<u>Exercise</u> Classify all rank 2 UMTCs.

Call the simple objects $\{1, X\}$.

We have $X \otimes X \cong 1 \oplus nX$ for some $n \in \mathbb{N}$.

(It's easy to see the fusion ring is associative for any n.)

The charge conjugation matrix C is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$\cancel{S^2 = \dim \mathcal{C} \cdot C \text{ and}}$

the unnormalized S-matrix.

~~Now~~ $\cancel{\hat{S}_{1i} = \dim X_i \text{ and } \hat{S}_{ij} = \hat{S}_{ji}}$, so

$$\cancel{S = \begin{pmatrix} 1 & \dim X \\ \dim X & \alpha \end{pmatrix}}$$

~~but~~ $\cancel{S^2 = \dim \mathcal{C} \cdot C, \text{ so } \alpha = -1.}$

Now $S_{ij} = \frac{1}{\sqrt{\dim \mathcal{C}}}$ 

so $S_{11} = \frac{1}{\sqrt{\dim \mathcal{C}}}$, $S_{1X} = S_{X1} = \frac{\dim X}{\sqrt{\dim \mathcal{C}}}$.

Then $S^2 = C \Rightarrow$

$$S = \frac{1}{\sqrt{\dim \mathcal{C}}} \begin{pmatrix} 1 & \dim X \\ \dim X & -1 \end{pmatrix}.$$

What is $\dim X$ anyway?

$$(\dim X)^2 = n \dim X + 1,$$

so $\dim X = \dfrac{n \pm \sqrt{n^2+4}}{2}$ (+, for positive dimensions)

$$\left(\begin{array}{c|ccccc} n & 0 & 1 & 2 & 3 & \cdots \\ \hline \dim X & 1 & \frac{1\pm\sqrt{5}}{2} & 1\pm\sqrt{2} & \frac{3\pm\sqrt{13}}{2} & \cdots \end{array} \right)$$

Does Verlinde's formula tell us anything?

$$N_{ij}^{\ \ k} = \sum_{\ell} \frac{S_{\ell j} S_{\ell i} \overline{S_{\ell k}}}{S_{\ell 0}}$$

$$N_{xx}^{\ \ x} = \sum_{\ell = 1, X} \frac{S_{\ell x} S_{\ell x} \overline{S_{\ell x}}}{S_{\ell 0}}$$

$$= \frac{1}{\dim \mathcal{C}} \left(\frac{(\dim X)^3}{1} + \frac{-1}{\dim X} \right)$$

$$= \frac{1}{1+d^2} \frac{d^4 - 1}{d} = d - d^{-1} = n.$$

No; we learn nothing new here. (This is a low-rank accident.)

What about T?

Up to an overall scalar, $T = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$.

Then $$(ST)^3 = \frac{1}{(\dim \mathcal{C})^{3/2}} \begin{pmatrix} 1 + d^2(2t - t^2) & d^3t^2 + dt(1-t+t^2) \\ d^3t + d(1-t+t^2) & d^2(1-2t)t - t^3 \end{pmatrix}$$

$$= C = I \qquad \text{(up to a scalar)}$$

so $d^3t + d(1-t+t^2) = 0$

$$|d^3| = |d|\,|1-t+t^2| \leq 3|d|$$

$$\Rightarrow n = 0 \text{ or } 1!$$

At $n=0$, $d=1$, $t = \zeta_4^{\pm 1} = \pm i$

at $n=1$, $d = \frac{1+\sqrt{5}}{2}$, $t = \zeta_5^{\pm 2}$, $d = -t-t^{-1}$

$\cancel{\text{or } d = \frac{1-\sqrt{5}}{2},\ t = \zeta_5^{\pm 1},\ d = t+t^{-1}}$

Although in this case finding T was straightforward, we also have the Anderson-Moore-Vafa theorem

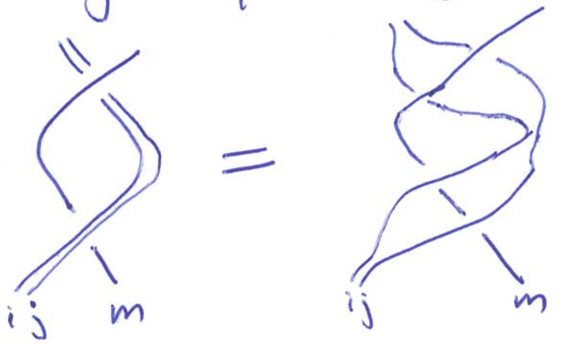
$$\left(T_i T_j T_m T_r\right)^{N_{ijm}^{\;r}} = \prod_p T_p^{\,N_{ij}^{\;p} N_{pm}^{\;r} + N_{im}^{\;p} N_{jp}^{\;r} + N_{jm}^{\;p} N_{ip}^{\;r}}$$

which in our example tells us $t^{n^2+4} = 1$

<u>Proof</u> $\beta^2 =$  acts on $\mathrm{Hom}(i \otimes j \to k)$ by the scalar

$T_i T_j / T_k$. 

