LYMPH-TOFU, Kauffman, and ...?

HOMFLY-PT-U

- Why do the HOMFLY and Kauffman polynomials exist?

  They are both two variable polynomial invariants of links.

  They are <u>explained</u> by <u>families of Lie algebras</u>, which can be coherently quantised.

  Homfly()$(a,z) =$  $\in \operatorname{Rep} U_q(\mathfrak{gl}_n)$

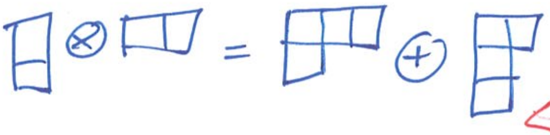
  when $a = q^n$, $z = q - q^{-1}$.

  It seems there may be another such polynomial.

Families of 'Lie algebras':

classical classical $\mathfrak{gl}_t$, O_t	classical exceptional Deligne's exceptional series.
quantum classical Homfly, Dubrovnik	quantum exceptional

- The Lie algebras $\mathfrak{gl}(n, \mathbb{C})$ show a variety of uniform behaviour
  - the f.d. irreps are indexed by partitions, n only enters through a restriction that there are at most n rows.
  - for a fixed partition, the dimension of the irrep is a rational function of n
  - the Littlewood-Richardson rule describes $\otimes$-products



vanishes when $n=2$

These are explained by Deligne's $\mathfrak{gl}_t$, which is essentially a <u>skein theory</u> based on (unembedded) oriented curves, modulo a relation:

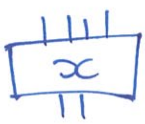
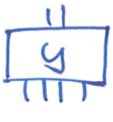
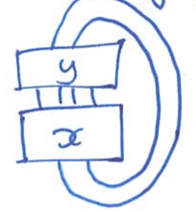
$\mathfrak{gl}_t = \mathbb{Q}[d]\{$  $\}/$  $= d$

We can think of this as a (pivotal) category

$\mathrm{Obj}(gl_t) = \{\text{words in } \{\uparrow, \downarrow\}\}$

~~[illegible]~~ $gl_t(\uparrow\downarrow\uparrow\downarrow \to \uparrow\downarrow\uparrow) = \mathbb{Q}[d]\Big\{$  $\Big\} \Big/ \bigcirc = d$

In any pivotal category, we have the negligible ideal

$\mathcal{N} = \Big\{$  $\Big|\ \forall$ ,  $= \text{zero} \Big\}$.

Theorem

- Generically $\mathcal{N}(gl_t) = \{0\}$
- Generically, the idempotent completion is semisimple, with simples labelled by partitions, satisfying the LR rule.
- At $t = n \geq 0$

$$gl_t/_{\mathcal{N}} \cong \mathrm{Rep}\, gl(n, \mathbb{C})$$

$$\uparrow \longmapsto \mathbb{C}^n.$$

This explains all the uniform behaviour!

Similarly there is an unoriented skein theory O_t explaing the uniform behaviour of $o(n)$ and $sp(2n)$.

Each of the Lie algebras $gl(n,\mathbb{C})$ has a quantisation $U_q gl(n,\mathbb{C})$,

and $\operatorname{Rep} U_q gl(n)$ is a braided $\otimes$-category.

We get link invariants

 $\in \operatorname{Rep} U_q gl(n)$

is a rational function in q, depending on n.

These link invariants all fit together into a 2-variable polynomial

Homfly()(a,z) $=$  $\in \operatorname{Rep} U_q gl(n)$

at $a=q^n$, $z=q-q^{-1}$.

