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Khovanov homology of  
rational tangles

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# Khovanov homology for tangles

(following Bar-Natan)

- Bar-Natan defined a planar algebra of categories (aka 'a 'canopolis')

$$\mathcal{B}_k = \left\{ \begin{array}{c} \text{[Diagram 1]} \\ \text{[Diagram 2]} \\ \text{[Diagram 3]} \\ \dots \end{array} \right\}$$

2k boundary points



$$\text{Hom}(D_1, D_2) = \mathbb{Z} \left[ \frac{1}{2} \right] \left\{ \begin{array}{c} \text{surfaces in} \\ \text{[Cylinder Diagram]} \\ \text{modulo relations} \end{array} \right\}$$

Surface relations:

$$\text{[Cap Diagram]} = \text{[Circle Diagram]}$$

$$\text{[Cup Diagram]} = 2 \text{[Circle Diagram]}$$

$$\text{[Torus Diagram]} = \frac{1}{2} \text{[Torus Diagram]} + \frac{1}{2} \text{[Circle Diagram]} + \text{[Cup Diagram]}$$

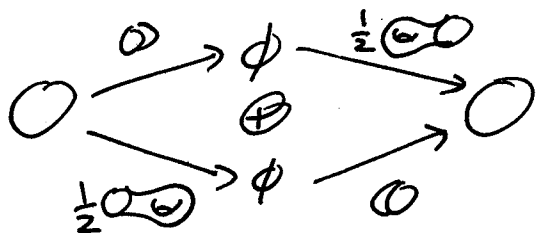
- The Khovanov invariant of a tangle is a complex in the category of matrixes over this category.

$$\text{Kh}(T \in \text{Tangles}_k) \in \text{Kom}(\text{Mat}(\mathcal{B}_k)).$$

# Isomorphisms and decategorification

(2)

In  $\mathcal{B}$ , there is an isomorphism  $0 \cong \phi \oplus \phi$



In fact  $K_0(\mathcal{B}_k) \cong TL_k(\delta=2)$ .

(And by keeping track of gradings we can get  $\delta = q + q^{-1}$ .)

## The structure of $\mathcal{B}_2$ and $\mathcal{B}_4$

Every object in  $\mathcal{B}_2$  is isomorphic to a direct sum of copies of  $(\cdot)$ .

$$\text{Hom}(\cdot \rightarrow \cdot) = \mathbb{Z}[\frac{1}{2}] \left\{ \square, \square \begin{array}{c} \oplus \\ \ominus \end{array}, \begin{array}{c} \square \\ \oplus \\ \square \end{array}, \dots \right\}$$

$$= \mathbb{Z}[\frac{1}{2}, \text{two circles}] \left\{ \square, \square \begin{array}{c} \oplus \\ \ominus \end{array} \right\}$$

(since  $\begin{array}{c} \square \\ \oplus \\ \square \end{array} = \frac{1}{2} \square \text{two circles} + \frac{1}{2} \square \begin{array}{c} \oplus \\ \ominus \end{array}$   
and  $\text{two circles} = 0$ .)

## Complexes in $B_2$ .

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Over  $\mathbb{Q}$ , every complex in  $B_2$  decomposes uniquely into a direct sum of the indecomposable complexes:

$$E = )$$
$$C_{k \geq 1} = ) \xrightarrow{\text{Diagram}} )$$

The diagram shows a square with a circle inside, and a horizontal arrow pointing to the right from the circle.

You can recover the  $s$ -invariant, and both the reduced and unreduced homologies from this decomposition.

There's an implementation (Morrison & Shumakovitch / Bar-Natan / Green)

```
Mathematica
<< KnotTheory`
sInvariant[Knot[8,19]]
6
UniversalKh[Knot[8,19]]
q^6 E + q^12 t^3 C_1 + q^16 t^5 C_2
```

