A gentle introduction to topological field theory.

What is a topological field theory?

At its most basic, it is a recipe, that for each description of the background, or context, in which physics is taking place, produces the set of all possible physical states on that background, and for each change (or transition, or transformation) in the background, produces a function from the set of possible states before the change to the set of possible states after the change.

$$\{\text{backgrounds}\} \longrightarrow \{\text{sets}\}$$

$$\{\text{changes}\} \longrightarrow \{\text{functions}\}$$

As mathematicians, we recognise this as a <u>functor</u> between categories.

(Recall, if necessary, the definitions of categories and functors.)

If we're doing quantum physics, we expect that the collection of all possible states should form a vector space (i.e. given any two states ψ and ϕ, any linear combination of them $a\psi + b\phi$ should also be a state), and the transition functions should always be linear maps.

Today we won't go into why this should be true, but just run with it!

Examples

① 1+1-dimensional topological field theory.

Let's consider the case where the possible "backgrounds" are closed 1-dimensional manifolds.

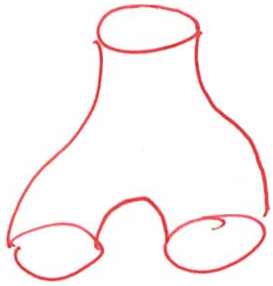
There aren't many of these: just disjoint unions of circles.

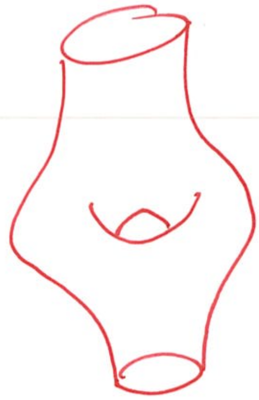
Let's consider the "changes" to be circles merging or splitting.

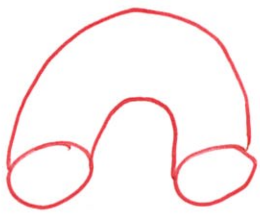
One way to say this is that a morphism in this category is a 2-manifold Σ with boundary, with that boundary divided into an "incoming" and "outgoing" part, both of which are disjoint unions of circles:

$$\partial\Sigma = \partial^{in}\Sigma \cup \partial^{out}\Sigma.$$

We'll say that such a morphism has source $\partial^{in}\Sigma$ and target $\partial^{out}\Sigma$.

 $\in \mathrm{Hom}(OO \to O)$

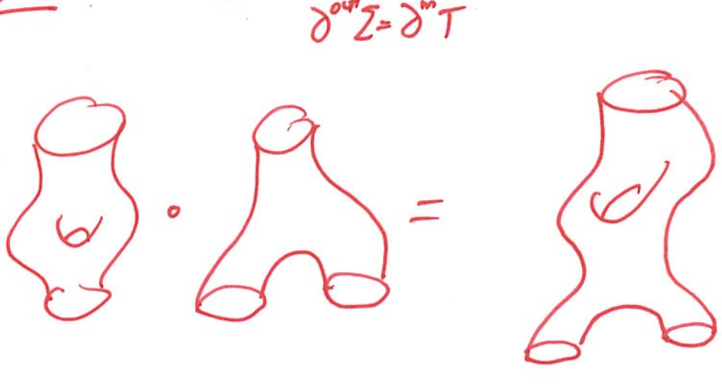
 $\in \mathrm{Hom}(O \to O)$

 $\in \mathrm{Hom}(OO \to \phi)$

When we have two composable morphisms Σ and T (i.e. the target of Σ is the same as the source of T) we can compose them by gluing:

$$T \circ \Sigma = T \bigcup_{\partial^{out}\Sigma = \partial^{in} T} \Sigma$$

e.g.



This is the 2-dimensional cobordism category: 2Cob.

(You can readily imagine how to generalise to other dimensions.)

