Khovanov homology of rational tangles

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joint work with
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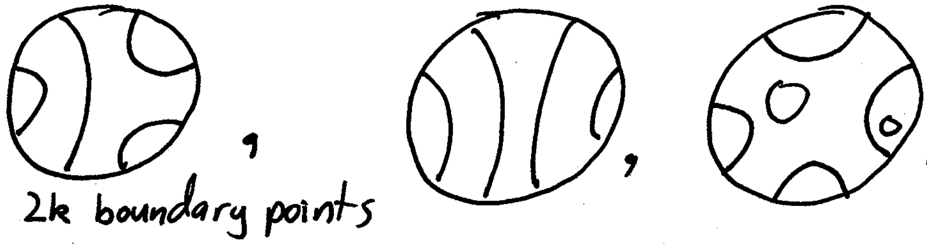
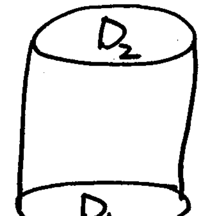
Khovanov homology for tangles

(following Bar-Natan)

- Bar-Natan defined a planar algebra of categories (aka 'a 'canopolis')

$\mathcal{B}_k = \{$ 

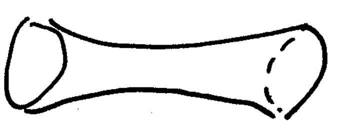
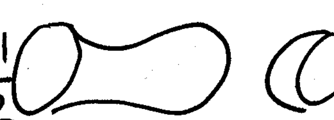
$, \dots \}$

$\mathrm{Hom}(D_1, D_2) = \mathbb{Z}[\tfrac{1}{2}]\{$ surfaces in 

modulo relations $\}$

Surface relations:

 $= 0$  $= 2$

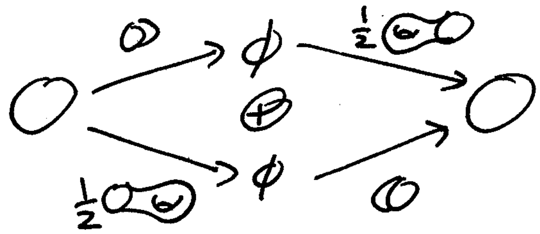
 $= \frac{1}{2}$  $+ \frac{1}{2}$ 

- The Khovanov invariant of a tangle is a complex in the category of matrixes over this category.

$$Kh(T \in \mathrm{Tangles}_k) \in \mathrm{Kom}(\mathrm{Mat}(\mathcal{B}_k)).$$

Isomorphisms and decategorification

In $\mathcal{B}$, there is an isomorphism $O \cong \phi \oplus \phi$



In fact $K_0(\mathcal{B}_k) \cong TL_k(\delta = 2)$.

(And by keeping track of gradings we can get $\delta = q + q^{-1}$.)

The structure of $\mathcal{B}_2$ and $\mathcal{B}_4$

Every object in $\mathcal{B}_2$ is isomorphic to a direct sum of copies of $)($.

$\mathrm{Hom}() \longrightarrow)) = \mathbb{Z}[\tfrac{1}{2}]\{\ \ , \ \ , \ \ , \dots\}$

$= \mathbb{Z}[\tfrac{1}{2}, \ \ ]\{\ \ , \ \ \}$

(since $\ = \frac{1}{2}\ \ + \frac{1}{2}\ \$

and $\ = 0$.)

Complexes in $\mathcal{B}_2$.

Over $\mathbb{Q}$, every complex in $\mathcal{B}_2$ decomposes uniquely into a direct sum of the indecomposable complexes:

$$E =)$$

$$C_{k\geq 1} =) \longrightarrow)$$

You can recover the s-invariant, and both the reduced and unreduced homologies from this decomposition.

There's an implementation (Morrison & Shumakovitch / Bar-Natan / Green)

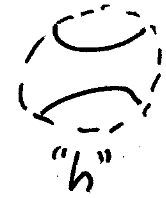
Mathematica

```
<< KnotTheory`
sInvariant[Knot[8,19]]
   6
UniversalKh[Knot[8,19]]
   q^6 E + q^12 t^3 C_1 + q^16 t^5 C_2
```

