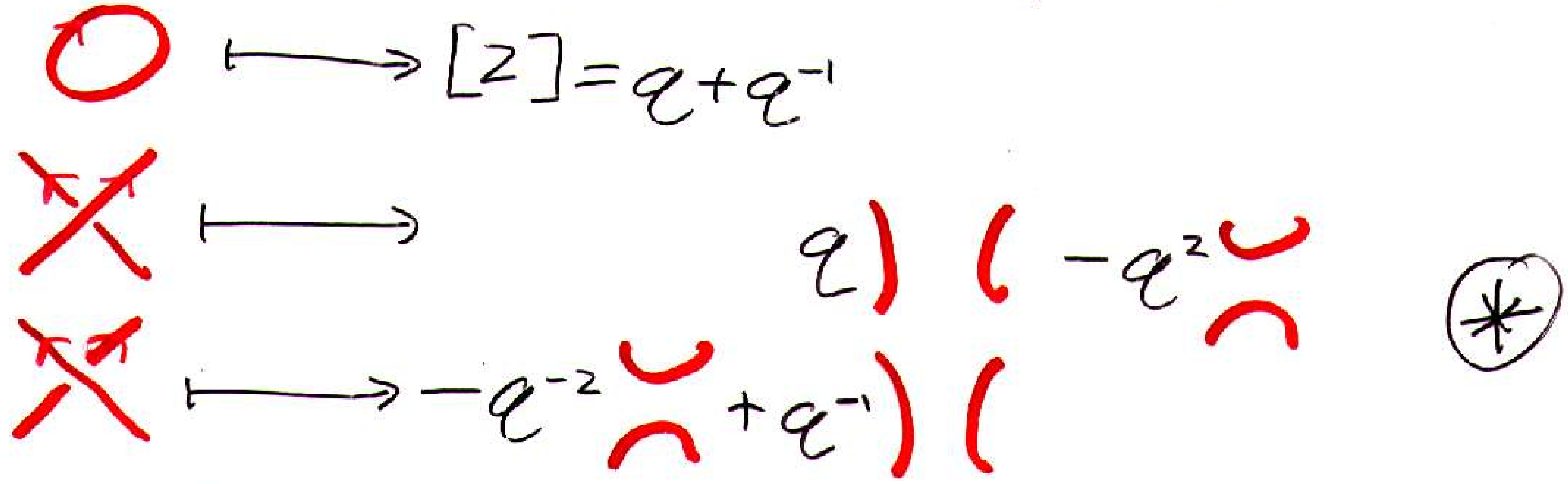
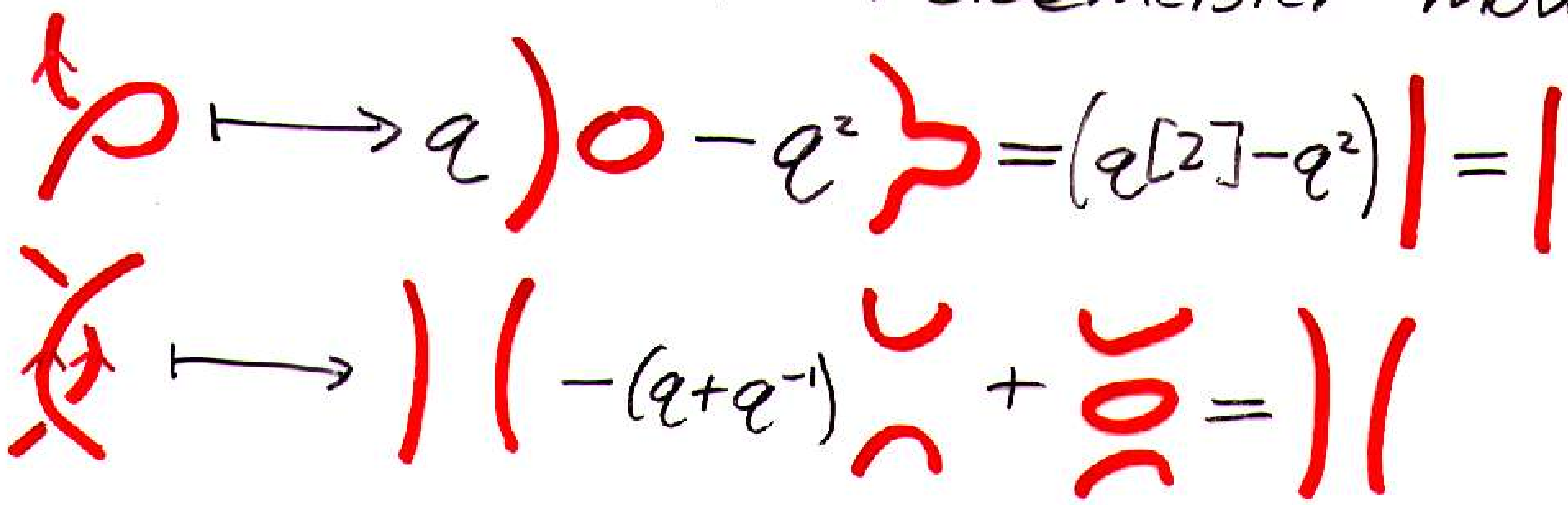
The Jones Polynomial

(a map from the planar algebra of tangles to the $\mathbb{Z}[q,q^{-1}]$ - Temperley-Lieb planar algebra.)



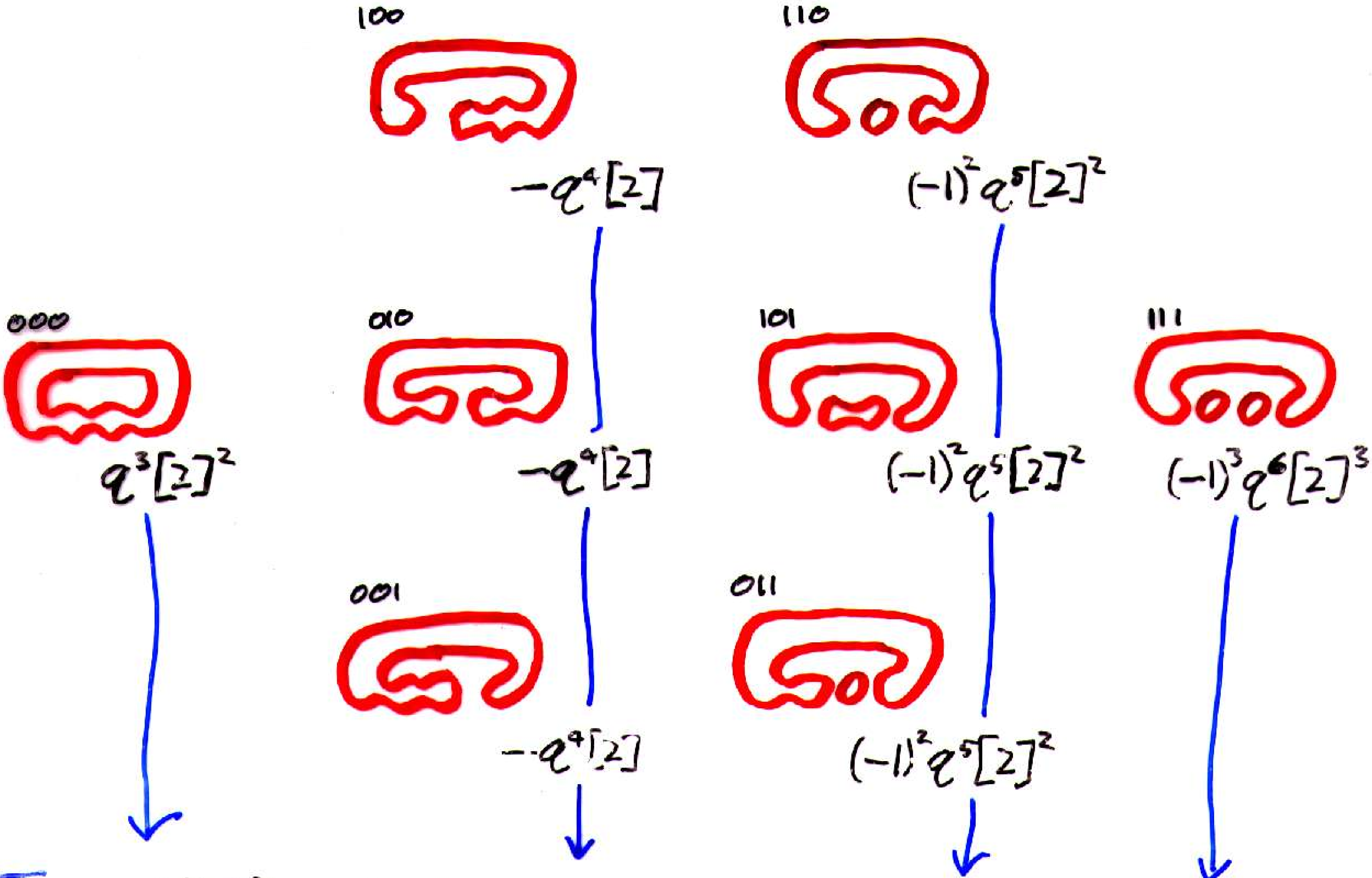
I.. riance under the first two Reidemeister moves—