A 1+1-dimensional topological quantum field theory (we'll explain why we write 1+1 instead of 2 in a bit!) is then a functor

$$\mathcal{F} : 2\mathrm{Cob} \longrightarrow \mathrm{Vec}.$$

Actually — that's not quite right: there's a little bit of extra structure in 2Cob that we want our functor to preserve.

Our functor $\mathcal{F}$ has to assign a vector space to each closed 1-manifold, so we have

$$\mathcal{F}(\phi),\ \mathcal{F}(\circ),\ \mathcal{F}(\circ\,\circ),\ \dots \quad \in \mathrm{Vec}$$

Should there be some relationship between these?

The vector space $\mathcal{F}(\bigcirc\bigcirc)$ describes the possible states on a background consisting of two circles. Since the circles are disjoint, it's natural to say that ~~so~~ to specify a state on two circles, we should just need to specify a state separately on each of the circles.

That is, $\mathcal{F}(\bigcirc\bigcirc)$ should be built out of two copies of $\mathcal{F}(\bigcirc)$.

Two plausible alternatives would be to ask

a) $\mathcal{F}(\bigcirc\bigcirc) = \mathcal{F}(\bigcirc) \otimes \mathcal{F}(\bigcirc)$ (tensor product)

b) $\mathcal{F}(\bigcirc\bigcirc) = \mathcal{F}(\bigcirc) \oplus \mathcal{F}(\bigcirc)$ (direct sum)

I'll leave it as an exercise to think about what goes wrong with b). (And indeed, why a) is the right answer!)

Our revised definition of a 1+1-dimensional topological field theory is a functor $\mathcal{F}: 2\mathrm{Cob} \to \mathrm{Vec}$ taking disjoint unions to tensor products.

(This applies both at the level of objects and morphisms, so $\mathcal{F}($  $) = \mathcal{F}($  $) \otimes \mathcal{F}($  $) : \mathcal{F}(\bigcirc) \otimes \mathcal{F}(\bigcirc) \otimes \mathcal{F}(\bigcirc) \to \mathcal{F}(\bigcirc) \otimes \mathcal{F}(\bigcirc)$.)

(We can also say this as "symmetric monoidal functor", meaning that both 2Cob and Vec have a "symmetric monoidal structure" given by the ~~products~~ operations $\sqcup$ and $\otimes$, and we want the functor to preserve this.)

Now, ϕ is the identity for disjoint union, so we must have $\mathcal{F}(\phi) = \mathbb{C}$, the identity for tensor product.

We see that at the level of objects, $\mathcal{F}$ is determined once we've specified the value of $\mathcal{F}(O)$.

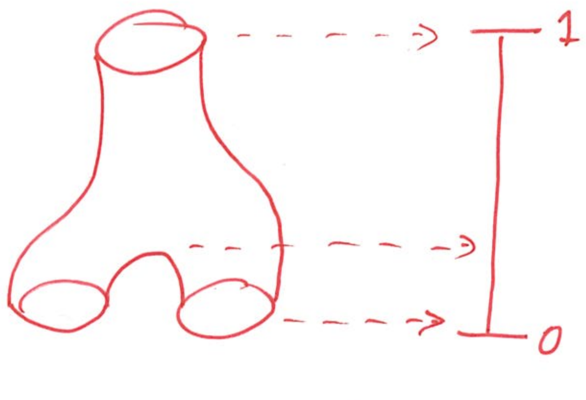
What about morphisms?

Let's consider a surface Σ, with $\partial\Sigma = \partial^{in}\Sigma \cup \partial^{out}\Sigma$.

Pick a 'generic' smooth function $f : \Sigma \to [0,1]$,

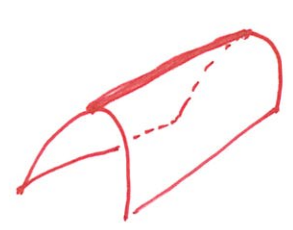
so $f^{-1}(0) = \partial^{in}\Sigma$, $f^{-1}(1) = \partial^{out}\Sigma$.

