Today I want to calculate for you the modular data for the extended Haagerup subfactor.

Before embarking on such a calculation, I need to convince you that this is an interesting thing to attempt.

$\boxed{\text{Why study fusion categories?}}$

$\boxed{\text{Why study } \mathcal{EH}\text{?}}$ $\boxed{\text{Why study modular data?}}$

$\boxed{\text{How can we compute the modular data for } \mathcal{EH}\text{?}}$

- Fusion categories are
  finitely semisimple $\otimes$-categories

Examples
- Rep G, G a finite group
- Vec G, G a finite group
- Rep $U_\xi \mathfrak{g}$, the representation category of a quantum group at a root of unity
- If $N \subset M$ is a II_1-subfactor, the category of N-N bimodules $\otimes$-generated by ${}_N M_N$ is a semisimple $\otimes$-category.
  It is finite iff $N \subset M$ is finite depth.
  - Conversely, every unitary fusion category can be realized in this way.

– Theorem (Deligne)

A symmetric fusion category with subexponential growth ~~[illegible]~~ (~ some Schur functor vanishes)

is the category of representations of a finite (super-) group.

Slogan "Fusion categories are non-symmetric finite groups".

– Fusion categories are the 'symmetries' of 2+1-dimensional topological field theories

Thm (Lurie) ~~[illegible]~~ Cobordism hypothesis

local $n+1$ dimensional field theories with values in $\mathcal{E}$ $\longleftrightarrow$ fully dualizable objects in $\mathcal{E}$

Thm (Douglas – Schommer-Pries – Snyder)

When $n=2$, $\mathcal{E} = \mathrm{TensorCat}/\mathbb{C}$,

fully dualizable objects are the nondegenerate fusion categories.

While many fusion categories arise from finite groups and quantum groups, there are certainly others.

Any fusion category 'coming from' finite groups or quantum groups will be defined over some cyclotomic field.

However

<u>Thm</u> (Morrison-Snyder)

The fusion categories $\mathcal{H}6$ and $\mathcal{E}\mathcal{H}6$ coming from the Haagerup and extended Haagerup subfactors cannot be defined over any cyclotomic field.

What do fusion categories look like?

- Are they mostly grotesque monsters?
- Or could we hope for some sort of classification?

It would be unreasonable to be optimistic.

However, we can today say:

"every known fusion category can be constructed from

- finite groups
- quantum groups at roots of unity
- quadratic categories,
  (with first examples coming from E_6 and Haagerup)
  to be discussed in Masaki's lectures
- $\mathcal{E}_{76}$."

This reflects our deep ignorance, much more than it reflects the world.

What should we do?

1. Try to understand $\mathcal{E}_{76}$ much better.
2. Try to construct many more examples.

Why study modular data?

Every fusion category $\mathcal{C}$ has a Drinfeld centre $Z(\mathcal{C})$, with objects (X, β_-), $X \in \mathrm{Obj}\,\mathcal{C}$, $\beta_Y : X \otimes Y \to Y \otimes X$.

$Z(\mathcal{C})$ is a modular tensor category.

Given a pair of unitary fusion categories $\mathcal{C}$ and $\mathbb{D}$,

$$Z(\mathcal{C}) \cong Z(\mathbb{D})$$

$$\Updownarrow$$

there is a finite depth subfactor $N \subset M$ with

$$\mathcal{C} = \langle {}_N M_N \rangle \subset N\text{-mod-}N$$

$$\mathbb{D} = \langle {}_M M \underset{N}{\otimes} M_M \rangle \subset M\text{-mod-}M. \iff \text{"}\mathcal{C} \text{ and } \mathbb{D} \text{ are Morita equivalent"}$$

Until a few years ago, $A H$ was also 'exotic', but ~~theorem~~ a recent theorem of Izumi-Grossman-Snyder shows it is Morita equivalent to a quadratic category.

Thus studying the Drinfeld centre can reveal new constructions.