And a miracle occurs—

as long as we choose the coefficients in * so we have invariance under R1 and R2, invariance under R3 comes for free!

$J\Big($  $\Big)$



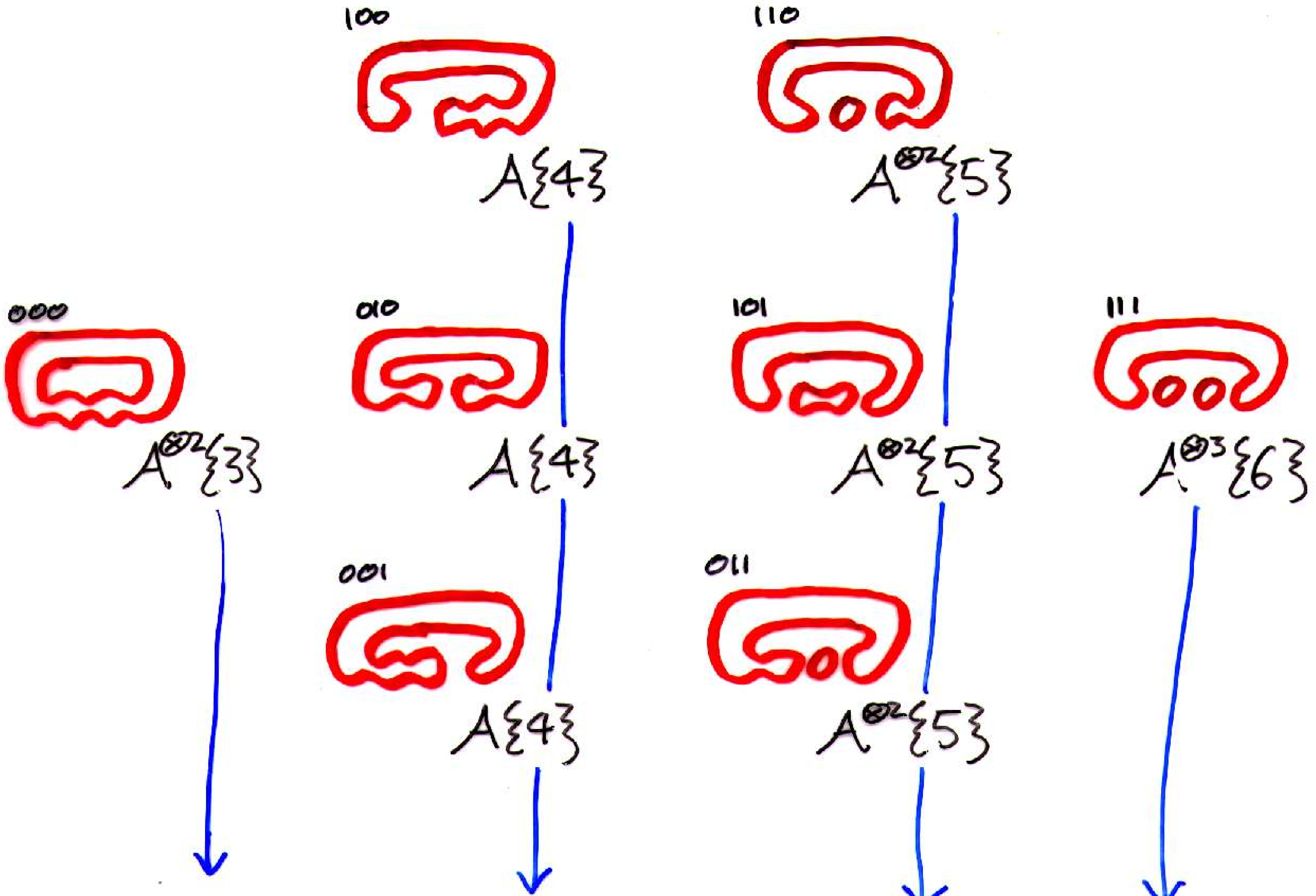
$$J = q^3[2]^2 - (q^4[2]+q^4[2]+q^4[2]) + (3q^5[2]^2) - q^6[2]^3$$
$$= q + q^3 + q^5 - q^9$$

Next, introduce a <u>graded</u> <u>module</u> $A = R\{1\} \oplus R\{-1\}$ (where $\{n\}$ denotes a grading shift)

Then $q\dim A = q + q^{-1} = [2]$,

and, for example, $q\dim A^{\otimes 2}\{3\} = q^3[2]^2$

$J\Bigg($  $\Bigg)$



$C^0 = A^{\otimes 2}\{3\}$ $\quad C^1 = \bigoplus_3 A\{4\}$ $\quad C^2 = \bigoplus_3 A^{\otimes 2}\{5\}$ $\quad C^3 = A^{\otimes 3}\{6$