We can think of this as a height function:



Since we've picked a 'generic' function, we'll only see 'generic' ~~Singula~~ critical values, i.e.

but never, e.g.

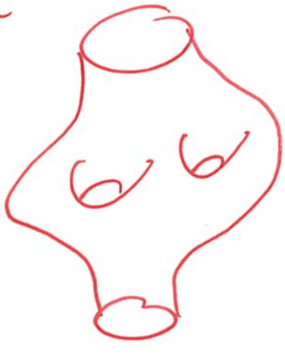
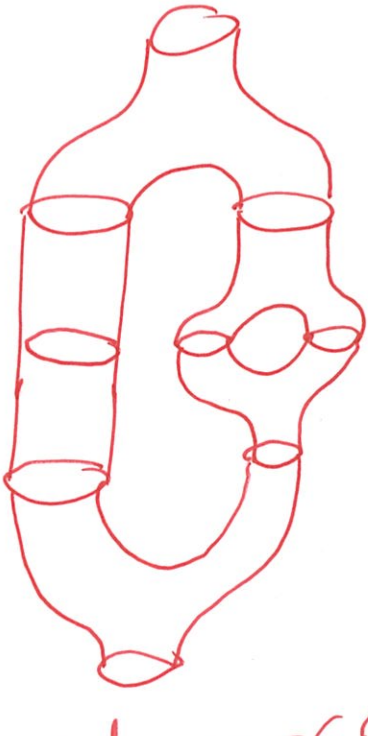


We may also assume that each critical value occurs at a different height.

Such a height function thus allows us to chop our surfaces into strips.

In each strip exactly one interesting thing happens, and that interesting thing is a cap, a saddle, or a cup.

<u>Example</u>

 $=$  $\overset{\text{or}}{=}$ 

Now $\mathcal{F}($  $) = \mathcal{F}($  $) \otimes \mathcal{F}($  $) = id_{\mathcal{F}(0)} \otimes \mathcal{F}($  $)$

and $\mathcal{F}($  $) = \mathcal{F}($  $) \circ \mathcal{F}($  $)$.

We thus see that $\mathcal{F}$ is completed determined by the vector space

$$\mathcal{F}(\bigcirc)$$

and the four linear maps

$\mathcal{F}($  $)$: $\mathcal{F}(\bigcirc) \to \mathbb{C}$

$\mathcal{F}($  $)$: $\mathcal{F}(\bigcirc) \otimes \mathcal{F}(\bigcirc) \longrightarrow \mathcal{F}(\bigcirc)$

$\mathcal{F}($  $)$: $\mathcal{F}(\bigcirc) \longrightarrow \mathcal{F}(\bigcirc) \otimes \mathcal{F}(\bigcirc)$

and $\mathcal{F}($  $)$: $\mathbb{C} \to \mathcal{F}(\bigcirc)$.

<u>Example</u>

Consider $\mathcal{F}(\bigcirc) = \mathbb{C}[x]/x^2=0$

$\mathcal{F}($  $)$: $1 \longmapsto 0$, $x \longmapsto 1$

$\mathcal{F}($  $)$: $1 \otimes 1 \longmapsto 1$, $1 \otimes x \longmapsto x$, $x \otimes 1 \longmapsto x$, $x \otimes x \longmapsto x^2 = 0$

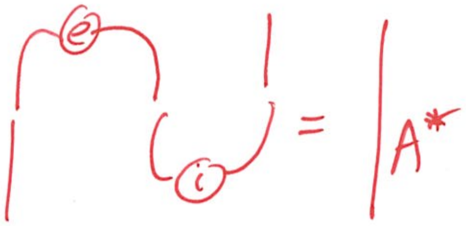
$\mathcal{F}($  $)$: $1 \mapsto 1 \otimes 1$, $x \longmapsto x \otimes 1 + 1 \otimes x$

$\mathcal{F}($  $)$: $1 \longmapsto 1$

Since $\mathcal{F}$ is symmetric monoidal, it also preserves <u>duals</u>.

