A quantum exceptional family?

Scott Morrison

AustMS, 2015 September 29
https://tqft.net/math/20150929-AustMS

Definition

A metric Lie algebra partition function is a function

$$Z : \left\{ \begin{array}{c} \textit{(vertex oriented)} \\ \textit{cubic graphs} \end{array} \right\} \to \mathbf{k}$$

satisfying:

[M] $Z(G_1 \sqcup G_2) = Z(G_1)Z(G_2)$

[AS] $Z\Big($  $\Big) = -Z\Big($  $\Big)$

[IHX] $-Z\big($  $\big) + Z\big($  $\big) + Z\big($  $\big) = 0$

An honest metric Lie algebra $\mathfrak{g}$ gives us one!

We interpret the building blocks of a cubic graph via the structure maps.

bracket:  $: \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$

pairing:  $: \mathfrak{g} \otimes \mathfrak{g} \to \mathbf{k}$

copairing:  $: \mathbf{k} \to \mathfrak{g} \otimes \mathfrak{g}$

switch:  $: \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$

An honest metric Lie algebra $\mathfrak{g}$ gives us one!

We interpret the building blocks of a cubic graph via the structure maps.

bracket:  $: \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$

pairing:  $: \mathfrak{g} \otimes \mathfrak{g} \to \mathbf{k}$

copairing:  $: \mathbf{k} \to \mathfrak{g} \otimes \mathfrak{g}$

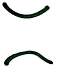
switch:  $: \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$

Conversely, from such a function Z we can recover a pivotal $\otimes$-category equipped with a metric Lie algebra object.

(By taking the category of cubic graphs with boundary, and quotienting out the kernel of Z.)

Conjecture (Deligne)

There is a family of Z_d of partition functions, uniquely determined by the relations

-  $= d$
-  $=$ *zero*
-  $= 6$ 
-  $= 3$ 
-  $=$  $+$  $+ \frac{30}{d+2} \Big($  $+$  $+$  $\Big)$

such that at $d = \dim(\mathfrak{g})$*, the associated* $\otimes$*-category recovers* $\operatorname{Rep}\mathfrak{g}$*, for each of the exceptional Lie algebras* $\mathfrak{g} = \mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6.2, \mathfrak{e}_7, \mathfrak{e}_8$.

Conjecture (Morrison-Snyder-Thurston)

There is a family of invariants $Z_{v,w}$ of knotted cubic graphs, uniquely determined by the relations

-  $= v^{12}$  *and*  $= -v^{6}$ 
-  $= -\frac{\{2\}[\lambda+5][\lambda-6]}{[\lambda][\lambda-1]} (=: d)$
-  $=$ *zero*
- v^{-3}  $- v^{-1}$  $+ v$  $+ \alpha \Big($  $+ v^{-4}$  $+ v^{4}$  $\Big)$

(with $\alpha = -\frac{[\lambda][\lambda-1]}{[1]}$, $[k\lambda+\ell] = w^k v^\ell - w^{-k} v^{-\ell}$, $\{k\lambda+\ell\} = w^k v^\ell + w^{-k} v^{-\ell}$) such that the associated braided $\otimes$-category at $d = \mathrm{qdim}(\mathfrak{g})$, $v = q^{\langle \lambda_{ad}, \lambda_{ad}+\rho \rangle}$ recovers Rep $U_q\mathfrak{g}$, for each of the exceptional Lie algebras.

Why might you believe this?

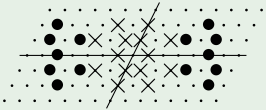
In the associated $\otimes$-category, we can compute the decomposition into irreducibles:

$$\mathfrak{g} \otimes \mathfrak{g} \cong 1 \oplus \mathfrak{g} \oplus X_2 \oplus Y_2 \oplus Y_2^*$$

$$\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \cong 1 \oplus 5\mathfrak{g} \oplus 4X_2 \oplus 3Y_2 \oplus 3Y_2^* \oplus X_3 \oplus Y_3 \oplus Y_3^* \oplus 3A \oplus 2C \oplus 2C^*$$

and compute the dimensions of the summands, e.g.

$$\dim(X_2) = \frac{[5][\lambda+5][\lambda+3][\lambda-4][\lambda-6][2\lambda+4][2\lambda-6]}{[1][\lambda+2][\lambda][\lambda-1][\lambda-3][2\lambda][2\lambda-2]}$$

$=$ 

which correctly specialise to all the exceptional Lie algebras.

- There are well defined representations of the braid groups $B_{n\leq 6}$ over $\mathbb{Q}(v,w)$.
- From these we can compute the invariants of most knots up to 11 crossings. In cases we can compute directly, these agree with all the quantum knot invariants for the exceptional Lie algebras.

$Z_{v,w}$ ()

$$= -v^{-32}(1+dv^2+v^4)^{-2}(d^3v^{54} - d^3v^{50} - d^3v^{48} - 2d^3v^{46} + 2d^3v^{40} + 2d^3v^{38}$$
$$+ 2d^3v^{36} + 2d^3v^{34} - d^3v^{30} - 2d^3v^{28} - d^3v^{26} - 2d^3v^{24} - d^2v^{72} - d^2v^{60}$$
$$- 2d^2v^{58} + 3d^2v^{52} + 4d^2v^{50} + 3d^2v^{48} + 6d^2v^{46} + 2d^2v^{42} - 5d^2v^{40} - 6d^2v^{38}$$
$$- 5d^2v^{36} - 6d^2v^{34} + 5d^2v^{28} + 2d^2v^{26} + 6d^2v^{24} - 2d^2v^{18} - 3d^2v^{16} - 2d^2v^{14}$$
$$- 2d^2v^{12} - 2dv^{74} - 3dv^{70} - dv^{66} + 3dv^{64} + 2dv^{62} + 4dv^{60} + 4dv^{58} + dv^{56} + 2dv^{54}$$
$$- 4dv^{52} - 6dv^{48} - dv^{46} - dv^{44} + dv^{42} + dv^{40} + 3dv^{38} + dv^{36} + dv^{34} - 2dv^{30}$$
$$- dv^{28} - 4dv^{26} - dv^{24} - 2dv^{22} + 2dv^{16} + dv^{14} + dv^{12} + dv^{10} - dv^8 - 2dv^4 - d)$$