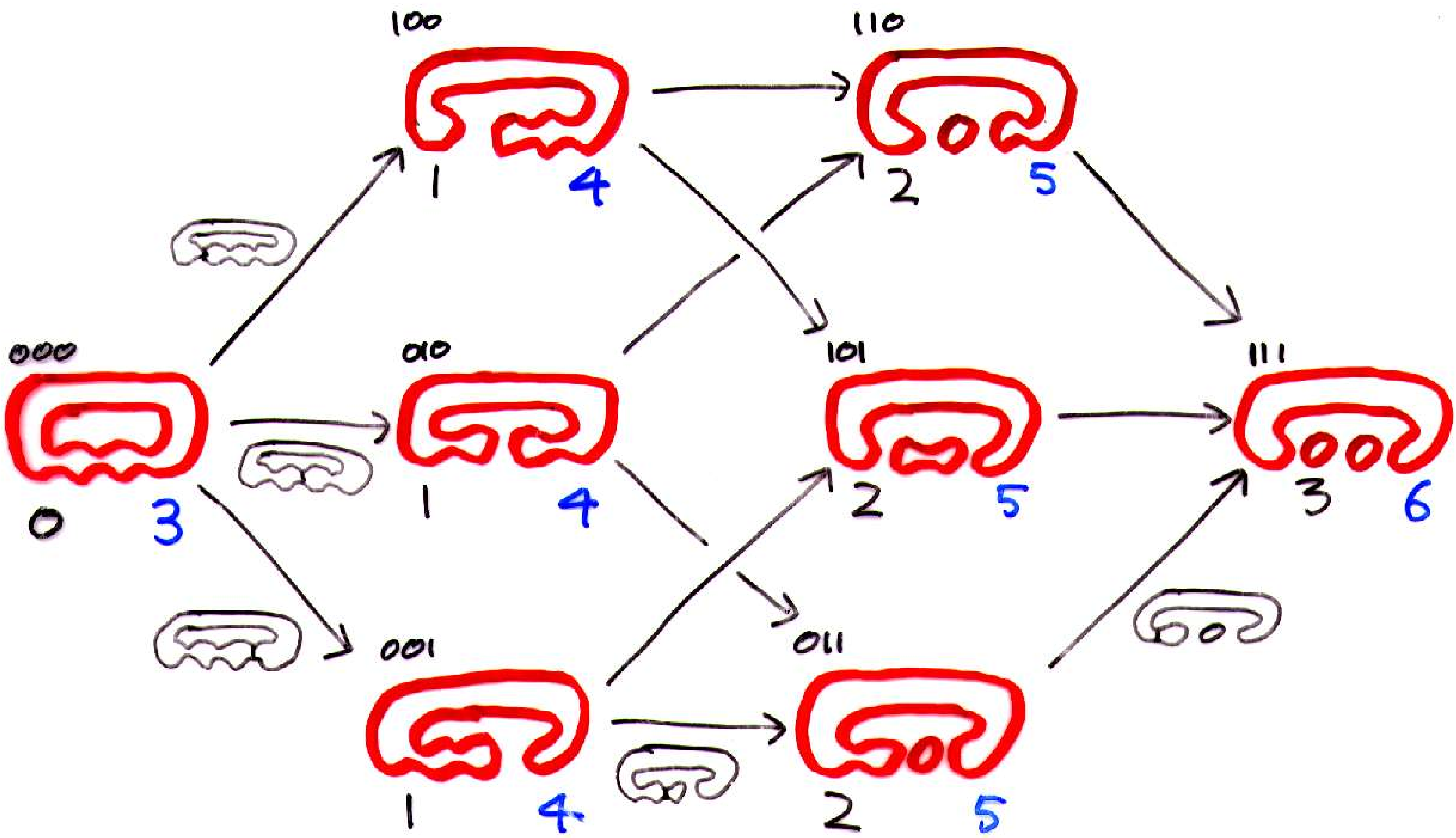
Then $J = \chi_q(C^*)$

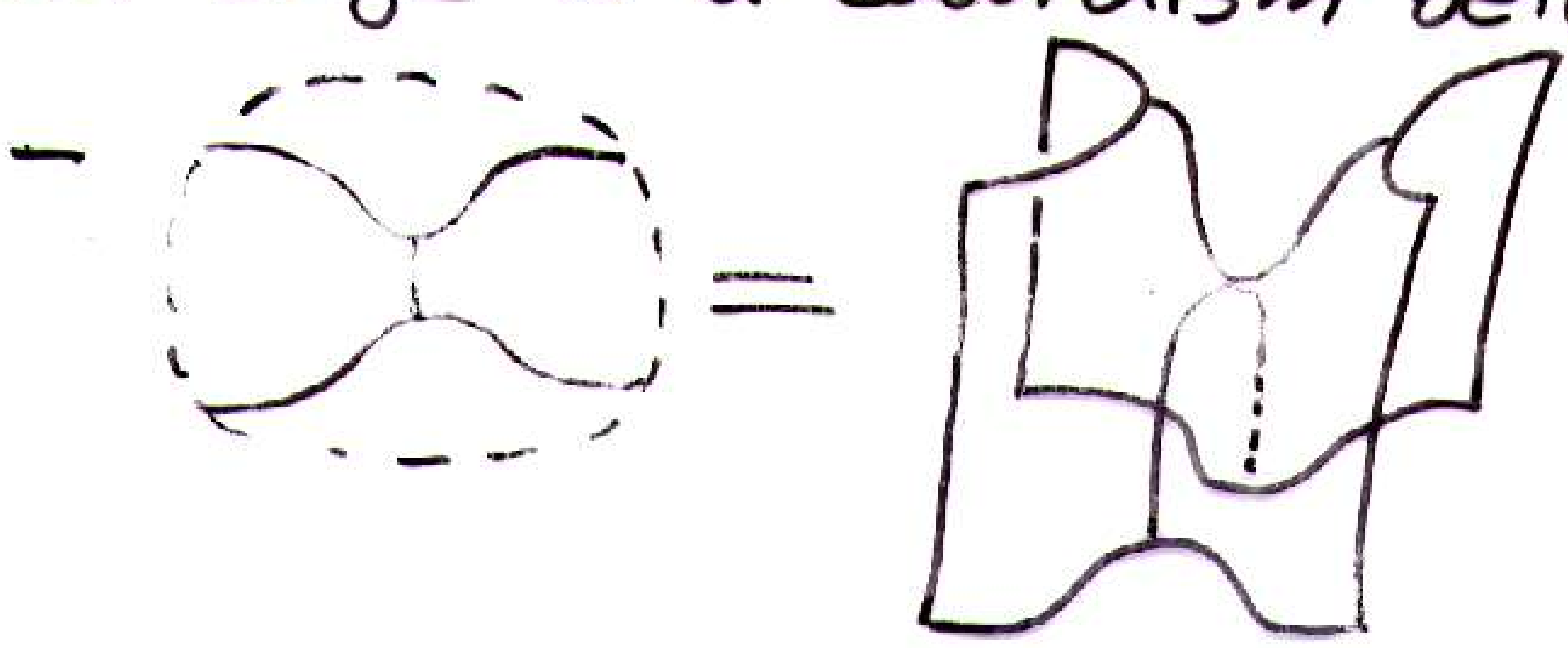
$$= q\dim C^0 - q\dim C^1 + q\dim C^2 - q\dim C^3$$

Writing J as an Euler characteristic gave the original motivation for Khovanov homology.

Kh( right handed trefoil)

Step 1: build the commutative cube



- This picture is a <u>commutative cube</u> in the category of 1+1 cobordisms
  - each vertex is a 1-manifold 
    - decorated with a degree and a grading
  - each edge is a cobordism between 1-manifolds
  - 

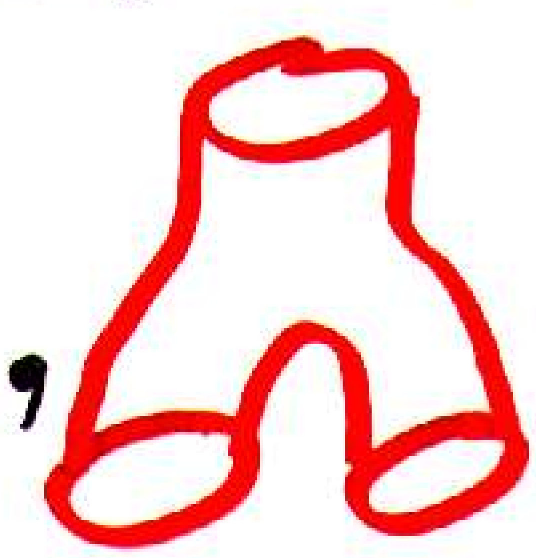
Kh() Step 2: Apply a TQFT

$$
\begin{tikzcd}
 & C_{100}\{4\} \arrow[r] \arrow[dr] & C_{110}\{5\} \arrow[dr] & \\
C_{000}\{3\} \arrow[ur] \arrow[r] \arrow[dr] & C_{010}\{4\} \arrow[ur] \arrow[dr] & C_{101}\{5\} \arrow[r] & C_{111}\{6\} \\
 & C_{001}\{4\} \arrow[ur] \arrow[r] & C_{011}\{5\} \arrow[ur] &
\end{tikzcd}
$$

- This is a commutative cube in the category of graded R-modules.
  - A TQFT takes 1-manifold to R-modules

$$\underbrace{\bigcirc\ \bigcirc \cdots \bigcirc}_{k} \longmapsto A^{\otimes k}$$