This is explained by the existence of a braided $\otimes$-category over $\mathbb{Q}[a,z]$:

HOMFLY $=$ $\{$ oriented tangles $\}$ /  $=z$ ,  $= a$ 

(as a consequence, $d =$  $= \dfrac{a-a^{-1}}{z}$

$$= \frac{q^n-q^{-n}}{q-q^{-1}} = q^{n-1}+q^{n-3}+\cdots+q^{-n+1}$$

$= n$ at $q=1$)

Theorem

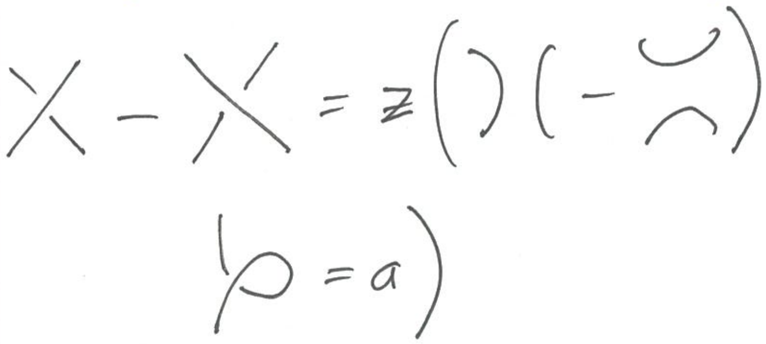
- HOMFLY is generically semisimple (after idempotent completing) (with simples labelled by partitions)

- At $a = q^n$, $z = q - q^{-1}$,

$$\mathrm{HOMFLY}/\mathcal{N} \cong \mathrm{Rep}\, U_q\, gl(n)$$

$$\uparrow \longmapsto \mathbb{C}(q)^n$$

Similarly, the Dubrovnik ~Kauffman skein relation explains the uniform behavour of $\mathrm{Rep}\, U_q\, o(n)$ and $\mathrm{Rep}\, U_q\, sp(n)$.



(and so $\bigcirc = \frac{a - a^{-1}}{z} + 1$.)

At $a = q^{4n}$, $z = q^2 - q^{-2}$, $\mathrm{Dubrovnik}/\mathcal{N} \cong \mathrm{Rep}^{\text{vector}}\, U_q\, so(2n+1)$

$a = q^{2n-1}$, $z = q - q^{-1}$, $\cong \mathrm{Rep}^{\text{vector}}\, U_q\, so(2n)$

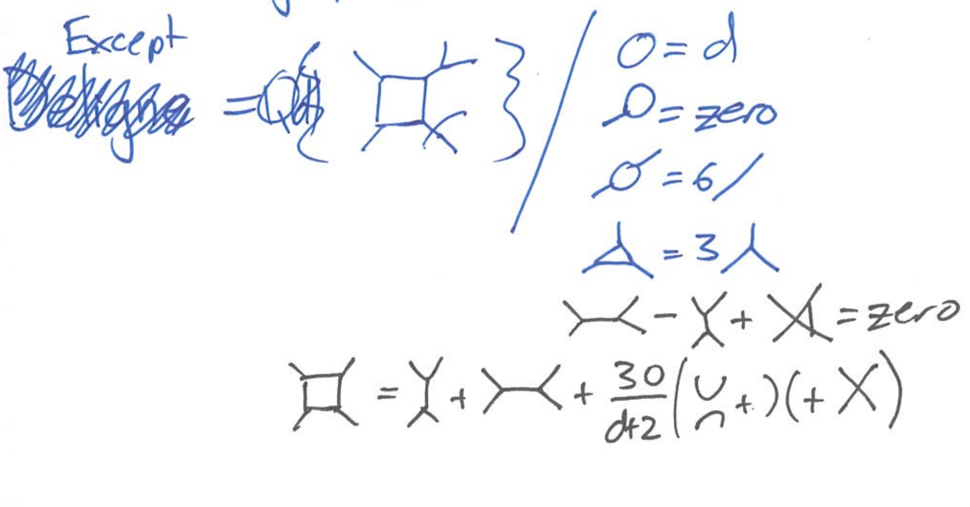
$a = -q^{2n+1}$, $z = q - q^{-1}$, $\cong \mathrm{Rep}\, U_q\, sp(2n)^{\text{unimodal}}$

Vogel, Cvitanovic, Cohen-de Mann, and Deligne have noticed certain <u>uniform behaviour</u> amongst the exceptional Lie algebras g_2, f_4, e_6, e_7, e_8.