A (right) dual of an object A is an object A^* equipped with maps
$i: \mathbb{1} \to A \otimes A^*$ (copairing/coevaluation)
$e: A^* \otimes A \to \mathbb{1}$ (pairing/evaluation

satisfying



(the object A^* is uniquely determined, if it exists, but the maps i and e are not).

In 2Cob under $\sqcup$, duals exist and are just the orientation reversal. (And left and right duals coincide.) The pairing and copairing maps are the "obvious" cobordisms:

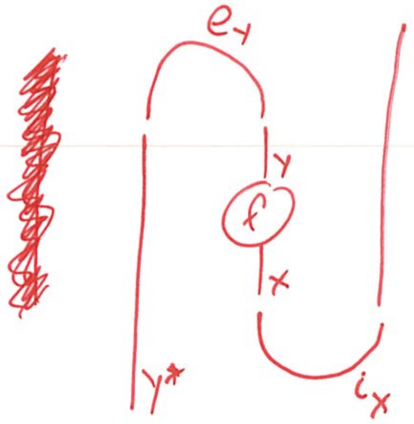
$i: \mathbb{1} \to S^1 \sqcup S^{1*}$ = 

$e: S^{1*} \sqcup S^1 \to \mathbb{1}$ = .

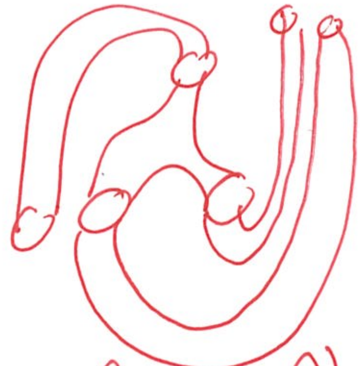
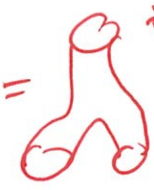
(Note $S^{1*} \cong S^1$)

In Vec under $\otimes$, duals are just dual vector spaces.

The dual of a morphism $f: X \to Y$ between dualizable objects X and Y is

$f^*: Y^* \to X^* =$ 

Notice  = , i.e  =  *

and  =  =  =  *.

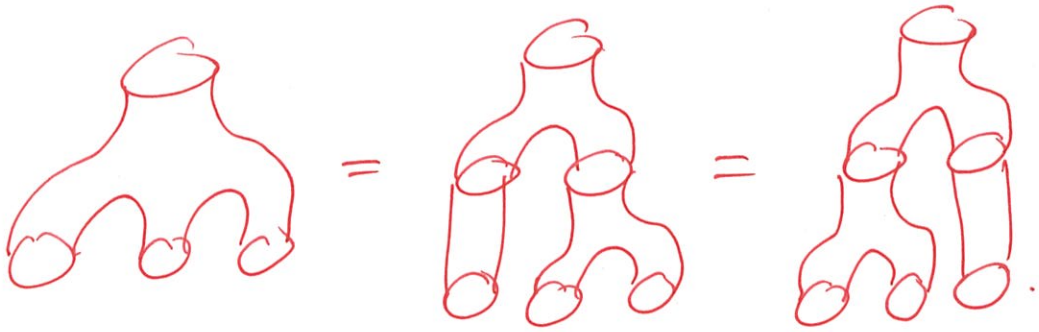
Thus we don't need to separately specify $\mathcal{F}$() and $\mathcal{F}$().

Moreover, since $S^{1*} \cong S^1$, $\mathcal{F}$() gives a bilinear pairing : $\mathcal{F}(\circ) \otimes \mathcal{F}(\circ) \to \mathbb{C}$, which moreover is non-degenerate, since $\mathcal{F}$() = $\mathcal{F}$().

Does that really define a functor?

Are there relations these linear maps must satisfy?

Yes — because the decomposition of a surface into elementary pieces is not unique.