  - and cobordisms to maps between them

 $\longmapsto \mathrm{id}_A$,  $\longmapsto (A\otimes A \xrightarrow{m} A)$, etc

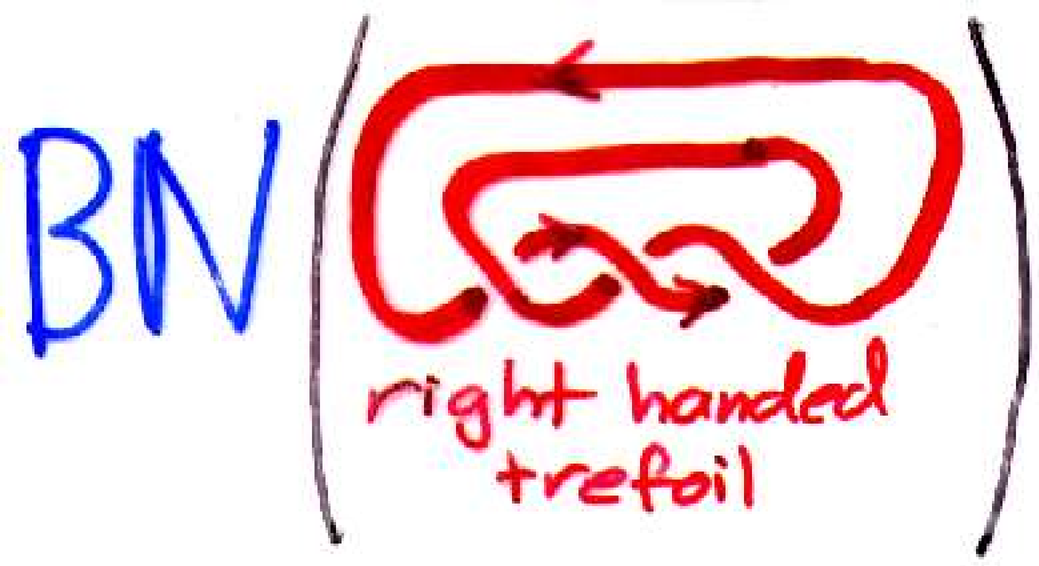
Kh()

Step 3: Sprinkle signs and collapse to a complex.

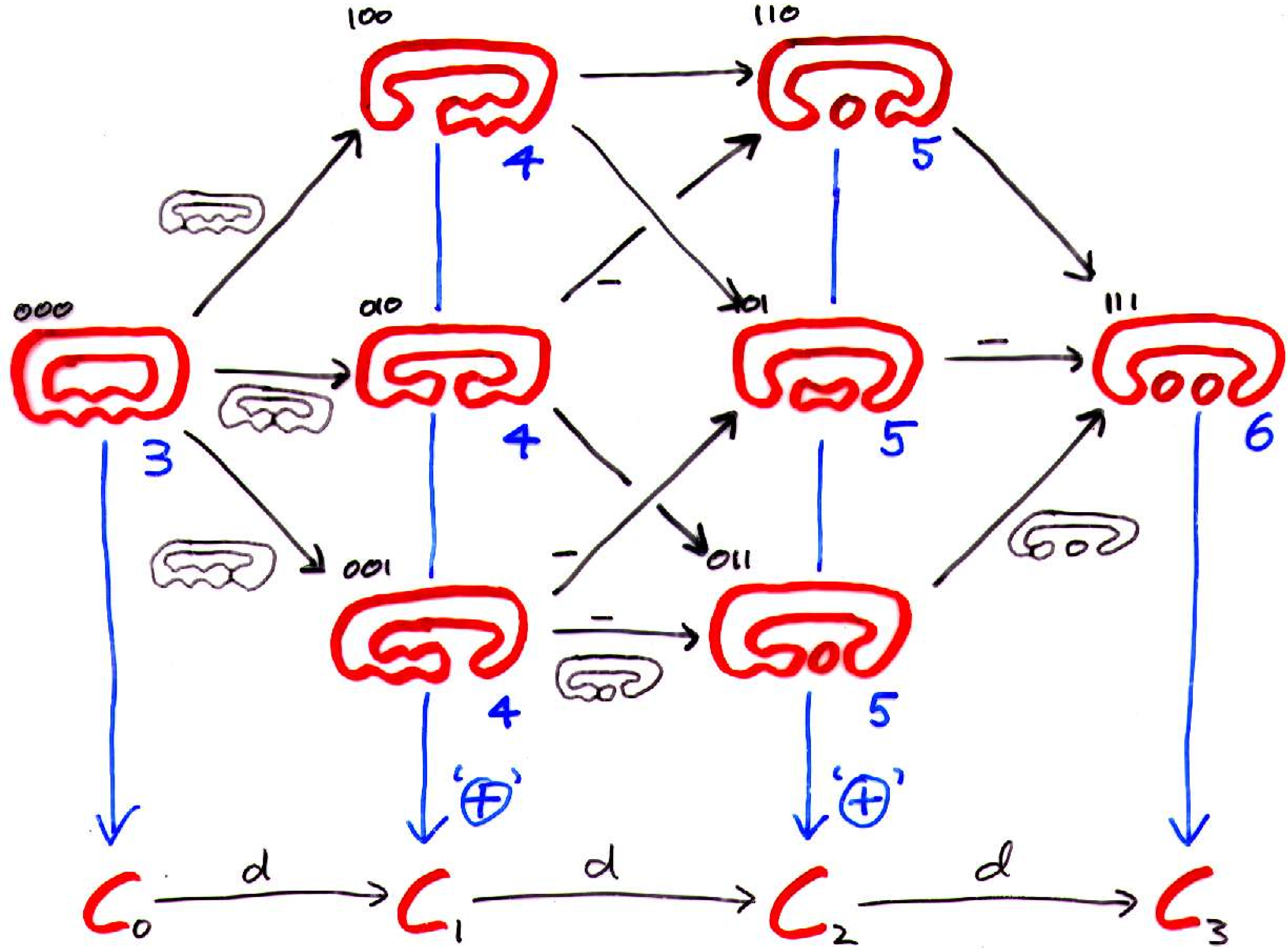
$$
\begin{tikzcd}
 & C_{100}\{4\} \arrow[r, "+"] \arrow[dr, "+"] \arrow[d, dash] & C_{110}\{5\} \arrow[dr, "+"] \arrow[d, dash] & \\
C_{000}\{3\} \arrow[ur, "+"] \arrow[r, "+"] \arrow[dr, "+"] \arrow[ddd] & C_{010}\{4\} \arrow[ur, "-"] \arrow[dr, "+"] \arrow[d, dash] & C_{101}\{5\} \arrow[r, "-"] \arrow[d, dash] & C_{111}\{6\} \arrow[ddd] \\
 & C_{001}\{4\} \arrow[ur, "-"] \arrow[r, "-"] \arrow[dd, "\oplus"] & C_{011}\{5\} \arrow[ur, "+"] \arrow[dd, "\oplus"] & \\
 & & & \\
C_0 \arrow[r, "d"] & C_1 \arrow[r, "d"] & C_2 \arrow[r, "d"] & C_3
\end{tikzcd}
$$

- This is a <u>complex</u> in the category of R-modules
  - $d^2=0$ since every square <u>anticommutes</u>

Theorem (Khovanov) The homology of this complex, Kh, is a knot invariant.



Forget the TQFT, sprinkle signs, and collapse to a complex using 'formal' direct sums.



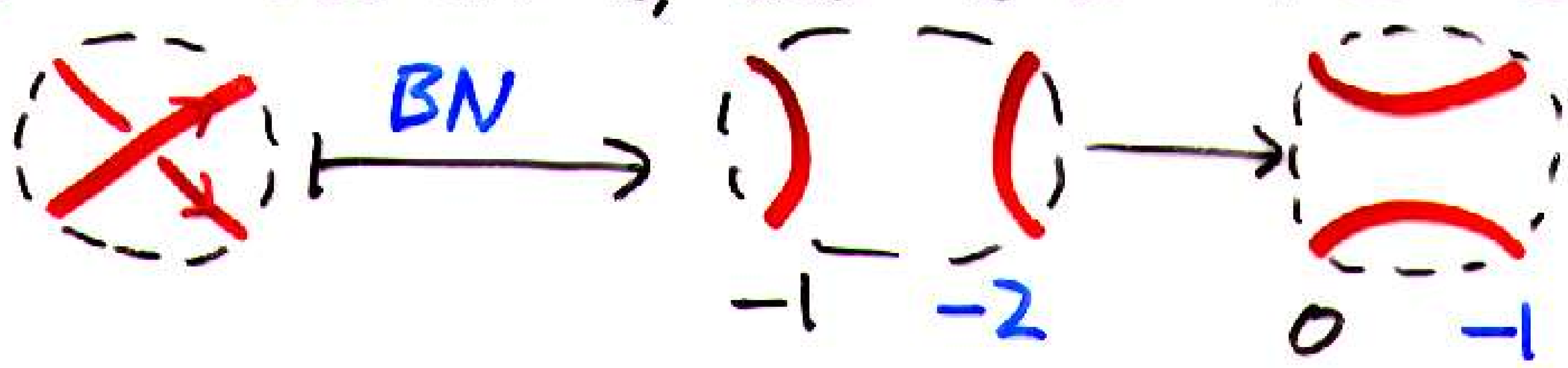
- This is a complex in the category of matrices of R-linear combinations of cobordisms.