Even better

<u>Conjecture</u> (physicists/Terry)

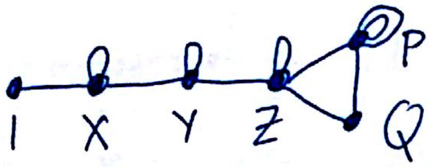
Every MTC arises as the representation category of some VOA,

so there ought to be a VOA that 'explains' $\mathcal{E}H6$.

To see this, we first need to know $Z(\mathcal{E}H6)$.

All we need is the fusion ring $K_0(\mathcal{E}H6)$,

which is uniquely determined by its principal graph



$$\left(\begin{aligned} &\text{so } X\otimes X = 1\oplus X \oplus Y \\ &P\otimes X = Z \oplus 2P \oplus Q, \text{ etc.}\end{aligned}\right)$$

There is a forgetful functor $R: Z(\mathcal{C}) \to \mathcal{C}$, with adjoint $I: \mathcal{C} \to Z(\mathcal{C})$, and

$$R(I(V)) \cong \bigoplus_{X \in \mathrm{Irr}\,\mathcal{C}} X^* \otimes V \otimes X.$$

At the level of fusion rings, we have a non-negative integer matrix A, so

$$K_0(I): K_0(\mathcal{C}) \to K_0(Z(\mathcal{C}))$$
$$v \longmapsto Av$$
$$K_0(R): K_0(Z(\mathcal{C})) \longrightarrow K_0(\mathcal{C})$$
$$v \longmapsto A^T v.$$

Thus $K_0(R)K_0(I) = A^T A$ is determined once we know the fusion ring.

This does not generally uniquely determine A.

However the dimension of any ~~object~~ simple object in $Z(\mathcal{C})$ must ~~be~~
1) be a d-number
2) divide the global dimension $\dim Z(\mathcal{C}) = (\dim \mathcal{C})^2$ as an algebraic integer.

Since R and I preserve dimensions, this is a condition on the rows of A.

Rather unreasonably, A is now uniquely determined.
(This is not a general phenomenon; eg. the 4442 fusion category has oodles of possible As.)

We learn that A has 22 rows, so $Z(\mathcal{E}36)$ must have 22 simple objects, and moreover we learn their dimensions:

1		ω_0
177.701		ω_1
49.396		ω_2
114.049	(7 times)	$\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4$
176.701	(4 times)	γ_i
128.304	(4 times)	δ_i
48.396	(4 times)	ε_i

These live in a degree 3 extension of $\mathbb{Q}$, sitting inside $\mathbb{Q}[\zeta_{13}]$.
We can compute the Galois group action.

'Recall' that for a unitary modular tensor category $\mathcal{C}=Z($[illegible]$)$, with

$$S = \frac{1}{\dim(\mathcal{C})}\begin{pmatrix} & \vdots & \\ \cdots & \quad & \cdots \\ & \vdots & \end{pmatrix}$$

$$T = \begin{pmatrix} \ddots & & \\ & \quad & \\ & & \ddots \end{pmatrix}$$

we have

- $S^2 = (ST)^3 = C, \quad C^2 = I$

  $\Rightarrow$ so S and T give a projective representation of $SL(2,\mathbb{Z})$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto S$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto T$

- $\mathbb{Q}[S_{ij}] = \mathbb{Q}[\zeta_N]$, with $N = \mathrm{ord}(T)$,

  and the representation factors through $SL(2,\mathbb{Z}/N\mathbb{Z})$.

- For ℓ coprime to N, $G_\ell = C S T^{1/\ell} S T^{\ell} S T^{1/\ell}$

  defines a signed permutation

  $$(G_\ell)_{xy} = \varepsilon_\ell(x)\, \delta_{y, x^\ell}$$

  matching the Galois action:

  $$\sigma_\ell(S_{xy}) = \varepsilon_\ell(x)\, S_{x^\ell, y}$$

- $T_{x^\ell x^\ell} = T_{xx}^{\ell^2}$

and Cauchy's theorem for MTCs: (Bruillard, Ng, Rowell, Wang)

The primes dividing N are the same as the primes dividing $\|\dim \mathcal{C}\|$.