Example: 

If we write $V \stackrel{\text{def}}{=} \mathcal{F}(\circ)$, $m \stackrel{\text{def}}{=} \mathcal{F}($$)$

and think of m as a 'multiplication' on V,

this says that m is associative:

$$m \circ (1\!\!1_V \otimes m) = m \circ (m \otimes 1\!\!1_V)$$

equivalently $m(a, m(b,c)) = m(m(a,b),c)$.

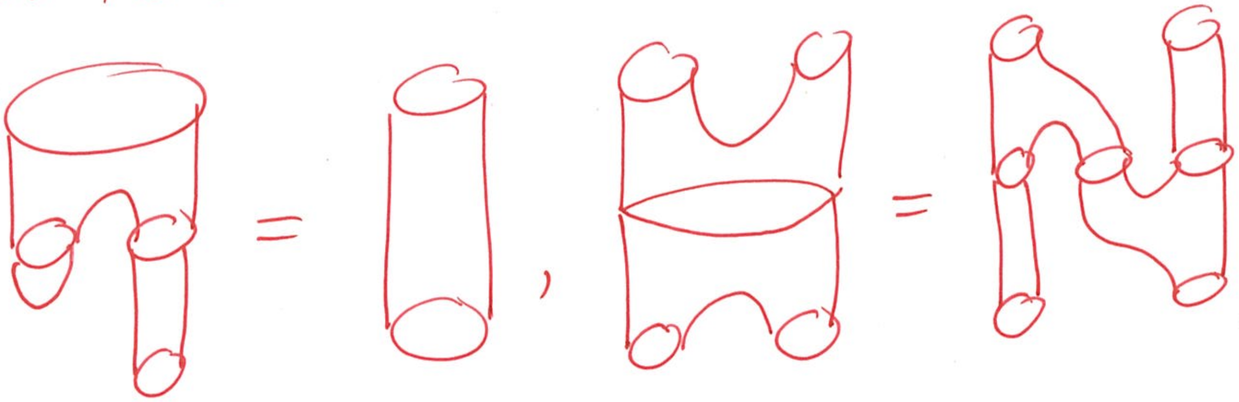
This condition holds in our example, because

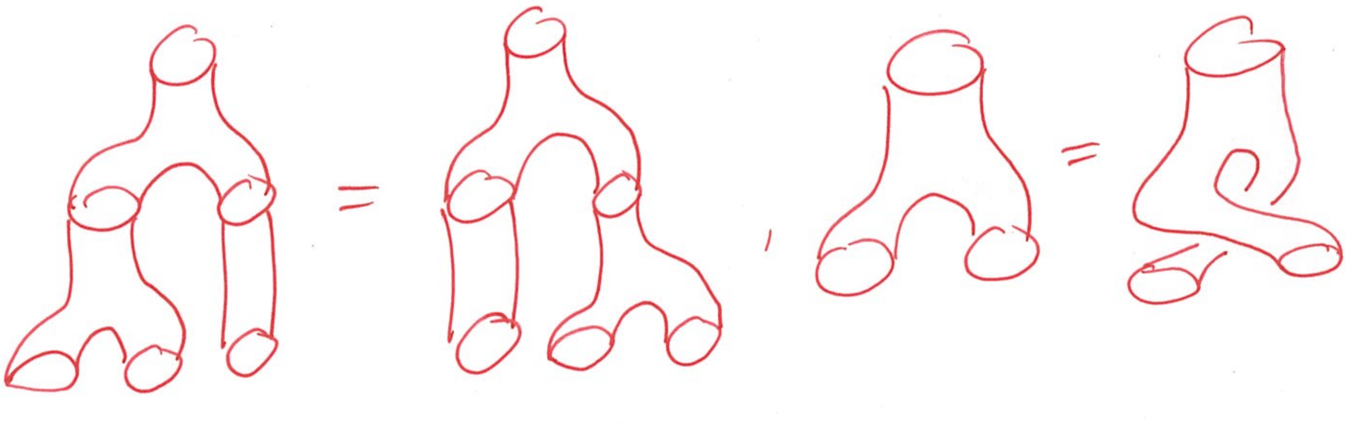
there $\mathcal{F}($$)$ was just defined to be multiplication

in the ring $\mathbb{C}[x]/x^2=0$.