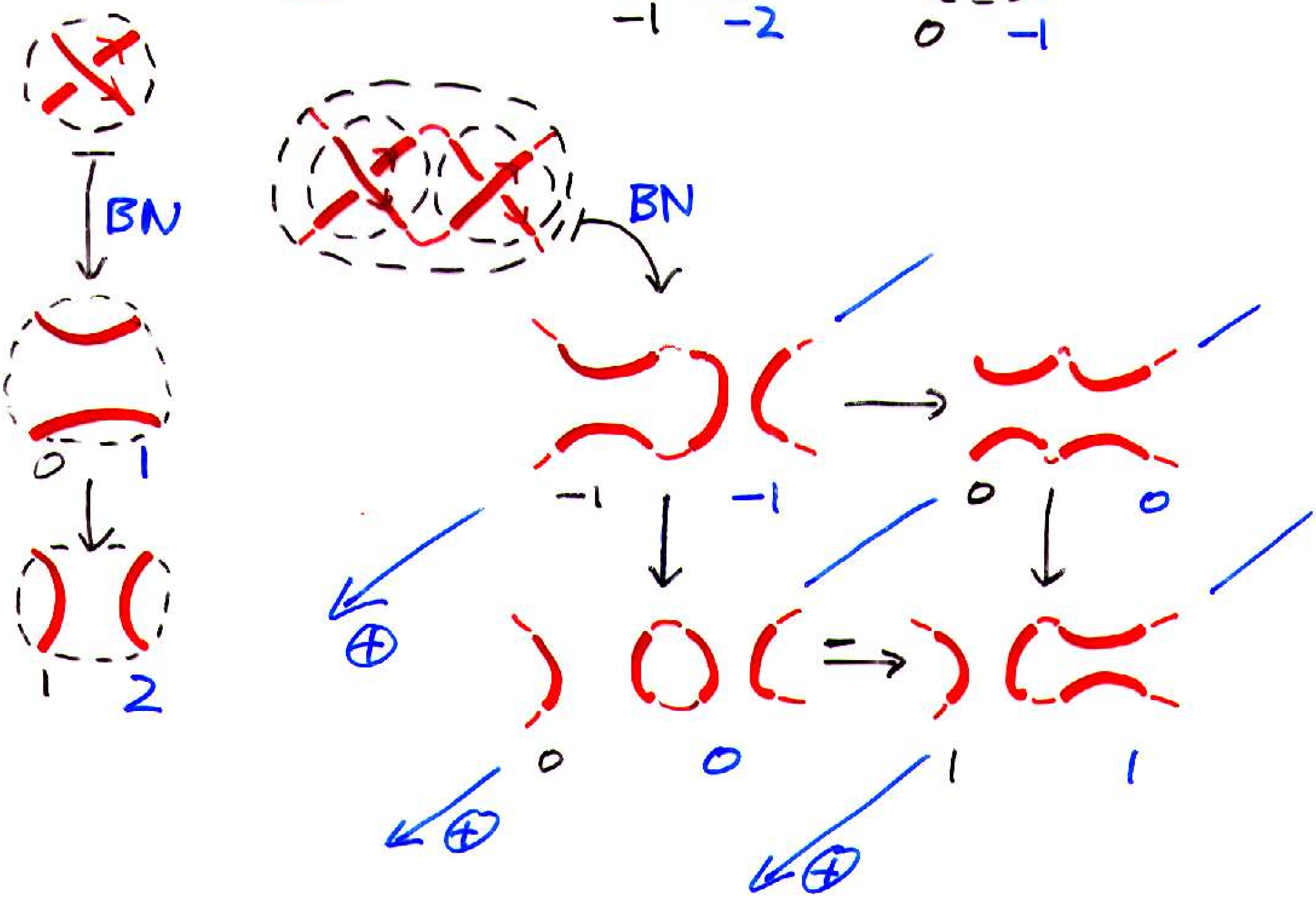
<u>Q</u>: Might the homotopy type of this complex be a knot invariant?

Bar-Natan's theory for tangles

- everything so far makes sense for tangles, as well a links.

- the theory is a map of planar algebras; to compose complexes of cobordisms, join up the cobordisms, and tensor the complexes.





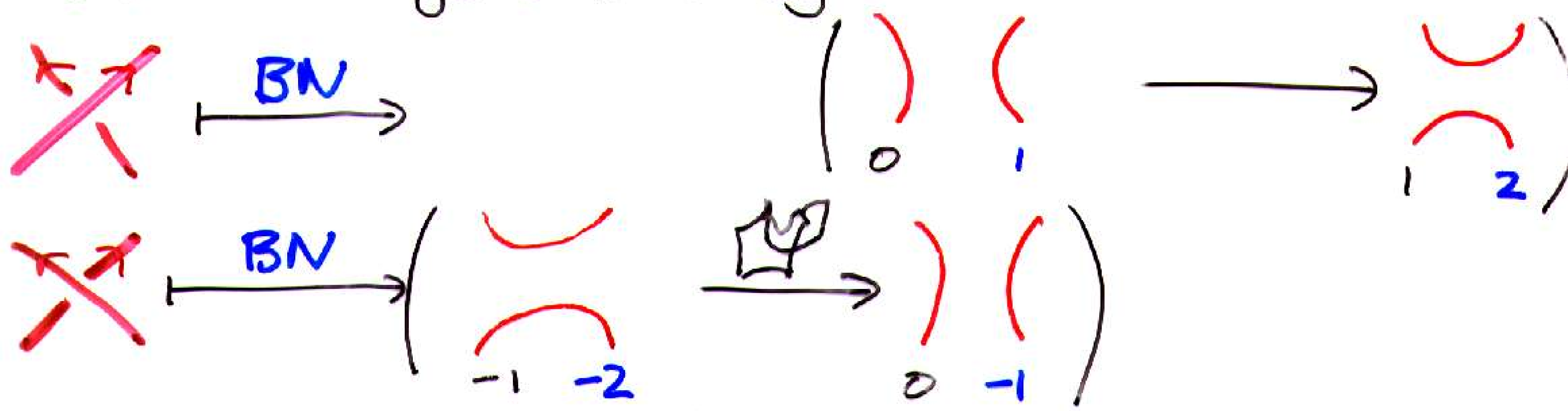
The story so far...

Bar-Natan's theory is a morphism of planar algebras

$$\text{Tangle-Diagrams} \xrightarrow{BN} \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}))$$

additive and graded!

defined on generators by



Kom(Mat(Cob)) is a planar algebra; take tensor products of complexes and matrices, and use the usual planar algebra operations on cobordisms.

We now need to show this descends to

$$\text{Tangles} \cong \text{Tangle-Diagrams}/\text{Reidemeister moves} \xrightarrow{BN} \mathrm{Kom}_{/\mathrm{htpy}}(\mathrm{Mat}(\mathrm{Cob}_{/\sim}))$$

Quick reminder on chain homotopies.