The simplest phenomenon is seen in $\dim \mathrm{Inv}(\mathfrak{g}^{\otimes k})$:

$\mathfrak{g}$	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7
g_2	1	0	1	1	5	15	80	436
f_4	1	0	1	1	5	15	80	436
e_6	1	0	1	1	5	17	90	542
e_7	1	0	1	1	5	16	80	436
e_8	1	0	1	1	5	16	79	421

Deligne proposed an explanation in terms of a skein theory based on (vertex-oriented) (unembedded) cubic graphs, over $\mathbb{Q}(d)$



(IHX)

(the exceptional relation)

We then have:

<u>Conjecture</u> (Deligne, more or less)

- Except is generically semisimple.
- $\mathrm{Except}(0 \to 0) = \mathbb{Q}(d) \cdot \phi$.

  $\overset{\shortparallel}{\text{the space of closed diagrams}}$

- At $d = \dim \mathfrak{g}$, for $\mathfrak{g} = \mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$,

$$\mathrm{Except}/_N \cong \mathrm{Rep}\,\mathfrak{g} \qquad \left(\text{or } \mathrm{Rep}\,\mathfrak{e}_6^{\mathbb{Z}/2\mathbb{Z}} \text{ when } \mathfrak{g} = \mathfrak{e}_6\right)$$

$$| \longmapsto \mathfrak{g}$$

$$\curlywedge \longmapsto [\,,\,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}.$$

(It is at least easy to see that there is a functor

$$\mathrm{Except}/_N \to \mathrm{Rep}\,\mathfrak{g}.)$$

(and not too hard to see it is an isomorphism)

This would depend on two subconjectures:

<u>Conjecture 1</u>: the relations suffice to evaluate any <u>closed</u> cubic graph

(e.g. one can show that any $(2n+1)$-gon can be rewritten in terms of 'simpler' diagrams)

(in fact, a <u>wild</u> conjecture is that this is even possible without the exceptional relation!)

<u>Conjecture 2</u>: this evaluation is consistent (i.e. gives a unique answer)

Both seem hard to prove!

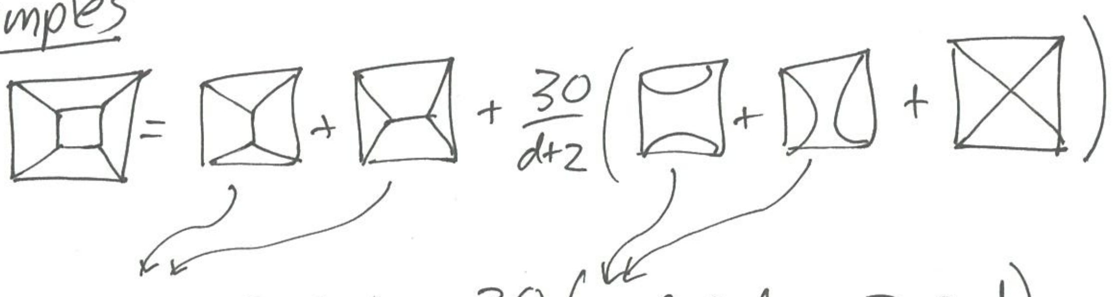
But conjecture 2 may be easy to disprove — it would suffice to exhibit some cubic graph, and two different ways to evaluate it.

This would give a polynomial in d, which must vanish. Certainly $d = 14, 52, \ldots$ must be roots.

(Notice we couldn't have done this to break gl_t; infinitely many points actually exist.)

Enough points on the exceptional curve actually exist to ensure ~~such a m~~ ~~this cubic graph must be rela~~ that a minimal example must be quite large....