Conjecture

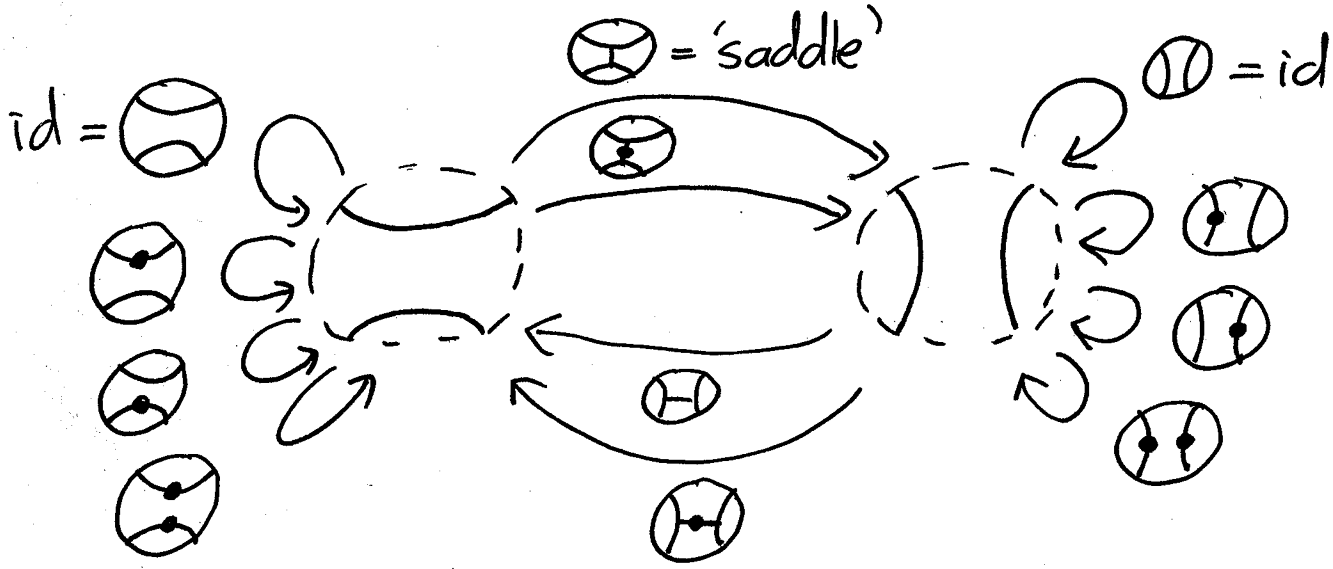
In the complex for a tangle T, only the summands E, C_1 & C_2 appear.

The structure of $\mathcal{B}_4$

Every object is a direct sum of 

& 

As $\mathbb{Z}[\frac{1}{2},$  $]$ modules, the morphisms are



<u>Question</u>: Can you describe the indecomposable complexes over $\mathcal{B}_4$?

<u>Question</u>: What are the chain maps between indecomposables? Equivalenty, how do tensor products

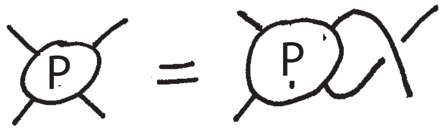
decompose?

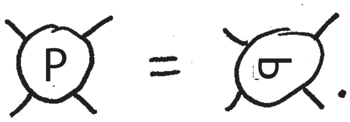
Rational tangles

PGL$(2,\mathbb{Z})$ acts on $\mathbb{Q}\cup\{\infty\}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \frac{av+b}{cv+d}$$

and also on tangles via

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ 

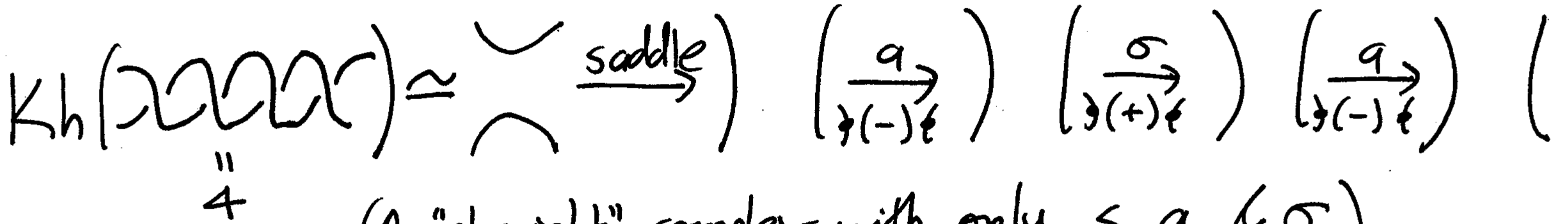
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 

Rational tangles are just the orbit PGL$(2,\mathbb{Z})\cdot$  and can be identified with $\mathbb{Q}\cup\{\infty\}$ via continued fractions. (Note $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v = v+1$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v = \frac{1}{v}$.)