A map between complexes $F: C^* \to D^*$ must satisfy the chain map condition: $dF = Fd$

$$\begin{tikzcd}
\cdots \arrow[r] & C^\bullet \arrow[r, "d"] \arrow[d, "F^\bullet"] & C^{\bullet+1} \arrow[r, "d"] \arrow[d, "F^{\bullet+1}"] & C^{\bullet+2} \arrow[d, "F^{\bullet+2}"] \arrow[r, dash] & \cdots \\
\cdots \arrow[r] & D^\bullet \arrow[r, "d"] & D^{\bullet+1} \arrow[r, "d"] & D^{\bullet+2} \arrow[r, dash] & \cdots
\end{tikzcd}$$

Two chain maps $F, G: C^* \to D^*$ are homotopic if there is a

so $F - G = hd + dh$

$$\begin{tikzcd}
\cdots \arrow[r] & C^\bullet \arrow[r, "d"] \arrow[d, shift right, "F"'] \arrow[d, shift left, "G"] & C^{\bullet+1} \arrow[r, "d"] \arrow[d, shift right, "F"'] \arrow[d, shift left, "G"] \arrow[dl, "h"'] & C^{\bullet+2} \arrow[d, shift right, "F"'] \arrow[d, shift left, "G"] \arrow[dl, "h"'] \arrow[r, dash] & \cdots \\
\cdots \arrow[r] & D^\bullet \arrow[r, "d"] & D^{\bullet+1} \arrow[r, "d"] & D^{\bullet+2} \arrow[r, dash] & \cdots
\end{tikzcd}$$

and two complexes are homotopic if there are chain maps $F: C^* \to D^*$ and $G: D^* \to C^*$ so $F \circ G$ and $G \circ F$ are each homotopic to the identity.

Gradings

What is a graded category?

- For objects A, B, $\mathrm{Hom}(A,B)$ must be a <u>graded</u> abelian group.
  (Recall it must be an abelian group anyway, because our categories are all additive.)
- There is a $\mathbb{Z}$-action on objects, called 'grading shift' $(A, m) \longmapsto A\{m\}$

Morphisms don't change: $\mathrm{Hom}(A\{m\}, B\{n\}) = \mathrm{Hom}(A,B)$
(as abelian groups!)

but gradings do:
if $f \in \mathrm{Hom}(A,B)$ has degree d then
$f \in \mathrm{Hom}(A\{m\}, B\{n\})$ has degree $d+n-m$.

If the morphisms are graded, but there's no grading shift defined, we can introduce it by adding 'fake' objects $A\{m\}$.

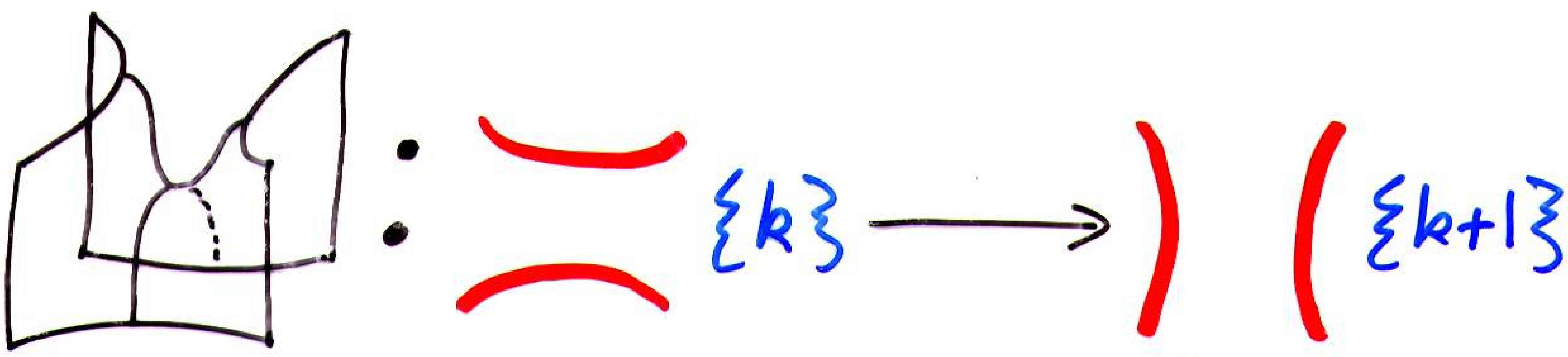
<u>Example</u> Graded R-modules
if $\mathrm{qdim}\, V = s(q)$, then $\mathrm{qdim}\, V\{m\} = q^m s(q)$

Complexes in graded categories should have differentials of degree zero.
(so we can hope for graded homology theories?)

Gradings

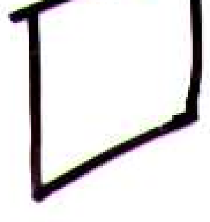
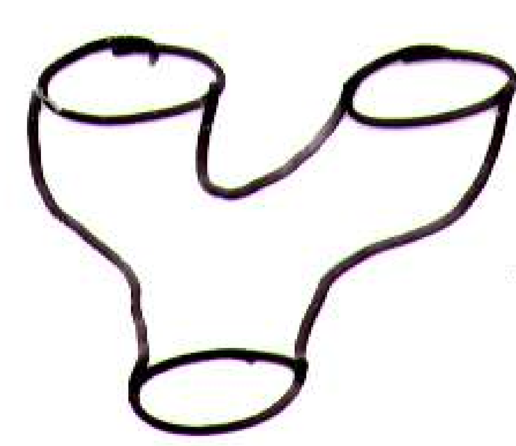
The differentials in this complex should be <u>homogeneous</u> (also the chain maps and homotopies).

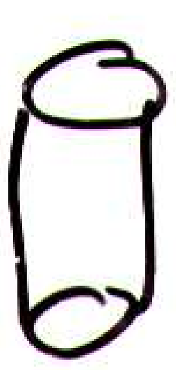
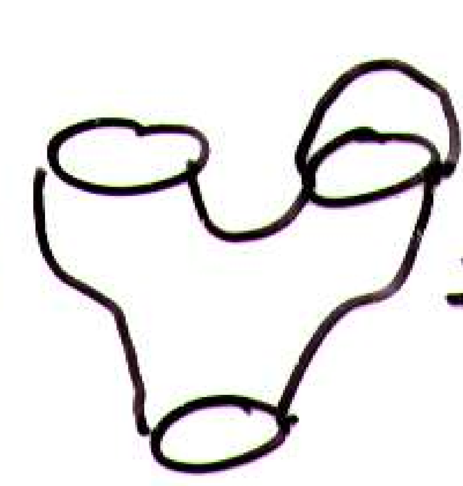
The differentials are all built out of saddle cobordisms



'so' the unshifted saddle  must have degree -1.

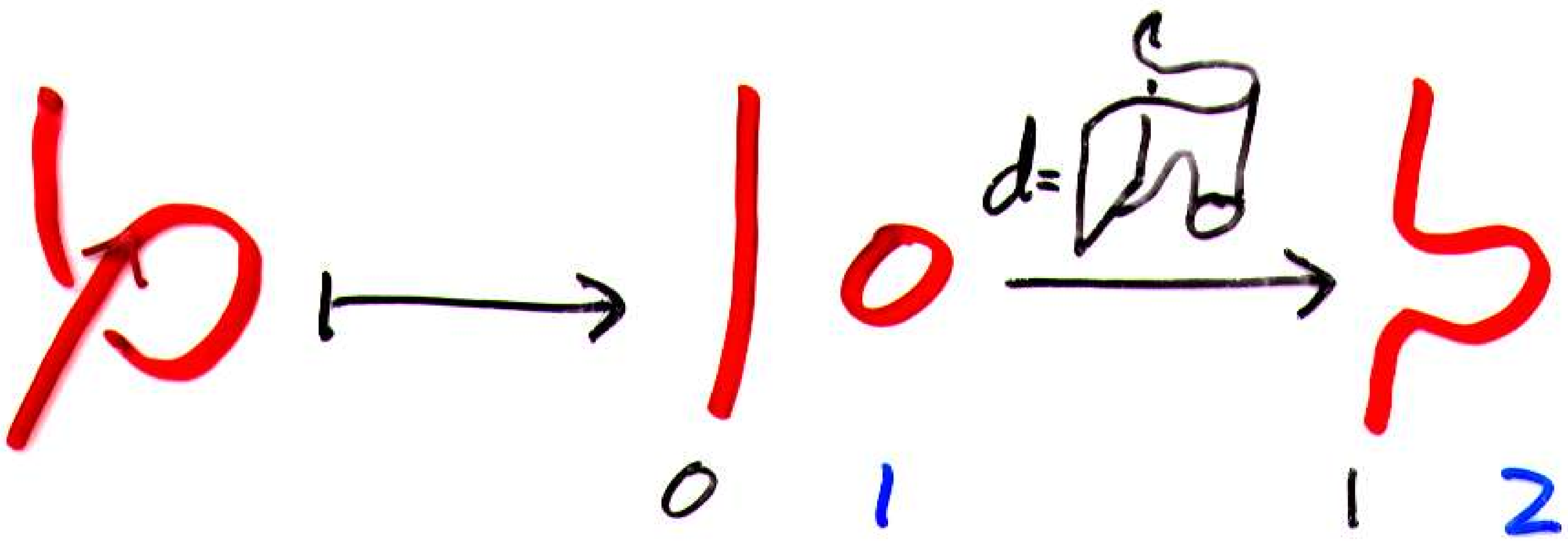
Further, we expect identity cobordisms to be of degree 0, and degree to be additive under composition and planar algebra operations.

