Today I'm going to talk about some new progress in understanding the Drinfeld centres of fusion categories.

I'll start, however, with a big picture look at why we study fusion categories and modular tensor categories.

(As some Nobel prizes were just awarded for something rather closely related, I'll try to briefly mention the connection with topological phases of matter, too!)

A <u>fusion category</u> is a finitely semisimple rigid monoidal category.

The fundamental example is

- Rep G, for G a finite group.

This example is <u>symmetric</u>: $V \otimes W \cong W \otimes V$ "in the obvious way", but this is not generally true.

In fact, you can think of fusion categories as a "noncommutative analogue" of the theory of finite groups:

<u>Theorem</u> (Deligne) A symmetric fusion category (in which some Schur functor vanishes) is the representation category of some finite (super) group.

There are plenty of non-symmetric examples.

The 'first' is the Fibonacci category,

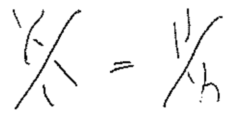
with two simple objects, 1 and X,

satisfying $X^{\otimes 2} \cong 1 \oplus X$ (notice this implies $\dim X = \frac{1+\sqrt{5}}{2} \notin \mathbb{N}$)

It naturally arises as a quantum group at a root of unity,

and many others arise this way.

These are all <u>braided</u>: $X \otimes Y \underset{\beta_{X,Y}}{\cong} Y \otimes X$,

satisfying 

but not 

There are still more beyond, which aren't even braided (indeed, $X \otimes Y \not\cong Y \otimes X$).

Mostly these come from classification programs (originally for subfactors), and some are very mysterious!

Why study fusion categories?

Recent results of Lurie, Douglas–Schommer-Pries–Snyder
<u>classify</u> 3210-dimensional TQFTs,
precisely in terms of fusion categories.

(The fusion category is the invariant of the point.)

These TQFTs provide a mathematical model for some classes of <u>topological matter</u>.

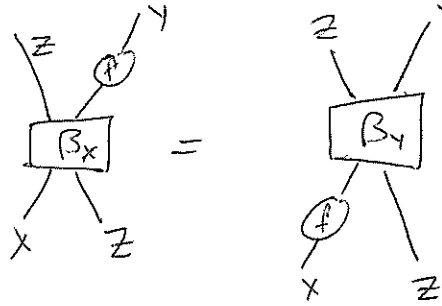
(Nobel prizes, woohoo!)

Indeed, the "Levin-Wen model" is a recipe producing a <u>commuting local Hamiltonian model</u> for the TQFT attached to a fusion category.

(That is, we can write down "physically realistic lattice models" for any of these topological phases.)

The 3|2|1-dimensional part of such a TQFT is entirely determined by the Drinfeld centre of the fusion category:

$$Z(\mathcal{C}) = \left\{ (X, \beta) \;\middle|\; \beta : X \otimes - \longrightarrow - \otimes X \right\}$$

$\{ f : X \to Y \mid$  $\}$

The Drinfeld centre is itself a braided fusion category, and when $\mathcal{C}$ is "spherical", $Z(\mathcal{C})$ is an example of a modular tensor category.

In fact, MTCs classify 3|2|1-dimensional TQFTs.

(Physicists are at this point in time are <u>much</u> more interested in MTCs than fusion categories. I don't think this is good!)

Why try to compute $Z(\mathcal{C})$, for $\mathcal{C}$ fusion?

1. To compute the TQFT invariants.

2. To find new MTCs, (for the entertainment of the physicists?)

3. Fusion categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are Morita-equivalent iff $Z(\mathcal{C}_1) \cong Z(\mathcal{C}_2)$.

   Often understanding the Morita equivalence class reveals a lot about a fusion category (e.g. 'explains' previously mysterious examples)

   Cain is working on this!

4. Many of the algebraic constructions on fusion categories are described in terms of data in the Drinfeld centre

   (in particular, group extensions, enriched quotients)

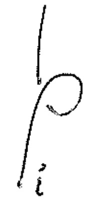
Sadly, computing Drinfeld centres is really hard.

For many purposes, the <u>modular data</u> is enough.

<u>Theorem</u> If $\mathcal{C}$ is an pseudo-unitary MTC, and we define matrices

$(S)_{i,j\in \mathrm{Irr}\,\mathcal{C}} =$ 

$\overline{\qquad \dim \mathcal{C} \qquad}$



$= T_{ii}$ 

,

then (S,T) forms 'modular data'.

<u>Definition</u> Modular data consists of

- an integer N. "the conductor"
- an integer n. "the rank"
- a diagonal $n\times n$ matrix T with order N.
- a $n\times n$ unitary, symmetric matrix

$$S \in M_{n\times n}(\mathbb{Q}[\zeta_N])$$

such that:

- $(ST)^3 = S^2 = C, \quad C^2 \doteq I$

  (and so $\begin{pmatrix}0&-1\\1&0\end{pmatrix} \mapsto S$, $\begin{pmatrix}1&1\\0&1\end{pmatrix} \mapsto T$ gives a representation of $SL(2, \mathbb{Z}/N\mathbb{Z})$)

- $S_{1x} > 0$ for all x

- $\rho\begin{pmatrix}\ell&0\\0&\ell^{-1}\end{pmatrix} = CST^{1/\ell}ST^{\ell}ST^{1/\ell}$ for ℓ coprime to N

  is a signed permutation matrix

  $$(G_\ell)_{xy} = \varepsilon_\ell(x)\,\delta_{y,x^\ell}$$

  for some action $x \mapsto x^\ell$ of $\mathrm{Gal}(\mathbb{Q}[\zeta_N])$

  on the irreducible ~~ob~~ objects

- $T_{x^\ell x^\ell} = T_{xx}^{\ell^2}$

- $\sigma_\ell(S_{xy}) = \varepsilon_\ell(x)\, S_{x^\ell, y}$

- Verlinde's formula:

  $$N_{xy}^{z} = \sum_w \frac{S_{xw} S_{yw} \overline{S_{zw}}}{S_{1w}} \in \mathbb{Z}_{\geq 0}$$

Modular data may be enough

- to determine parts of the TQFT
  (it tells you everything about the torus)
  - Corey is working on extracting braid group representations
- to identify subcategories, or the collection of invertibles
  (enough to undertake constructions...)
- to provide vector valued modular forms, which may help determine possible CFT realisations of the MTC

the fusion ring of $\mathcal{C}$

$\boxed{\text{induction matrices for } \mathcal{C} \to Z(\mathcal{C})}$ $\longrightarrow$ $\boxed{\text{dimensions of simple objects}}$ $\longrightarrow$ $\boxed{\text{conductors } N = \mathrm{ord}(T)}$

$\boxed{SL(2, \mathbb{Z}/N\mathbb{Z}) \text{ representation type } S', T'}$ $\longrightarrow$ $\boxed{T \text{ matrices}}$ $\longrightarrow$ $\boxed{\text{Frobenius-Schur indicators } \nu^k(X)}$

$\boxed{\text{Galois actions}}$ $\longrightarrow$ $\boxed{\text{change of basis } QS = S'Q \text{ satisfying linear conditions}}$

modular data for $Z(\mathcal{C})$