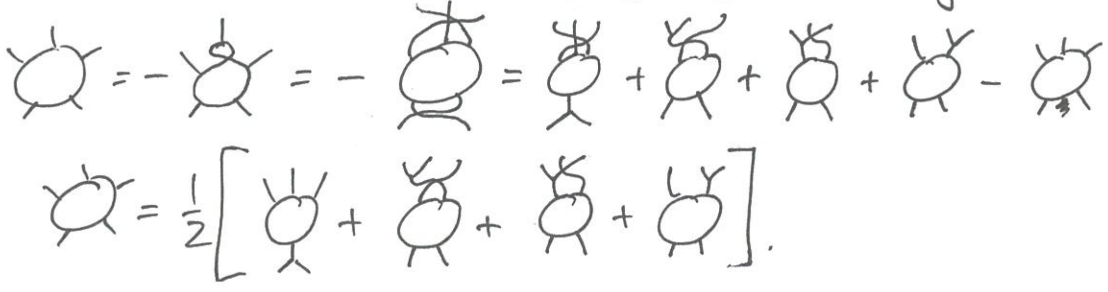
Examples

1) 

$$= 2\cdot3\cdot3\cdot6\cdot d + \frac{30}{d+2}\left(2\cdot6\cdot6\cdot d + 3\cdot6\cdot d\right)$$

$$= 108d + \frac{2700d}{d+2}$$

) The Jacobi relation lets us simplify any $(2n+1)$-gon:



Evidence (Cohen de-Mann)

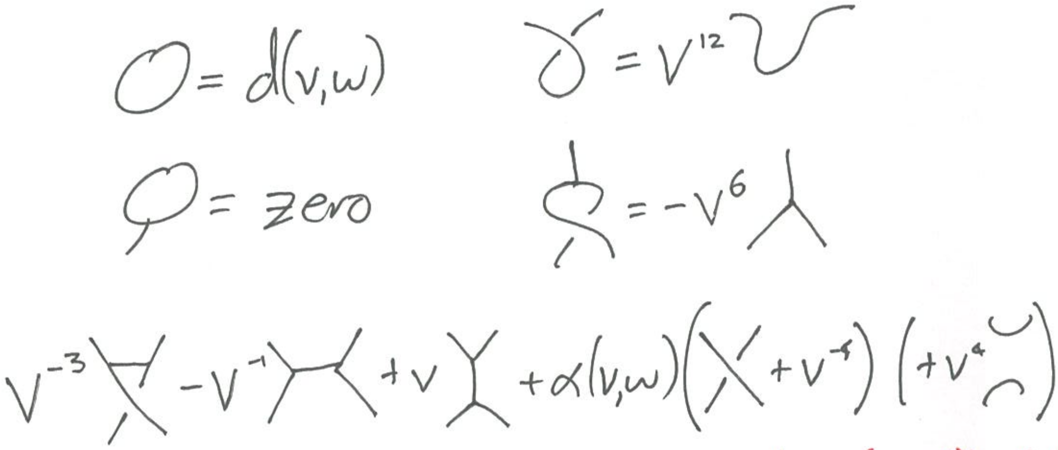
One can compute how $\mathfrak{g}^{\otimes k}$ decomposes into minimal idempotents for $k \leq 3$.

This agrees with the decomposition for the exceptional Lie algebras, and gives the dimensions of the irreps as rational functions in d.

Dissatisfied by the number of things we don't know how to prove, we decided to make some more conjectures:

~~Conjecture~~

Let QExcept $= \mathbb{Q}(v,w)\{\text{cubic tangles}\}/$



$\alpha = -\dfrac{(w-w^{-1})(wv^{-1}-w^{-1}v)}{v-v^{-1}}$

$d = -(v^2+v^{-2}) \times \dfrac{wv^5-w^{-1}v^{-5}}{w-w^{-1}} \times \dfrac{wv^{-6}-w^{-1}v^{6}}{wv^{-1}-w^{-1}v}$.

<u>Theorem</u> (MST) A skein theory for knotted trivalent graphs with $\dim P_k \leq 1, 0, 1, 1, 5$ is either small and well-understood, one other 1-parameter family, or QExcept.