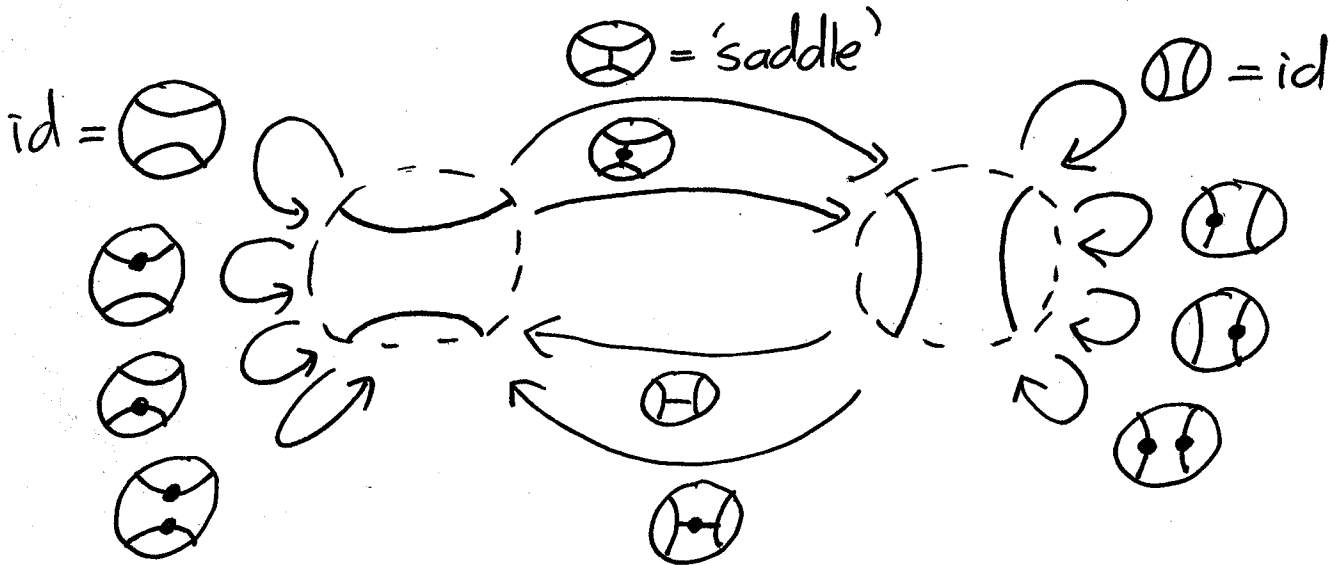
## Conjecture

In the complex for a tangle  $\textcircled{T}$ , only the summands  $E$ ,  $C_1$  &  $C_2$  appear.

# The structure of $B_4$

Every object is a direct sum of  & 

As  $\mathbb{Z}[\frac{1}{2}, \text{circle with two arcs}]$  modules, the morphisms are



Question: Can you describe the indecomposable complexes over  $B_4$ ?

Question: What are the chain maps between indecomposables?  
Equivalently, how do tensor products



decompose?

# Rational tangles

(5)

$\text{PGL}(2, \mathbb{Z})$  acts on  $\mathbb{Q} \cup \{\infty\}$  via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \frac{av+b}{cv+d}$$

and also on tangles via

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \textcircled{P} = \textcircled{P} \text{ with a crossing on the right}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \textcircled{P} = \textcircled{P} \text{ with a crossing on the top}$$

Rational tangles are just the orbit  $\text{PGL}(2, \mathbb{Z}) \cdot \frown$  and can be identified with  $\mathbb{Q} \cup \{\infty\}$  via continued fractions. (Note  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v = v+1$ ,  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v = \frac{1}{v}$ .)

## Theorem (Clark-Morrison-Walker)

The Khovanov complex of a rational tangle has an up-to-homotopy representative which is a "snake" (see below), and we can efficiently describe the action of  $\text{PGL}(2, \mathbb{Z})$  on snakes.

# Examples

$$Kh(\underbrace{\text{[Diagram of a straight complex with 4 saddles]}_4) \approx \text{[Diagram of a saddle]} \xrightarrow{\text{saddle}} \left( \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{(-) \sigma} \end{array} \right) \left( \begin{array}{c} \xrightarrow{\sigma} \\ \xrightarrow{(+)\sigma} \end{array} \right) \left( \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{(-)\sigma} \end{array} \right) \left( \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{(-)\sigma} \end{array} \right) \left( \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{(-)\sigma} \end{array} \right)$$

(A "straight" complex with only s, a &  $\sigma$ .)