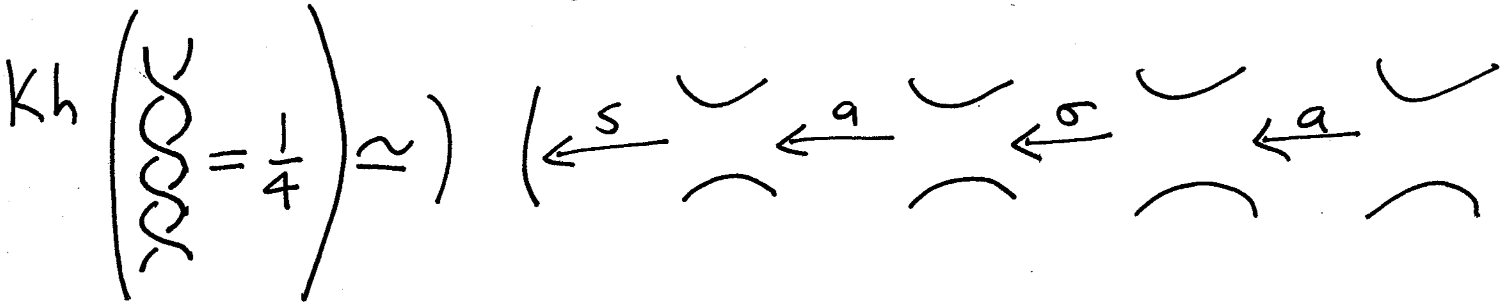
Theorem (Clark-Morrison-Walker)

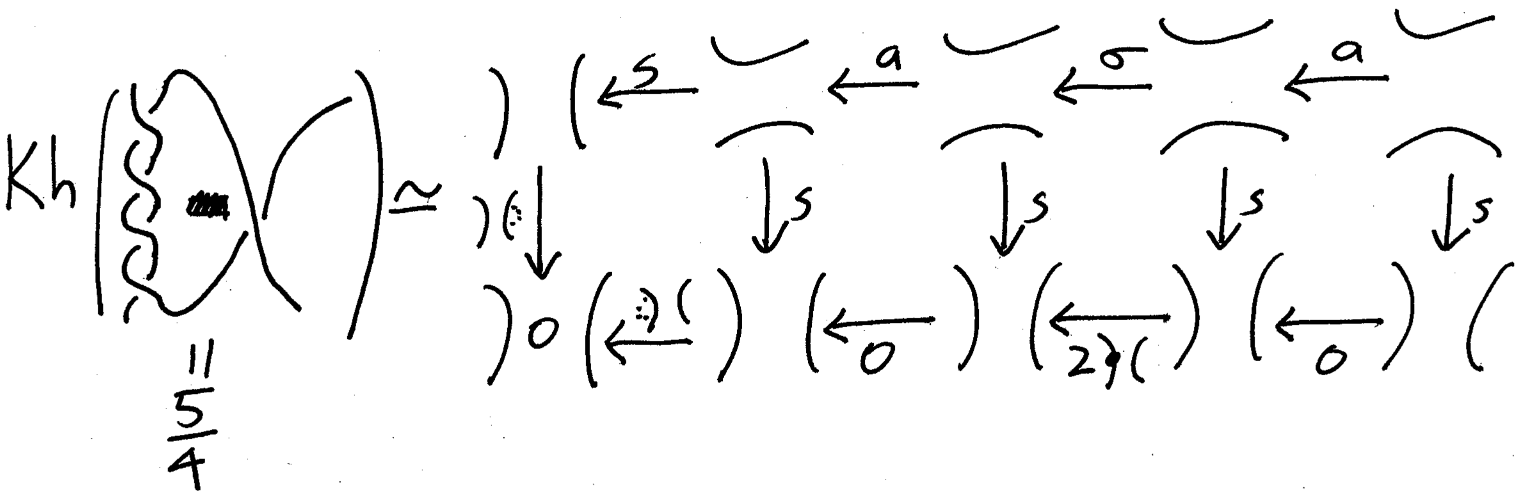
The Khovanov complex of a rational tangle has an up-to-homotopy representative which is a "snake" (see below), and we can efficiently describe the action of PGL$(2,\mathbb{Z})$ on snakes.