What are *all* the relations that must be satisfied? To discover these, we'd have to do some Cerf theory. This is the study of families of Morse functions.

Here's the answer:





~~Lemma (an exercise!)~~

<u>Definition</u> a commutative Frobenius algebra is (V, m, ι, τ) where

- V is a vector space
- m is a commutative associative multiplication on V
- $\iota : \mathbb{C} \to V$, is a unit for m
- $\tau : V \to \mathbb{C}$, and $\langle x, y\rangle = \tau(xy)$ is a nondegenerate pairing on V.

<u>Lemma</u> (an exercise!)

If you define $\mathcal{F}(\bigcirc) = V$,

$\mathcal{F}($$) = m$, $\mathcal{F}($$) = \iota$

$\mathcal{F}($$) = \left(V \xrightarrow{x \mapsto \langle x, -\rangle} V^* \xrightarrow{\iota^*} \mathbb{C}\right)$

$\mathcal{F}($$) = \left(V \longrightarrow V^* \xrightarrow{m^*} V^* \otimes V^* \longrightarrow V \otimes V\right)$

then $\mathcal{F}$ satisfies the relations coming from Cerf theory, and so is a 1+1-dimensional topological field theory.

Theorem

1+1-dimensional topological field theories are in one-to-one correspondence with commutative Frobenius algebras.

(Further reading, and much more careful detail, in Lecture 23 of Dan Freed's course notes "Bordism: Old and new".)

Next, let's go both up and down in dimension.

Baby examples

- The trivial "background" category, $\mathbb{1}$
  with just one object, and only the identity morphism.

  A functor $\mathbb{1} \to \mathrm{Vec}$ is just a vector space.

- The "time evolution" category, with one object $\bullet$,
  and $\mathrm{Hom}(\bullet \to \bullet) = \mathbb{R}$ under addition.

  A functor $\mathcal{F} : \mathbb{R} \to \mathrm{Vec}$
  is a vector space $V = \mathcal{F}(\bullet)$, and a one-parameter
  group of invertible maps
  $$\mathcal{F}(t) \in GL(V),$$
  so
  $$\mathcal{F}(t) \circ \mathcal{F}(s) = \mathcal{F}(s+t).$$

- A functor $\mathcal{F} : 1\mathrm{Cob} \to \mathrm{Vec}$ is just a <u>finite dimensional</u>
  vector space.

  $\mathcal{F}(\bullet^+) = V$ for some vector space V

  $\mathcal{F}(\bullet^-) = V^*$

  $\mathcal{F}(\cap) : V \otimes V^* \to \mathbb{C}$, $\quad \mathcal{F}(\cup) : V^* \otimes V \to \mathbb{C}$

  This shows that $V \cong V^{**} \implies V$ is finite dimensional.

- Central functions on a finite group, under convolution,
  $f(gh) = f(hg)$
  $$(f_1 * f_2)(x) = \sum_{x_1 x_2 = x} f_1(x_1) f_2(x_2)$$
  give a commutative Frobenius algebra.

The associated TQFT gives
$$\mathcal{F}(\Sigma^{\text{closed}}) = \#\text{ of principal } G\text{-bundles on } \Sigma.$$

Computing the values of a TFT can be difficult.

~~nice situ~~

We can evaluate an n+1-dimensional TFT on a ~~n~~ (n+1)-dimensional* manifold by cutting the manifold into elementary pieces, ~~and~~ finding the associated linear maps, and then forming the appropriate composition of tensor products.

What about evaluating the TFT on n-manifolds?
(Once $n>1$, and especially once $n\geq 3$, there are lots of these!)

In nice situations, we can cut up the n-manifold too, and then 'glue together' the results of local computations.