Turning to $\mathcal{EH}$, we see $N = 5^a 13^b$.

If $b \geq 2$, $T_{x^\ell x^\ell} = T_{xx}^{\ell^2}$ would require more than 22 simple objects.

We have $\pi : Gal(\mathbb{Q}[\zeta_N]/\mathbb{Q}) \twoheadrightarrow Gal(\mathbb{Q}_{dim}/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$, and we know how this acts on objects.

~~From this we determined all $\mathcal{E}_x$~~

From this we determine $\mathbb{Z}_N^\times$ permutes $\{\alpha_1, \dots, \alpha_3, \beta_1, \dots, \beta_4\}$,

$$\omega_i^\ell = \omega_{i+\pi(\ell)}$$

$$\ell \text{ sends } \begin{tikzcd} \{\gamma_i\} \arrow[r] & \{\delta_i\} \arrow[r] & \{\varepsilon_i\} \arrow[ll, bend left] \end{tikzcd}$$

and if $\pi(\ell) = 1$, $\gamma_i^\ell = \delta_i$, $\delta_i^\ell = \varepsilon_i$, $\varepsilon_i^\ell = \gamma_i$

These objects form between one and four Galois orbits, with sizes multiples of 3.

Now we see $a = 1$. Otherwise, there is an object x with $25 \mid \operatorname{ord}(T_{xx})$, and so the Galois orbit of x has ~~at least 10 elements~~ size a multiple of 10, which is impossible.

Our representation is thus a representation of ~~SL(2,Z/65Z)~~

$$SL(2,\mathbb{Z}/65\mathbb{Z}) \cong SL(2,\mathbb{Z}/5\mathbb{Z}) \times SL(2,\mathbb{Z}/13\mathbb{Z})$$

and we would like to determine how it splits into irreducibles:

$$\rho = \bigoplus_{i\in I} \rho_i, \qquad \rho_i = \rho_{i,5} \otimes \rho_{i,13}$$

Write $T(\rho_i)$ for the T-eigenvalues in ρ_i.

Lemmas

- If $S_{xy} \neq 0$, $t_x, t_y \in T(\rho_i)$ for some $i \in I$.
- If $t_x = \frac{m_5}{5} + \frac{m_{13}}{13}$, there is some $i_x \in I$

  so $0, \frac{m_5}{5} \in T(\rho_{i_x,5})$

  $0, \frac{m_{13}}{13} \in T(\rho_{i_x,13})$

These facts, together with the character tables for

$SL(2,\mathbb{Z}_5)$ and $SL(2,\mathbb{Z}_{13})$

(and sometimes some explicit details about these irreps)

eventually give $\rho = \rho_{14}^{(13)} \oplus \rho_8^{(5)} \oplus 1 \oplus 1 \oplus 1$,

and we come up with explicit matrix realisations of those.

We thus have explicit matrices S', T', and some unknown invertible matrix Q so

$$QS = S'Q, \quad QT = T'Q$$

and we know T and the first 3 rows/columns of S.

Further S is symmetric, $SA_1 = A_1$, and $S^2 = C = I$ (since all these reps are even).

A little work produces a unique solution for S.

We verify via Verlinde's formula

$$N_{xy}^{z} = \sum_{w} \frac{S_{xw} S_{yw} S_{z^* w}}{S_{1w}}$$

gives non-negative integers, compatible with induction and restriction.

Finally, one can compute the possible character vectors of a VOA producing this MTC.

A $c=8$ we find just four possibilities:

$$q^{\frac{1}{3}}\mathbb{X}(\tau)_{W_0} = 1+12q+73q^2+346q^3+1390q^4+\cdots$$

$$q^{\frac{1}{3}}\mathbb{X}(\tau)_{W_0} = 1+13q+\cdots$$

$$q^{\frac{1}{3}}\mathbb{X}(\tau)_{W_0} = 1+3q+\cdots$$

$$q^{\frac{1}{3}}\mathbb{X}(\tau)_{W_0} = 1+4q+\cdots$$

Terry will tell us more about these tomorrow!