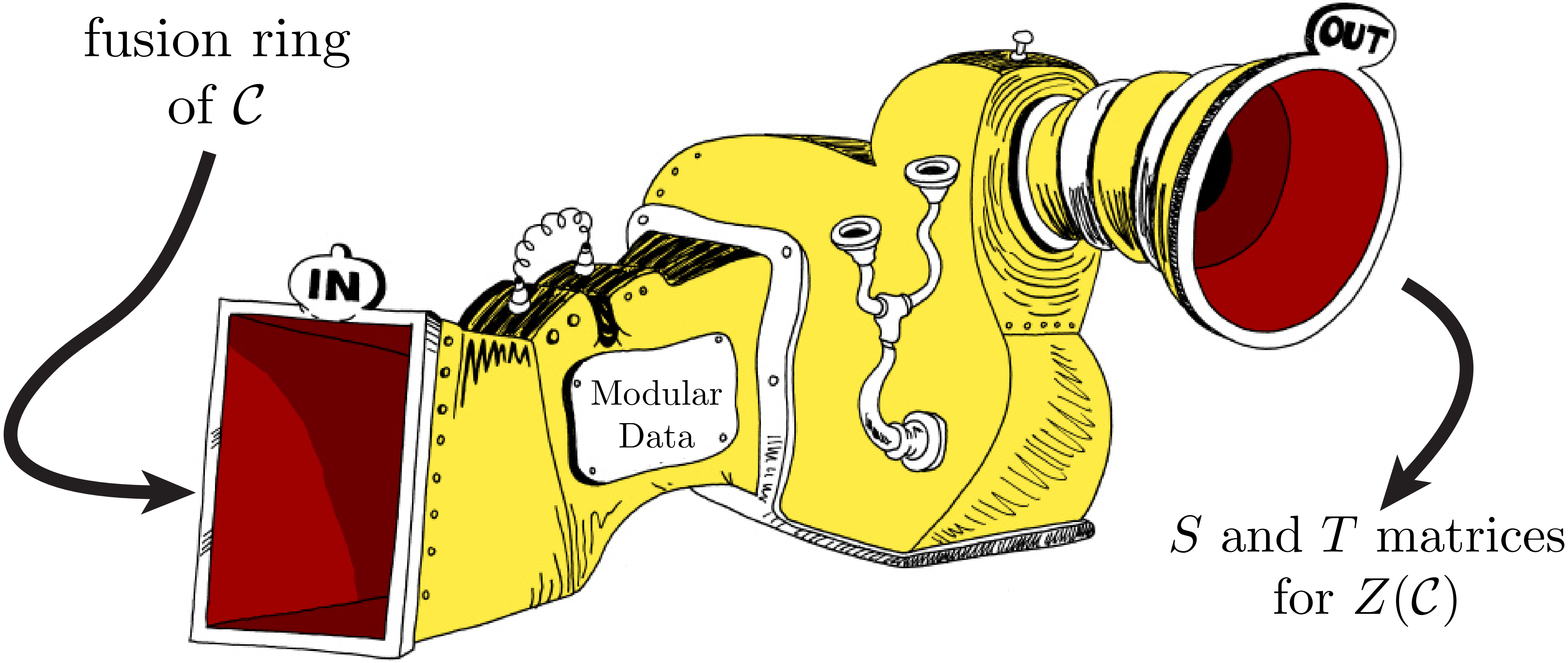

fusion ring of $\mathcal{C}$
IN
Modular Data
OUT
S and T matrices for $Z(\mathcal{C})$

Today I'm going to talk about the Drinfeld centres of some of the 'exotic-looking' fusion categories discovered during the classification of small index subfactors, in particular the 'extended Haagerup' categories.

~~But also talk~~

Really the point of this talk is to show you

'the modular data machine'

a program which takes as input a fusion ring, R and calculates all possible modular data (S and T matrices) for $Z(\mathcal{C})$, $\mathcal{C}$ categorifying R.

On many examples it is effective.

This is a little dangerous, as already my recent paper with Terry, arXiv:1606.07165 is 'obsolete'.