The hexagon equation gives



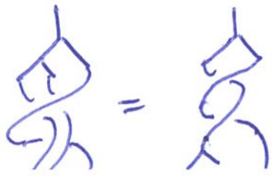
so $\det\left(\beta^2_{ij,m}\Big|_{\mathrm{Hom}((ij)m \to r)}\right) = \det\left(\beta^2_{i,m}\Big|_{\mathrm{Hom}(jim \to r)}\right) \det\left(\beta^2_{j,m}\Big|_{\mathrm{Hom}(ijm \to r)}\right)$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\prod_p \left(T_p T_m / T_r\right)^{N_{ij}^{\;p} N_{pm}^{\;r}} \qquad \prod_p \left(T_i T_m / T_p\right)^{N_{im}^{\;p} N_{jp}^{\;r}} \times \prod_p \left(T_j T_m / T_p\right)^{N_{jm}^{\;p} N_{ip}^{\;r}}$$

Here we use $\mathrm{Hom}(ijm \to r) = \bigoplus_p \mathrm{Hom}(ij \to p) \otimes \mathrm{Hom}(pm \to r)$

and



Are there categories realising these modular data?

For $n=1$, we need a category with a morphism 

In general,  $= \omega$  for some $\omega^3 = 1$,

but braiding implies $\omega = 1$:

 $= \alpha$ ,  $= \beta$ 

 $=$  $=$ 

$\|$ $\|$ $\|$

ω  $\alpha\beta^{-2}$ 

α^{-1} 

so $\omega = \omega^{-1}$, $\omega = 1$.

$\|$

$\alpha^{-1}\beta$ 

$\|$

$\alpha^{-1}\beta^2$ 

Since $\dim \operatorname{End}(X^{\otimes 2}) = 2$,

either $)(= \pm \genfrac{}{}{0pt}{}{\smile}{\frown}$ (in which case $d = \pm 1$)

or $\Upsilon\!\!\curlywedge = \alpha)(+ \beta \genfrac{}{}{0pt}{}{\smile}{\frown}$

and $)(, \genfrac{}{}{0pt}{}{\smile}{\frown}$ span $\operatorname{End}(X^{\otimes 2})$

Then $\times = A)(+ B\genfrac{}{}{0pt}{}{\smile}{\frown}$ and a standard argument in Temperley-Lieb gives $B = A^{-1}$; $-A^2 - A^{-2} = \dim X = \zeta_{10} + \zeta_{10}^{-1}$

so $A = \zeta_{10}^{l}$, $l = 2, 3, 7, 8$.

Now 

$= d^3 + 2d = -1$,

so the S-matrix is good.

$\left(\rho = A)\circ + A^{-1} \ldots = -A^3 \right)$.

so $T_X = \zeta_{10}^{3l+5}$ which only agrees with $t = \zeta_5^{\pm 2}$ for $l = 3, 7$.

What goes wrong otherwise?

We haven't checked this is really a braiding!

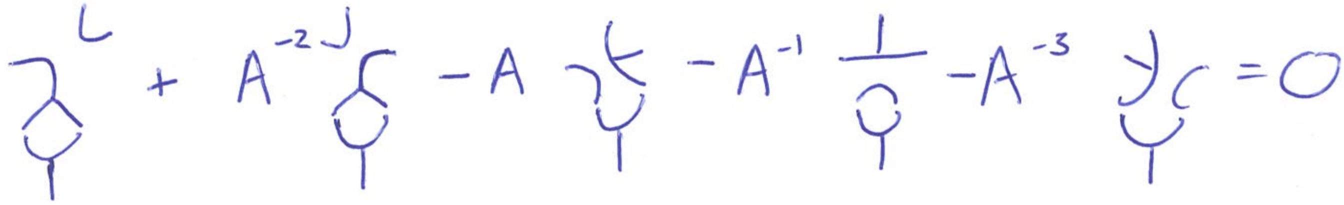
$$A\rho^0(\lambda^L)+A^{-1}\rho^1(\lambda^L)-A^2\rho^2(\lambda^L)-\rho^3(\lambda^L)-A^{-2}\rho^4(\lambda^L)=0$$

These coefficients are

	ξ^{l},	ξ^{-l},	ξ^{2l+5},	ξ^{5},	ξ^{-2l+5}
$l=2$	2	8	9	5	1
$l=3$	3	7	1	5	9
$l=7$	7	3	9	5	1
$l=8$	8	2	1	5	9

Thinking about the rotation (relations amongst these diagrams must form a representation of the cyclic group C_5) already shows $l=2$, $l=8$ are sick.