$$Kh\left(\underbrace{\text{[Diagram of a complex with 4 saddles]}_{= \frac{1}{4}}\right) \approx \left( \begin{array}{c} \xleftarrow{s} \\ \xleftarrow{a} \end{array} \right) \left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{\sigma} \end{array} \right) \left( \begin{array}{c} \xleftarrow{\sigma} \\ \xleftarrow{a} \end{array} \right) \left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{\sigma} \end{array} \right) \left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{\sigma} \end{array} \right)$$

$$Kh\left(\underbrace{\text{[Diagram of a complex with 5 saddles]}_{= \frac{5}{4}}\right) \approx \left( \begin{array}{c} \xleftarrow{s} \\ \xleftarrow{a} \end{array} \right) \left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{\sigma} \end{array} \right) \left( \begin{array}{c} \xleftarrow{\sigma} \\ \xleftarrow{a} \end{array} \right) \left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{\sigma} \end{array} \right) \left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{\sigma} \end{array} \right)$$

$$\downarrow \quad \downarrow s \quad \downarrow s \quad \downarrow s \quad \downarrow s$$

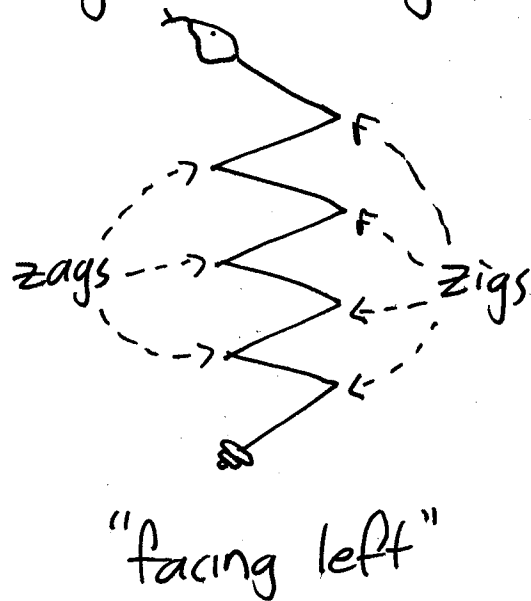
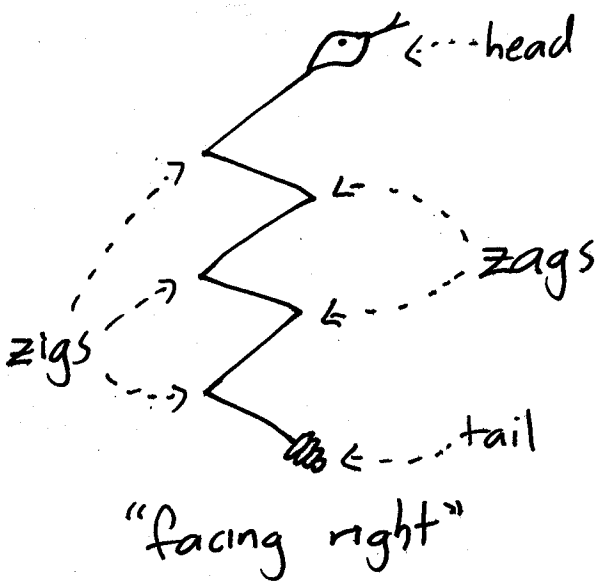
$$\left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{0} \end{array} \right) \left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{0} \end{array} \right) \left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{2\sigma} \end{array} \right) \left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{0} \end{array} \right) \left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{0} \end{array} \right)$$

$$\approx \left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{0} \end{array} \right) \left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{0} \end{array} \right) \left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{2\sigma} \end{array} \right) \left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{0} \end{array} \right) \left( \begin{array}{c} \xleftarrow{a} \\ \xleftarrow{0} \end{array} \right)$$

(a "kinky" complex with only s, a &  $\sigma$ .)

# Snakes

- All snakes are "kinky" complexes; each summand has at most two differentials starting or ending at it
- All differentials are either  $s$ ,  $a$  or  $\sigma$ .
- There are two types of snakes "facing left" or "facing right". Each consists of a head, a tail, and an alternating sequence of zigs and zags.



zigs  $\in \{+, -\}$ , + represents  $\begin{matrix} \curvearrowright \xrightarrow{s} \end{matrix} \begin{matrix} \curvearrowright \xrightarrow{a} \end{matrix} \begin{matrix} \curvearrowright \xrightarrow{s} \end{matrix} \begin{matrix} \curvearrowright \end{matrix}$