Definition Modular data consists of

- an integer N, "the conductor"
- an integer n, "the rank"
- a diagonal $n \times n$ matrix T with order N.
- a $n \times n$ unitary, symmetric matrix $S \in M_{n\times n}(\mathbb{Q}[\zeta_N])$
- a permutation matrix C with $C^2 = I$

such that:

- $(ST)^3 = C = S^2$ (and so $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto S$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto T$ gives a representation of $SL(2, \mathbb{Z}/N\mathbb{Z})$)
- $S_{1x} > 0$ for all x
- $\rho\begin{pmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{pmatrix} = C S T^{1/\ell} S T^{\ell} S T^{1/\ell}$ for ℓ coprime to N

  is a signed permutation matrix: $(G_\ell)_{xy} = \varepsilon_\ell(x)\, \delta_{y, x^\ell}$
- $T_{x^\ell x^\ell} = T_{xx}^{\ell^2}$
- $\sigma_\ell(S_{xy}) = \varepsilon_\ell(x)\, S_{x^\ell, y} = \varepsilon_\ell(y)\, S_{x, y^\ell}$
- Verlinde's formula:

$$N_{xy}^{\ z} = \sum_w \frac{S_{xw} S_{yw} \overline{S_{zw}}}{S_{1w}} \in \mathbb{Z}_{\geq 0}$$

(and then automatically they define a based ring).

Theorem The S and T matrices for a unitary modular tensor category

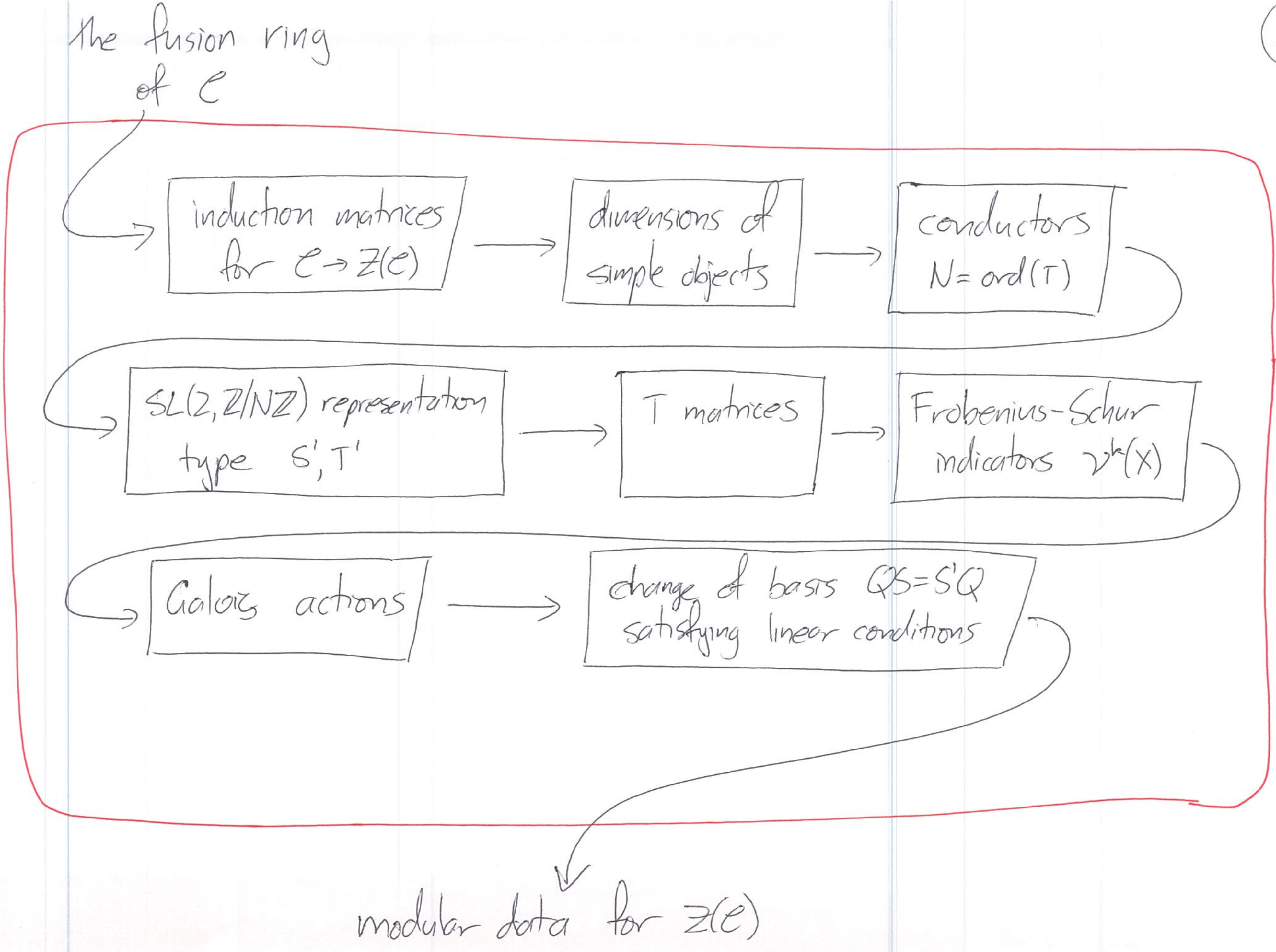
$$(S)_{xy} = \frac{1}{D}\, {}_{x}\,\,{}_{y}. \quad (T)_{xx} = {}_{x}$$

give modular data in this sense.