Here's a prototypical example:

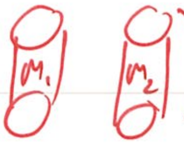
$$\begin{tikzcd}
\{\text{3-manifolds}\} \arrow[r, "\mathcal{F}"] \arrow[d, bend left] & \{\text{linear maps}\} \arrow[d, bend left] \\
\{\text{2-manifolds}\} \arrow[r] \arrow[d, bend left] & \{\text{modules}\} \arrow[d, bend left] \\
\{\text{1-manifolds}\} \arrow[r] & \{\text{rings}\}
\end{tikzcd}$$

Here, we ask

$\mathcal{F}\Big($ 

$\Big) = \mathcal{F}(M_1) \circ \mathcal{F}(M_2)$

and

$\mathcal{F}\Big($ 

$\Big) = \mathcal{F}(M_1) \otimes \mathcal{F}(M_2)$, as usual

but then $\mathcal{F}$ also takes values on 2-manifolds with boundary, and on 1-manifolds,

so that $\mathcal{F}(S^1) = R$, some ring,

and $\mathcal{F}\Big($ 

$\Big)$ is some module over R,

and we can compute

$\mathcal{F}\Big($  $\Big) = \mathcal{F}\Big($  $\Big) \underset{\mathcal{F}(S^1)}{\otimes} \mathcal{F}\Big($  $\Big)$

$\|$def

$\mathcal{F}\Big($  $\Big) \otimes \mathcal{F}\Big($  $\Big) \Big/ ma \otimes n = m \otimes an.$

A <u>fully local</u> $(n+1)$-dimensional TFT is then a functor

$$\begin{array}{ccc}
\{n+1\text{-manifolds}\} & \longrightarrow & \{\text{linear maps}\} \\
\curvearrowright & & \curvearrowright \\
\{n\text{-manifolds}\} & \longrightarrow & \{\text{vector spaces}\} \\
\curvearrowright & & \curvearrowright \\
\{n-1\text{-manifolds}\} & \longrightarrow & \{\text{categories}\} \\
\curvearrowright & & \curvearrowright \\
\{n-2\text{-manifolds}\} & \longrightarrow & \{2\text{-categories}\} \\
\curvearrowright & & \curvearrowright \\
\vdots & & \vdots \\
\curvearrowright & & \curvearrowright \\
\{0\text{-manifolds}\} & \longrightarrow & \{n\text{-categories}\}
\end{array}$$

satisfying all the possible "gluing formulas" you could ask for.

The Cobordism hypothesis

(due to Baez-Dolan, initially thought of as a criterion for good definitions of higher categories, more recently proved by Lurie)

says that

- a fully local TFT is entirely determined by its value on a point
- ~~these~~ gives a one-to-one correspondence between fully local TFTs and "fully dualizable" n-categories.

We won't come close to explaining this today.

Let's note, however, that "fully dualizable" is in part a pun. It abbreviates to "f.d.", just as "finite dimensional".

Indeed, as we saw in TFTs $1Cob \to Vec$, the fully dualizable 0-categories are exactly finite-dimensional vector spaces.

In every setting where ~~its known~~ the explicit meaning of "fully dualizable" has been spelled out, it has a "finite dimensional" flavour.

Example

A fully local 2-dimensional TFT is in particular a 1+1-dimensional TFT, so there is an associated commutative Frobenius algebra.

A theorem of Schommer-Pries says the TFT is fully local iff the Frobenius algebra is separable. ($\exists$ a central $\sum_i x_i \otimes y_i \in A \otimes A$ so $\sum x_i y_i = 1$)

(and... I think... over $\mathbb{C}$ this says it is semisimple)

Example

A fully local $(2+\varepsilon)$-dimensional TFT $\mathcal{F}$ is determined by $\mathcal{F}(\bullet)$, which may be any pivotal category.

For a fully local $(2+1)$-dimensional TFT, that pivotal category must be a fusion category, i.e. finitely semisimple.

Fusion categories are "quantum" analogues of finite groups.

I'm really interested in them!