mathcal{C}$, a pivotal fusion category, the "Levin-Wen" model associates to any cellulation of a surface Σ a 'microscopic' Hilbert ~~'big vector~~ space $\mathcal{H}$, and a local Hamiltonian H, such that the ground state for H is the Turaev-Viro vector space $\int_\Sigma \mathcal{C}$, and the point-like excitations are indexed by

- objects of $\mathcal{C}$, for excitations on $\partial\Sigma$, and
- objects of $Z(\mathcal{C})$, for excitations in the bulk.

To date we have only seen two fusion categories 'in the wild':

- the $\mathrm{Rep}\, U_{\zeta_8} sl_2$ category with 3 simple objects

, $\sigma^2 = 1 + \psi$

$\dim \sigma = \sqrt{2}$

$\dim \psi = 1$

realized by the $\frac{5}{2}$ FQHE

- (tentatively) the $\mathrm{Rep}\, U_{\zeta_{10}} so_3$ category with 2 simples.

, $\tau^2 = 1 + \tau$

$\dim \tau = \frac{1+\sqrt{5}}{2}$.

realized?? by the $\frac{12}{5}$ FQHE state.

The Levin-Wen model suggests the possibility of <u>engineering</u> desired topological phases of matter.

What do fusion categories look like?

We certainly have

- Rep G, G a finite group
- Rep $U_q \mathfrak{g}$, $\mathfrak{g}$ a complex semisimple Lie algebra, q a root of unity

(and Rep denotes the semisimplified category of tilting modules)

along with a number of constructions

(graded extensions, (de-)equivariantization, ...)

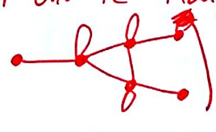
(which Noah will discuss).

Beyond these we have

- quadratic categories

(a group of invertible objects, along with one other orbit of objects under this group)

(the previous examples, and the Haagerup, all fit into this class



- fusion categories from exotic subfactors.

It's hard to know what we should try to prove.

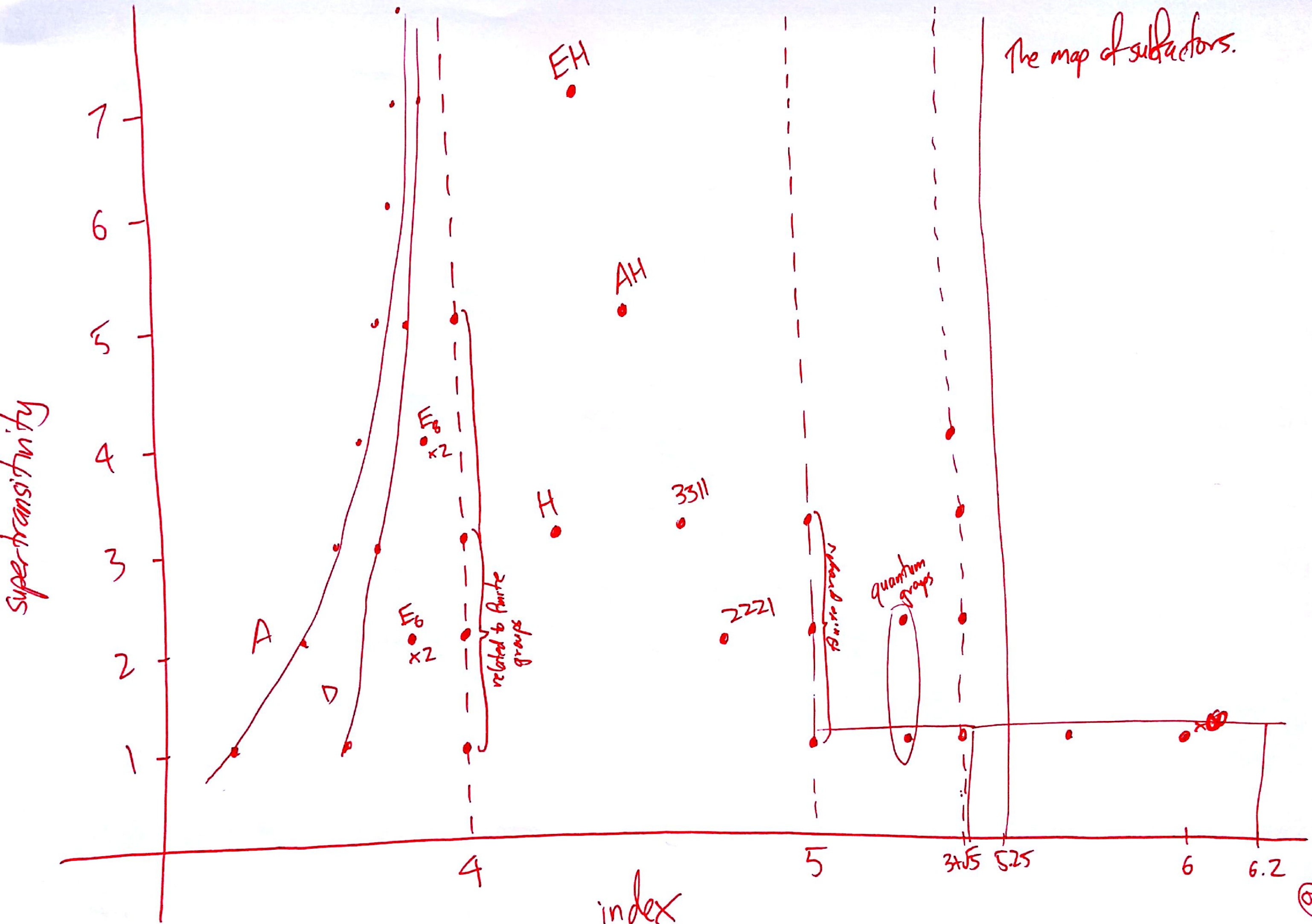
- are the exotic examples subsumed into regular families?
- could there just be a few (finitely many?!) 'sporadic' special cases?
- or are the exotic examples just the first sign of a hopeless complicated mess?