Question What are the modular tensor categories associated (as the Drinfeld centre) to the 'exotic' unitary fusion categories, e.g.

$$\mathcal{H}_1, \quad \mathcal{E}\mathcal{H}_1, \quad \mathcal{AH}_1.$$

Easier question What about just the modular data of $Z(\mathcal{C})$?


the fusion ring of $\mathcal{C}$
induction matrices for $\mathcal{C} \to Z(\mathcal{C})$
dimensions of simple objects
conductors $N = \mathrm{ord}(T)$
$SL(2, \mathbb{Z}/N\mathbb{Z})$ representation type S', T'
T matrices
Frobenius-Schur indicators $\nu^k(X)$
Galois actions
change of basis $QS = S'Q$ satisfying linear conditions
modular data for $Z(\mathcal{C})$

Induction and restriction

There are adjoint functors $I : \mathcal{C} \underset{\otimes}{\rightleftarrows} Z(\mathcal{C}) : R$,

which at the level of Grothendieck group gives

$$A^T : K_0(\mathcal{C}) \rightleftarrows K_0(Z(\mathcal{C})) : A$$

$R(I(X)) = \bigoplus_{V \in \mathrm{Irr}\,\mathcal{C}} V X \overline{V}$, so the fusion ring of $\mathcal{C}$ determines AA^T

(a symmetric non-negative integer matrix)

<u>Lemma</u> There are finitely many such A.

Now $\dim(Y \in Z(\mathcal{C})) = \sum_{X \in \mathrm{Irr}\,\mathcal{C}} A_{XY} \dim(X \in \mathcal{C})$,

and $\dim(Y) \mid \dim(Z(\mathcal{C}))$, $\dim(Y)$ is an Ostrik d-number.

<u>Fact</u> With these restrictions, and a trick when AA^T is not full rank, it becomes plausible to enumerate all possible A, and hence the dimensions of objects in $Z(\mathcal{C})$.

<u>Example</u>

left out: Galois/Verlinde, formal codegrees

Conductors

If $N = \prod p_i^{n_i}$, we must have an T-eigenvalue λ_i with object x_i with

$p_i^{n_i} \mid \operatorname{ord} \lambda_i$.

Then $T_{x^\ell x^\ell} = T_{xx}^{\ell^2}$ tells us the Galois orbit of x_i has

size a multiple of $p_i^{n_i - 1}(p_i - 1)/2$ (or $2^{n_i - 3}$ if $p_i = 2$).

$\Longrightarrow$ for a given rank, there are finitely many possible conductors.

Moreover, since $S_{1x} = \frac{\dim x}{D}$,

and $\sigma_\ell(S_{1x}) = \pm S_{1x^\ell}$,

we often have restrictions on the sizes of Galois orbits.

Example

Fact Using these, we often have a plausibly small set of possible conductors.

$SL(2, \mathbb{Z}/N\mathbb{Z})$ representations

We now enumerate possible (abstract) representation types.

If $N = \prod_i p_i^{n_i}$, $SL(2, \mathbb{Z}/N\mathbb{Z}) = \prod_i SL(2, \mathbb{Z}/p_i^{n_i}\mathbb{Z})$,

and GAP can compute character tables in the relevant ranges.

By identifying the conjugacy classes of $T^l \in SL(2, \mathbb{Z}/p^n\mathbb{Z})$, we can find the set of T-eigenvalues of any irrep, written $T(\rho)$.

We know:

- $\mathrm{lcm}(\mathrm{order}(T(\rho))) = N$
- if λ appears in some irrep, $\{0, \lambda\}$ both appear in some irrep (cf. Lemma 4.1)
- irreps have signs; $\#(\lambda \text{ in } T(\rho_{even})) \geq \#(\lambda \text{ in } T(\rho_{odd}))$
- traces of Galois group elements may be constrained
- $\#(1 \text{ in } T(\rho)) \geq \#$ of simples in the induction of 1_e.

Write T' for the T-matrix in the abstract representation type.

T-matrices

At this point we know T', so we know the (multi-)set of T-eigenvalues, but not how they correspond to columns of the induction matrix.

<u>Fact</u> We "can" enumerate all such bijections compatible with the limited information we have about the Galois action $\left(\sigma_\ell(S_{1x}) = \pm S_{1x^\ell},\ T_{x^\ell x^\ell} = T_{xx}^{\ell^2}\right)$,

<u>up to</u> permutations of the ~~columns of A~~. objects fixing A.