<u>Theorem</u> (MST) With $d = q\dim(\mathfrak{g})$, $v = q^{\langle \lambda_{ad}, \lambda_{ad}+\rho \rangle}$

$$\text{QExcept}/N \xrightarrow{\cong} \text{Rep}^{adj}\, U_q\mathfrak{g}$$ $\mathbb{Z}/2\mathbb{Z}$ fixed points, when $\mathfrak{g} = \mathfrak{e}_6$.

for each exceptional $\mathfrak{g}$

 $\longmapsto$ bracket

 $\longmapsto$ R-matrix

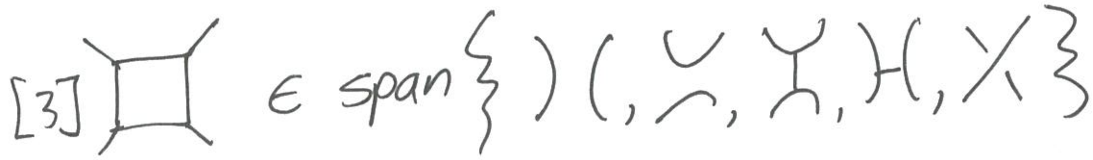
Conjecture (MST)

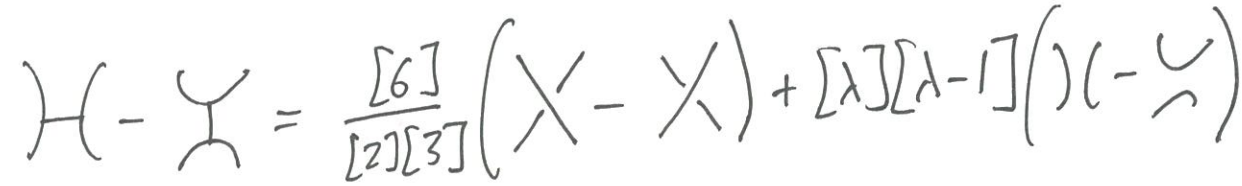
In QExcept, the space of (closed) knotted trivalent graphs is one-dimensional, spanned by the empty diagram.

We need an algorithm to evaluate (and, ambitiously, a consistent one)!

Examples

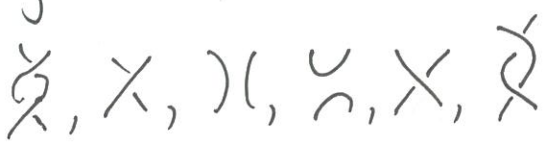
- adding an H in 3 different ways, and taking a linear combination,



- 

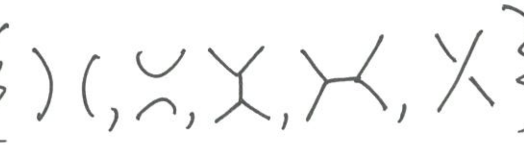
(here $[k\lambda + l] = \omega^{k} v^{l} - \omega^{-k} v^{-l}$)

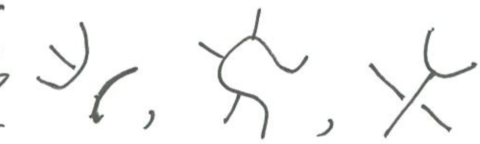
(Using this we can find a relation between



and use this to evaluate knots up to 8 crossings.)