While it would be unreasonable today to hope for anything except the 3rd option, there are tempting hints towards the first two possibilities.

- In recent years, previously exotic examples have been connected to families.
  - Masaki realized the Haagerup as an example of a quadratic category.
  - The Asaeda-Haagerup subfactor is also related to a quadratic category (Noah will talk about this later today).

  Of the 'small index' subfactors, only <u>extended Haagerup</u> remains, isolated from all the rest of mathematics.

- Since Haagerup's initial breakthrough — that the Haagerup subfactor at index $\frac{5+\sqrt{13}}{2} \approx 4.30$ has the minimal index above 4 amongst finite depth subfactors — there have been a succession of improvements to the classification of small index subfactors



The map of subfactors.
super-transitivity
1
2
3
4
5
6
7
index
4
5
$3+\sqrt{5}$
5.25
6
6.2
A
D
E_6
×2
E_8
×2
related to finite groups
H
EH
AH
3311
2221
quantum groups

The first observation to make is that at every step there's been less than we expected (or hoped for?).

How do these classification theorems work?

The outline is always:

1. enumerate families of graphs which could be the principal graph of a subfactor with index at most L.

2. argue, somehow, that most of these families can not be realized (skein theory, connections, number theory)

3. give constructions, and uniqueness results, for the finitely many graphs remaining.

Recall a subfactor $A \subset B$ is an inclusion of von Neumann factors.

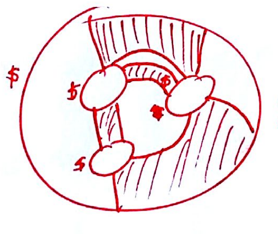
Let's denote B, as an A-B bimodule, by V. Its dual, V^*, is a B-A bimodule.

We begin by defining a collection of vector spaces

$$P_{n,+} = \mathrm{Inv}_{A-A}(\underbrace{V \otimes V^* \otimes \cdots \otimes V \otimes V^*}_{2n \text{ tensor factors}})$$

$$P_{n,-} = \mathrm{Inv}_{B-B}(V^* \otimes V \otimes \cdots \otimes V^* \otimes V)$$

These form a planar algebra: planar tangle act.



$: P_{2,+} \otimes P_{3,+} \otimes P_{3,-} \longrightarrow P_{2,+}$

From a unitary planar algebra, we can reconstruct a subfactor, and in many interesting settings (eg. finite depth subfactors of the hyperfinite II_1) the planar algebra is a complete invariant.

Notice that we can perform the same construction starting with any object V in a tensor category.

The planar algebra structure is extraordinarily rigid, and by now we have many algebraic obstructions.

We next need the principal graph.

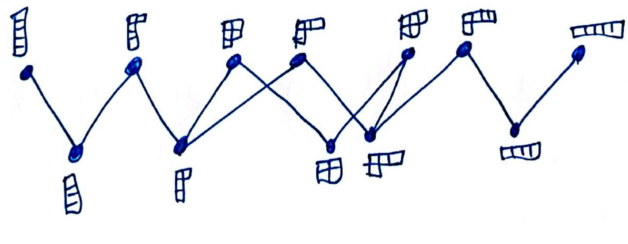
It has four classes of vertices, the simple A-A, A-B, B-B, and B-A bimodules (or rather, those $\otimes$-generated by V).

It has $\dim \mathrm{Hom}(P \otimes V^{\pm} \to Q)$ edges from P to Q, with Hom the space of (appropriate) bimodule maps.

(This data is all encoded in the planar algebra.)

We've seen some examples already.

<u>Examples</u> $R^{S_5} \subset R^{S_4}$



The two fundamental results behind our classification efforts are:

- $\dim V = [B:A]^{\frac{1}{2}} = \|\Gamma\|$ (the largest eigenvalue of the adjacency matrix of the principal graph).

- If $\Gamma' \subsetneq \Gamma$, then $\|\Gamma'\| < \|\Gamma\|$.