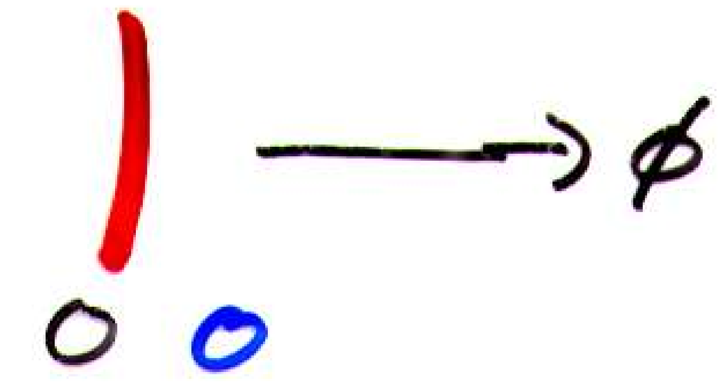
Thus deg  $= 0$, deg  $= -1$, so

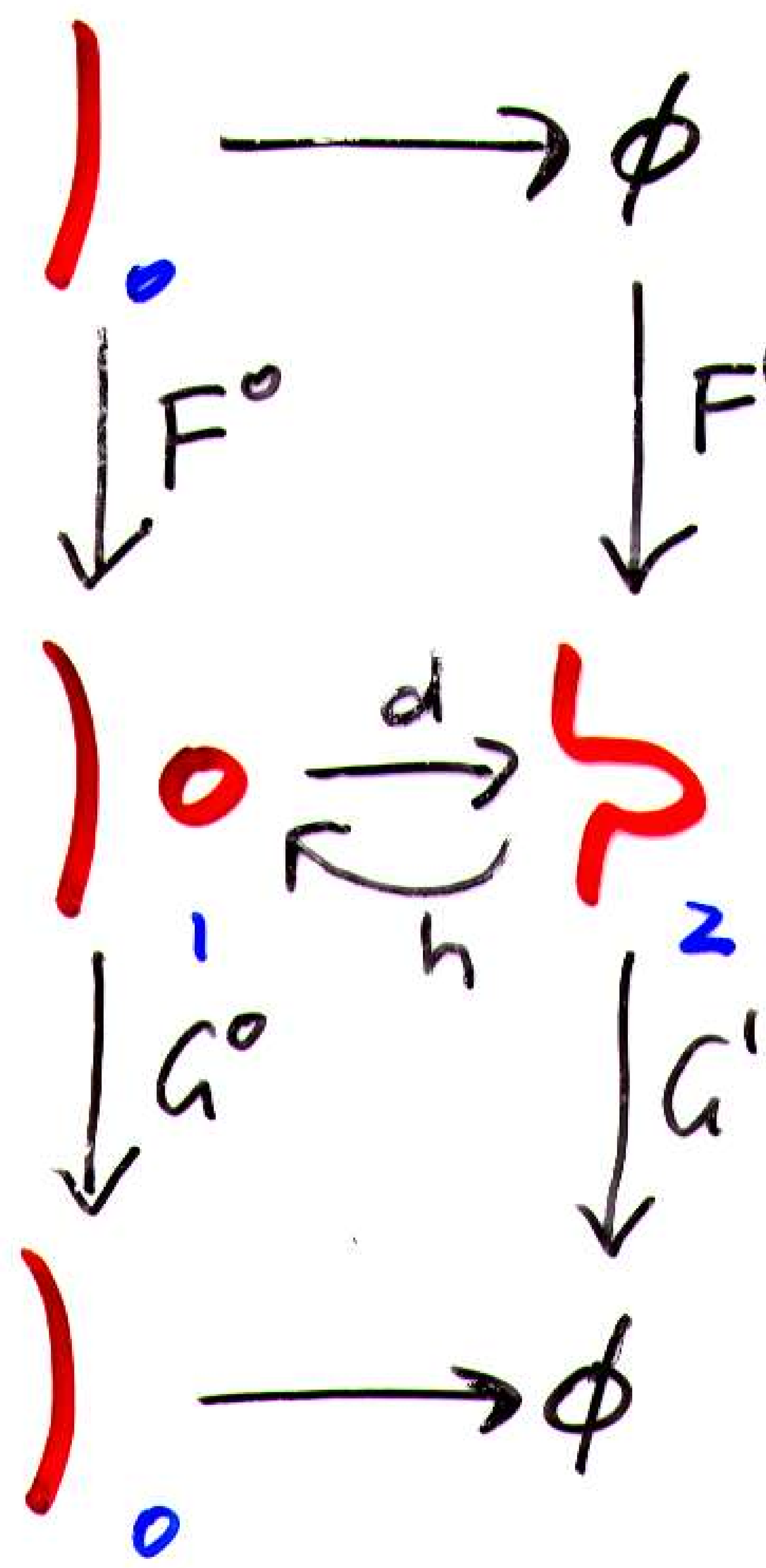
deg  $=$ deg  $= 0 \Longrightarrow$ deg  $= 1$.

This gives $\deg C = \chi(C) - \frac{1}{2}B$, where B is the number of vertical bdy components

Reidemeister moves.



We want this to be homotopic to 



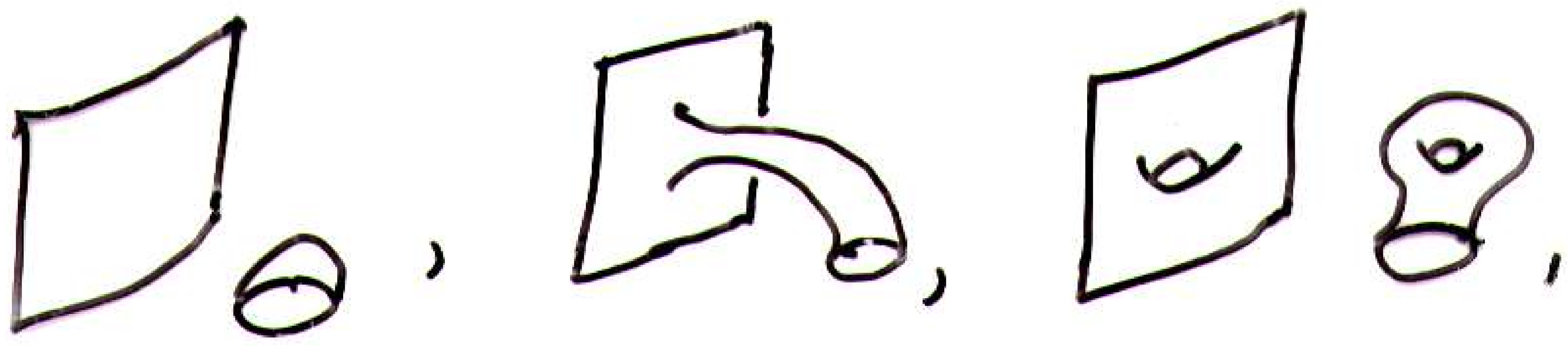
Clearly F' and G' are zero. But what might F° and G° be?