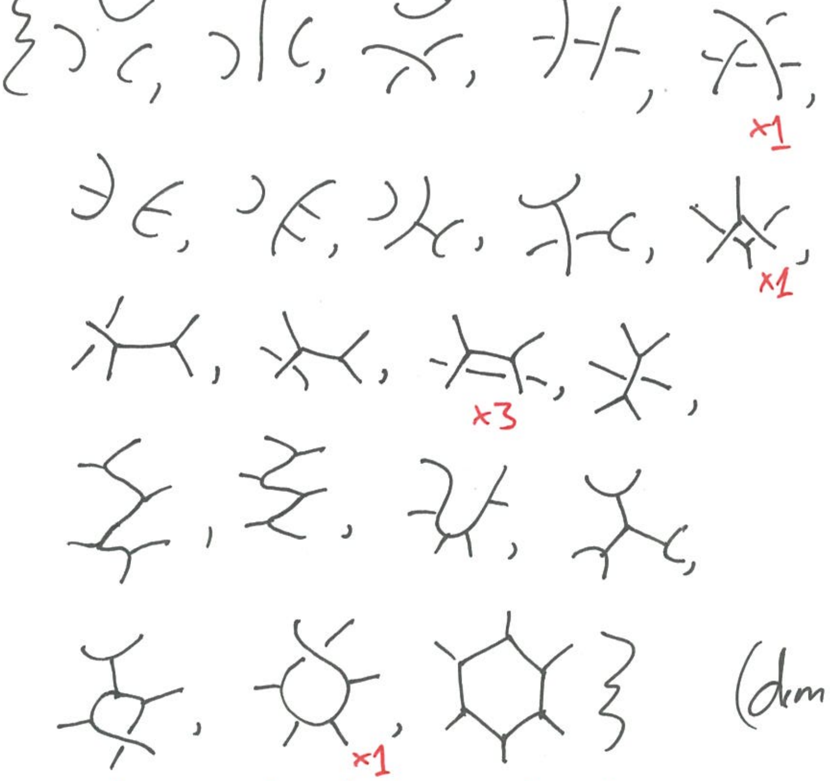
We have conjectural bases for

$QExcept_4 = \mathbb{Q}(v,w)\{$  $\}$ (dim=5)

$QExcept_5 = \mathbb{Q}(v,w)\{$  and rotations of these,

and 

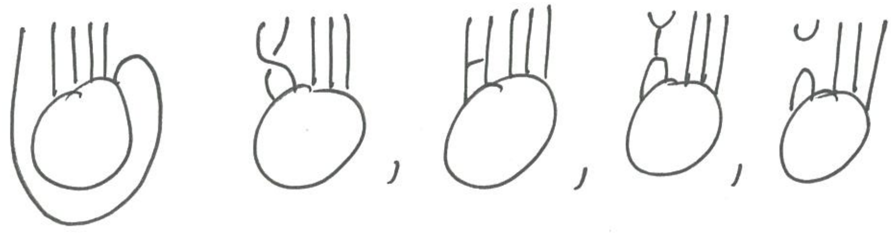
$\}$ (dim=16)

$QExcept_6 = \mathbb{Q}(v,w)\{$ 

$\}$ (dim=80)

but sometimes not all the rotations!

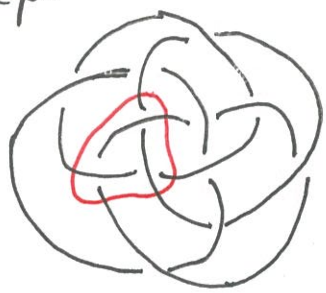
These have the nice property that they are closed under the operations



$\Rightarrow$ we get representations (over $\mathbb{Q}(v,w)$) of the $n \le 6$ strand braid groups.

$\Rightarrow$ we can calculate the quantum exceptional polynomial for any link with "Conway width" at most 6.

$\Rightarrow$ in particular, all prime knots with ≤ 14 crossings and almost all prime links up to 12 crossing, except



repeatedly collapse digons (i.e. identify maximal rational tangles) then measure the width.

Example

Unconditionally, this specialises to give $U_q(\mathfrak{g})$ knot invariants, which were previously too hard to compute.

It appears that $L \in$ QExcept is of the form

$$d^k + \frac{[3][4][\lambda-6][\lambda+5]}{[1][2][\lambda-1]^{k+1}[\lambda]^{k+1}} \mathcal{E}(L)$$

where $\mathcal{E}(L) \in \mathbb{Z}[v^{\pm 1}, w^{\pm 1}]$.

<u>Questions</u>

- finite type?
- a tree-width FPT algorithm?
- a quantum Vogel plane?