- represents  $\begin{matrix} \curvearrowright \xrightarrow{a} \end{matrix} \begin{matrix} \curvearrowright \xrightarrow{s} \end{matrix} \begin{matrix} \curvearrowright \xrightarrow{a} \end{matrix} \begin{matrix} \curvearrowright \xrightarrow{s} \end{matrix} \begin{matrix} \curvearrowright \end{matrix}$

zags  $\in \{(2n-1)^\pm\}$  eg.  $5^+$  represents  $\begin{matrix} \curvearrowright \xrightarrow{a} \end{matrix} \begin{matrix} \curvearrowright \xrightarrow{\sigma} \end{matrix} \begin{matrix} \curvearrowright \xrightarrow{a} \end{matrix} \begin{matrix} \curvearrowright \xrightarrow{\sigma} \end{matrix} \begin{matrix} \curvearrowright \xrightarrow{a} \end{matrix} \begin{matrix} \curvearrowright \xrightarrow{\sigma} \end{matrix} \begin{matrix} \curvearrowright \end{matrix}$

(for today, I'll ignore heads & tails)



# The action of $PSL(2, \mathbb{Z})$ on snakes.

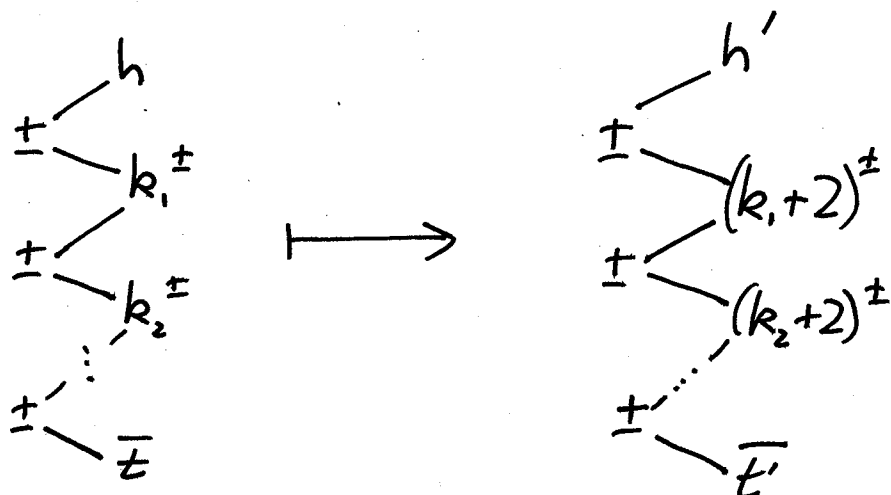
(8)

$$(0 \ 1) : \textcircled{T} \longmapsto \textcircled{\bar{T}}$$

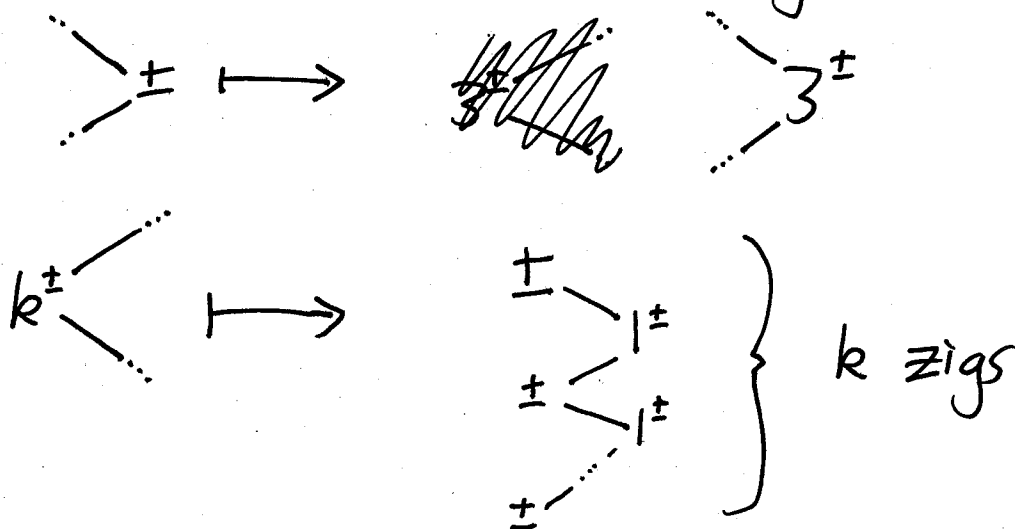
interchanges left-facing and right-facing snakes  
(and also  $\pm \leftrightarrow \bar{\pm}$ ).

$$(1 \ 0) : \textcircled{T} \longmapsto \textcircled{T}$$

is easy to describe on right-facing snakes:



and a bit harder on left-facing snakes:



(again, some messy details for heads  
and tails omitted.)