(It's essential we avoid permutations in the enumeration, not just quotient them out afterwards.)

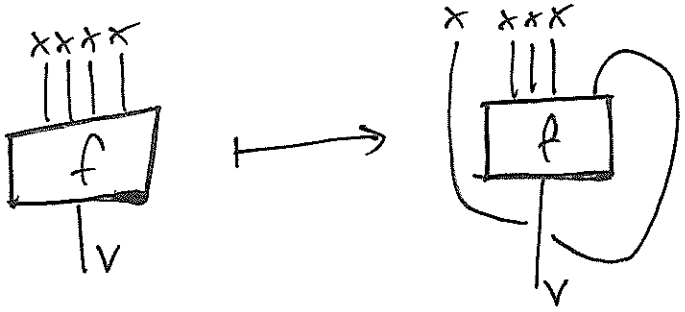
The top left entry of $STS = T^{-1}ST^{-1}C$

gives $\sum_i d_i^2 t_i = \sqrt{\sum_i d_i^2}$.

If we use this (via the Δ-inequality), it's actually plausible to do this enumeration.

Frobenius-Schur indicators

Define $\underset{\substack{\cap \\ \mathrm{Irr}}}{\rho_{X}},{}_{k,}\underset{\substack{\cap \\ \mathrm{Irr}\,Z(\mathcal{C})}}{V} : \mathrm{Hom}(V \to X^{\otimes k}) \circlearrowleft$



Knowing the T-matrix and the dimensions in the centre, we can compute

$$\mathrm{tr}(\rho_{X,k,V}) = {}_V(S T^k A^T)_X$$

for $V = 1$ (or any V in the Galois orbit of 1).

Since $\rho^k_{X,k,V}$:

so $\rho_{X,k,V}$ has order dividing $k \cdot \mathrm{ord}(t_V) \overset{\mathrm{def}}{=} k \cdot n_V$,

and hence eigenvalues $\{\zeta^{i}_{kn_V}\}$

If ℓ is coprime to kn_V, then $\mathrm{tr}(\rho^{\ell}_{X,k,V}) = \sigma_{\ell}(\mathrm{tr}\,\rho_{X,k,V})$,

Now assume $V=1$

$g = \gcd(\ell, k)$,

$\rho^{\ell}_{X,k,1} = \rho^{\ell/g}_{X^{\otimes g}, k/g, 1}$; then as $\mathrm{tr}(\rho_{X\oplus Y,k,1}) = \mathrm{tr}(\rho_{X,k,1}) + \mathrm{tr}(\rho_{Y,k,1})$

we know the traces of all powers of $\rho_{X,k,1}$.

Newton's identities tell us the eigenvalues of $\rho_{X,k,1}$, which must all lie in $\sqrt[k]{1}$.

$\Longrightarrow$ There is no "$c=2$" category in Larson's classification of non-self-dual pseudo-unitary rank 4 fusion categories.

Galois actions

We (finally!) determine the details of the Galois group action.

Recall $$\sigma_\ell(S_{1x}) = \varepsilon_\ell(x) S_{1x^\ell} = \varepsilon_\ell(x)\frac{\dim x^\ell}{D},$$
$$\sigma_\ell(S_{1x}) = \sigma_\ell\left(\frac{\dim x}{D}\right)$$

and $T_{x^\ell x^\ell} = T_{xx}^{\ell^2}$.

These constrain the possible Galois actions. Enumerating them all is still a mess, but doable — the problem is that we need to avoid listing Galois actions which only differ by a symmetry of A and T.

The change of basis

Now we obtain explicit S' and T' matrices for the chosen $SL(2,\mathbb{Z}/N\mathbb{Z})$ representation type. (From the RepS_n package in gap.)

There's some change of basis Q so $T'Q = QT$, $S'Q = QS$, with Q and S mostly unknown, Q invertible.

We first solve all the linear equations available:

$$\begin{cases} T'Q = QT \\ STA^t = T^{-1}A^t \iff S'QTA^t = QT^{-1}A^t & \text{(Generalised FP-indicators)} \\ S_{1x} = \frac{\dim x}{\cancel{\dim}\, D}, \quad S_{1^\ell x} = \varepsilon_\ell(1)\,\sigma_\ell(S_{1x}) \\ S^2 = C \iff S'^2 Q = QC & C = \begin{pmatrix} -1 & 0 \\ 0 & -1^{-1} \end{pmatrix} \text{ so determined by the Galois action} \\ \rho'\begin{pmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{pmatrix} Q = Q G_\ell \end{cases}$$

Now $S'Q = QS$ is a system of quadratic equations (in Q_{ij}, S_{xy} jointly), which if we're lucky we can solve (away from $\det Q = 0$).