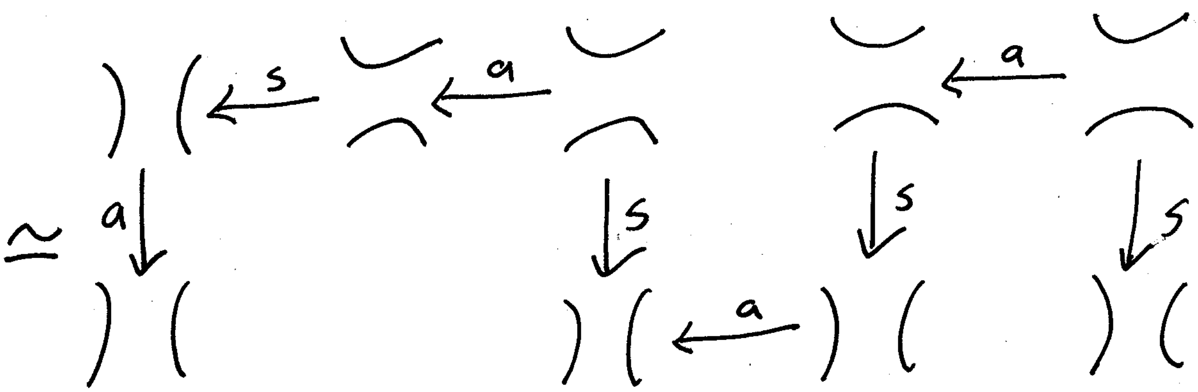
Examples



(A "straight" complex with only s, a & σ.)



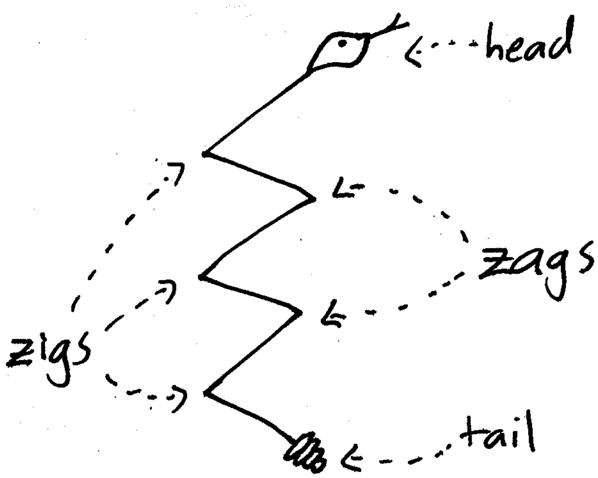




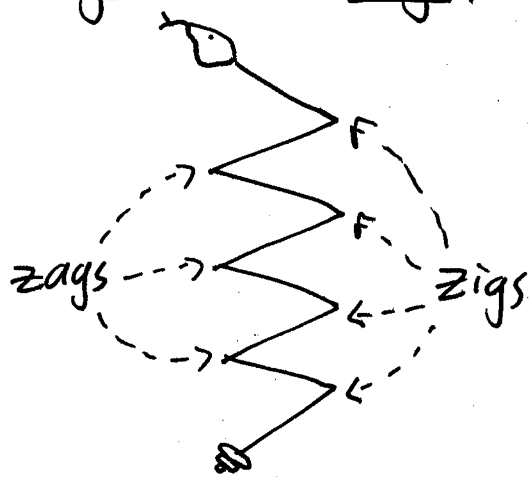
(a "kinky" complex with only s, a & σ.)

Snakes

- All snakes are "kinky" complexes; each summand has at most two differentials starting or ending at it
- All differentials are either s, a or σ.
- There are two types of snakes "facing left" or "facing right". Each consists of a head, a tail, and an alternating sequence of zigs and zags.

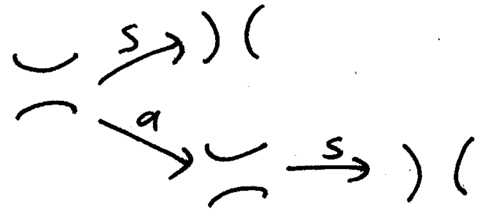


"facing right"