We'll use another approach.



$\cancel{-A}\ -A - A^{-1}(-A^{2} - A^{-2}) - A^{-3} = 0$

Now we know  $= \alpha\,)(+ \beta \smile\frown$, but we haven't found α, β.

$b) =$  $= \alpha d\,) + \beta$, so $b = \alpha d + \beta$

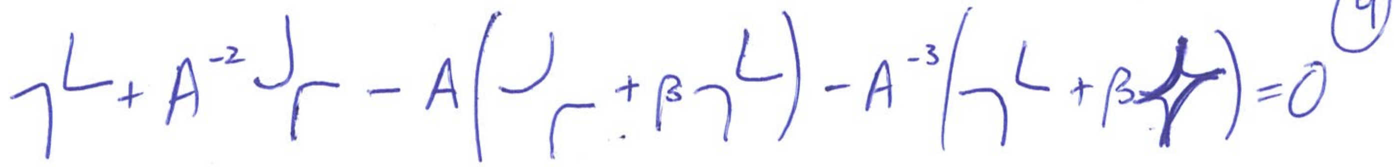
zero $=$  $= \alpha \frown + \beta d \frown$ so $\alpha = -\beta d$. $b = -\beta d^{2} + \beta = \beta(1-d^{2}) = -\beta d$

We can rescale the vertex freely, so let's take $\alpha = b = 1$, $\beta = -d^{-1} = \frac{1-\sqrt{5}}{2}$

 $= \cancel{\ }\,)(+ \beta \smile\frown$, $\cancel{\ }$

~~Then~~ $\cancel{b \,\urcorner^{\llcorner} + bA^{-2}\,\lrcorner\ulcorner - (A + A^{-3})(\ \ + \ \ ) = 0}$

~~So~~ $\cancel{b = -A - A^{-3} = \zeta_{10}^{\ell+5} + \zeta_{10}^{-3\ell+5}}$

Then 

so $1 - A\beta - A^{-3}\beta = 0$

$$\|$$

$$\cancel{1 + \xi_{10}^{l+5} + \xi_{10}^{-3l+5}\left(\xi_{10}^{3} + \xi_{10}^{-3}\right)} = 0$$

$$\cancel{\|}$$

$$\cancel{\xi_{10}^{0} + \xi_{10}^{l+5} + \xi_{10}^{-3l+8} + \xi_{10}^{-3l-3}}$$

$$\|$$

$$\xi^{0} + \xi^{l+5}\left(\xi^{3} + \xi^{-3}\right) + \xi^{-3l+5}$$

$$\|$$

$$\xi^{0} + \xi^{l-2} + \xi^{l+2} + \xi^{-3l+5}$$

$l=2$: $0, 0, 4, 9$ $\neq 0$

$l=3$: $0, 1, 5, 6$ $= 0$

$l=7$: $0, 5, 9, 4$ $= 0$

$l=8$: $0, 6, 0, 1$ $\neq 0$

Does these actually exist?

We could just invoke $\operatorname{Rep} U_q \mathfrak{g}_2$ at the appropriate root of unity now...

Alternatively, observe

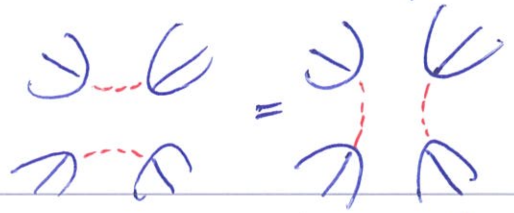
we can make this zero!

and then everything else lands in Temperley-Lieb.

Proposed evaluation algorithm:

- choose a pair of vertices.
- if they are not already near each other, chose a path, and drag them together along the path, using the braiding
- use the relation above to reduce the number of vertices.
- repeat until there are no vertices: Temperley-Lieb is easy to evaluate.

There is finitely much to check to see this is independent of choices: essentially only



We then need to verify the evaluation is consistent with the relations.