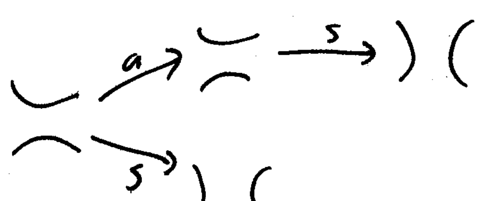


"facing left"

zigs $\in \{+, -\}$, + represents



− represents

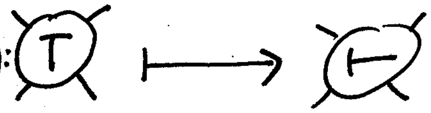


zags $\in \{(2n-1)^{\pm}\}$ eg. 5^+ represents $)(\xrightarrow{a})(\xrightarrow{\sigma})(\xrightarrow{a})($

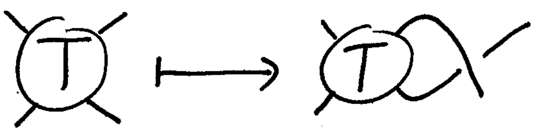
$)(\xrightarrow{a})(\nearrow^{\sigma}$

(for today, I'll ignore heads & tails)

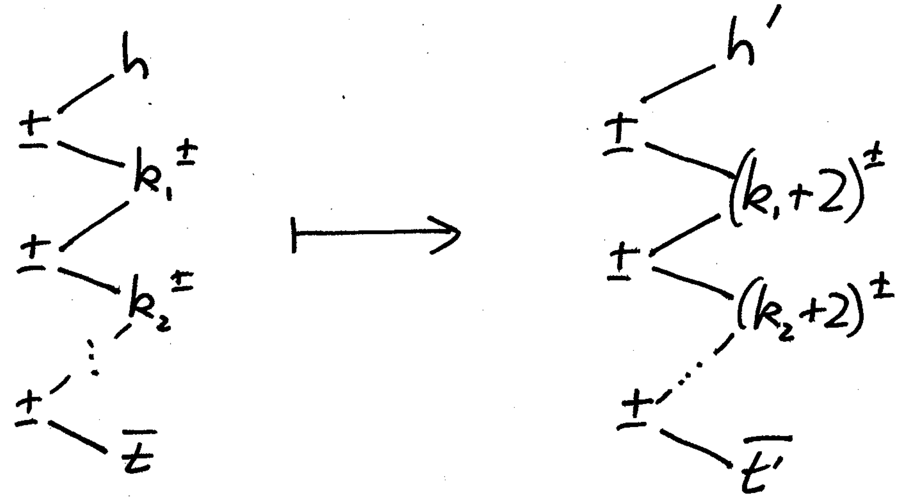
The action of PSL(2, Z) on snakes.

$\begin{pmatrix}0&1\\1&0\end{pmatrix}$: 

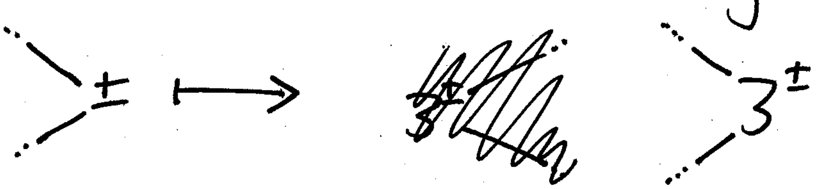
interchanges left-facing and right-facing snakes (and also $\pm \leftrightarrow \mp$).

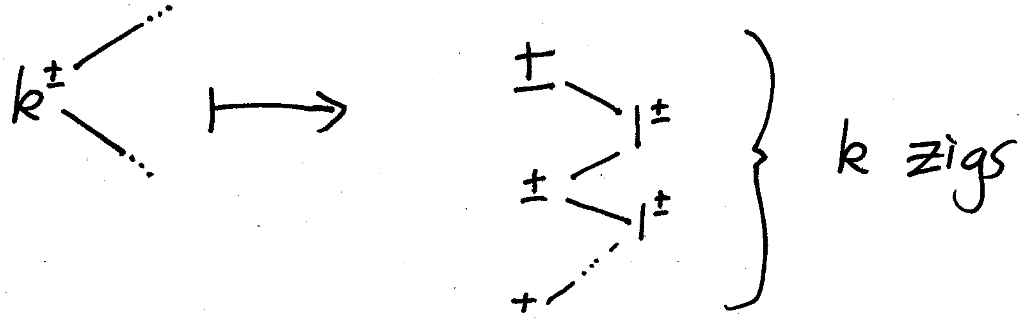
$\begin{pmatrix}1&1\\0&1\end{pmatrix}$: 

is easy to describe on right-facing snakes:



and a bit harder on left-facing snakes:





(again, some messy details for heads and tails omitted.)