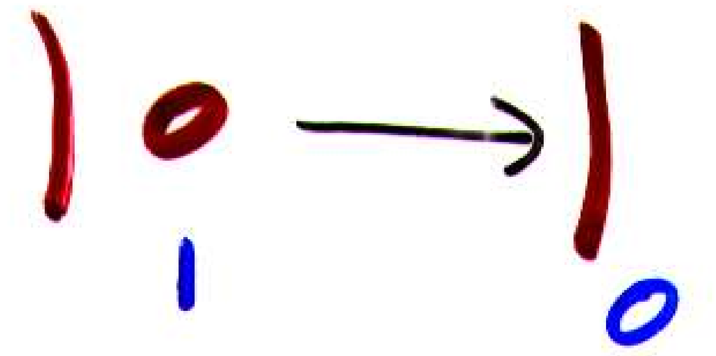
- They must be homogeneous.
- They must form chain maps. ($\Rightarrow dF^\circ = 0$)
- They must be homotopy inverses.

$$F^\circ G^\circ - I = hd$$

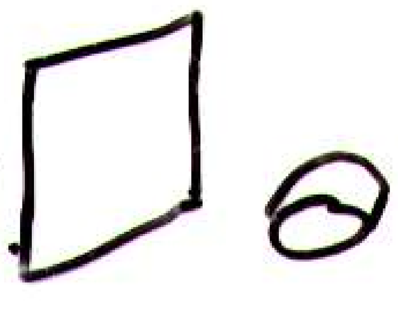
$$G^\circ F^\circ = I$$

Candidates for G° include

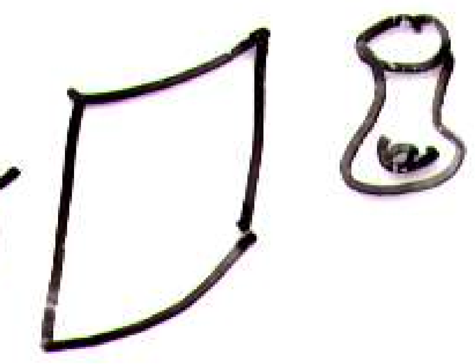
 etc. (and linear combos!)

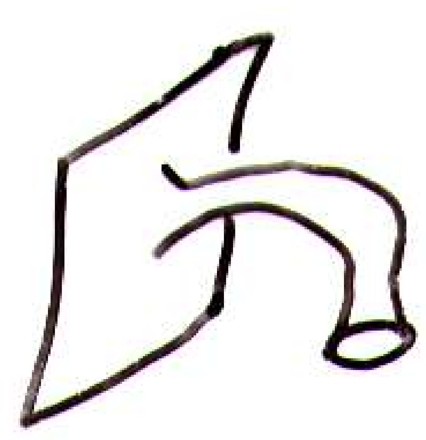
Since G°:  its 'bare' grading must be $+1$.

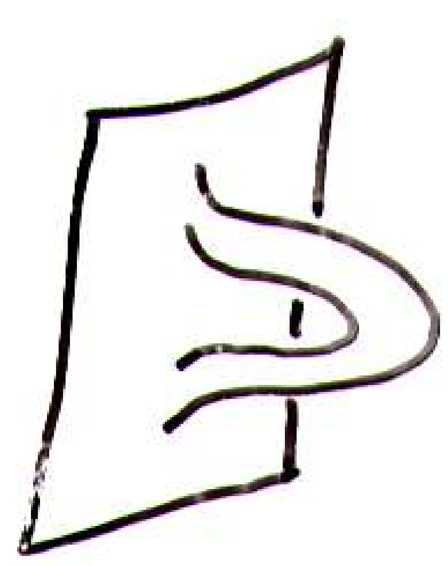
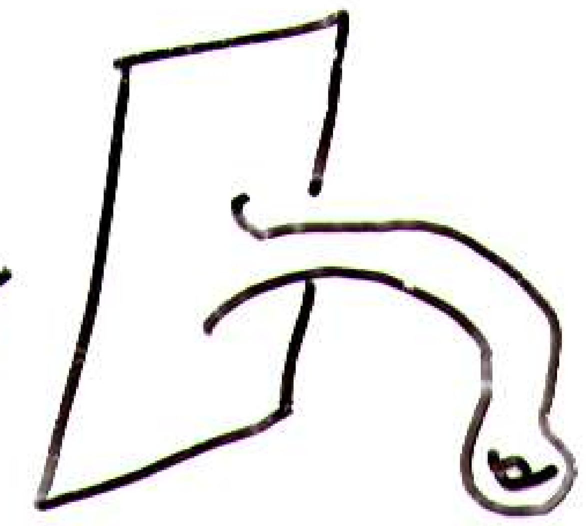
$$\deg G^\circ = \chi(G^\circ) - \tfrac{1}{2}B(G^\circ) = \chi(G^\circ) - 1, \text{ so } \chi(G^\circ) = 2.$$

Thus $G^\circ = \alpha$  for some $\alpha \in R$. Let's fix $\alpha = 1$.

Similarly, degree considerations force $\chi(F^\circ) = 0$,

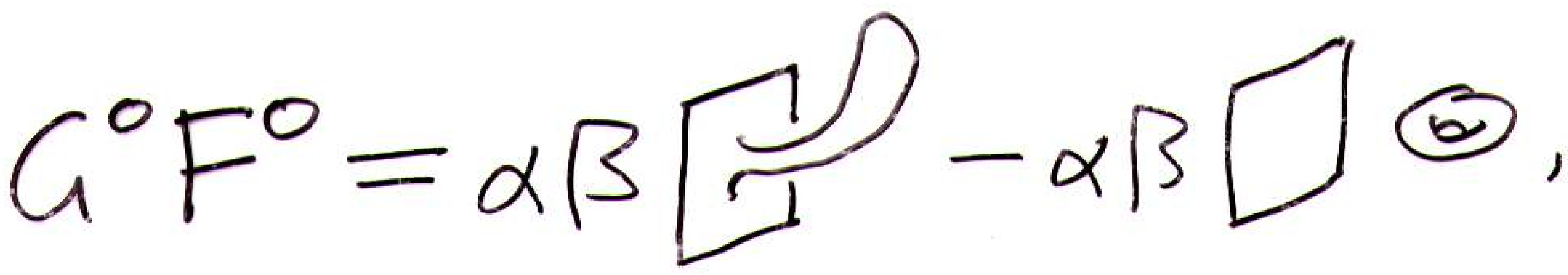
so $F^\circ = \beta$  $+\, \gamma$  for $\beta, \gamma \in R$

What about $dF^\circ = 0$? $d =$ 

$dF^\circ = \beta$  $+\, \gamma$ 

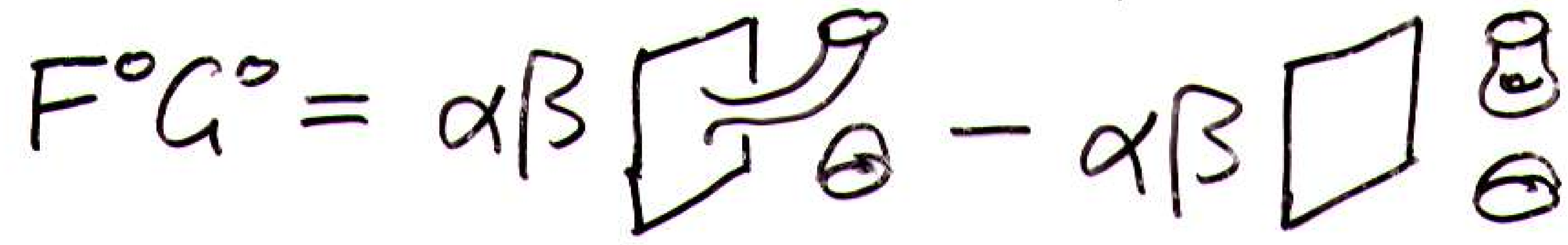
Thus $\beta = -\gamma$.

By now F and G are essentially determined!
Are they homotopy inverses? Fingers crossed!

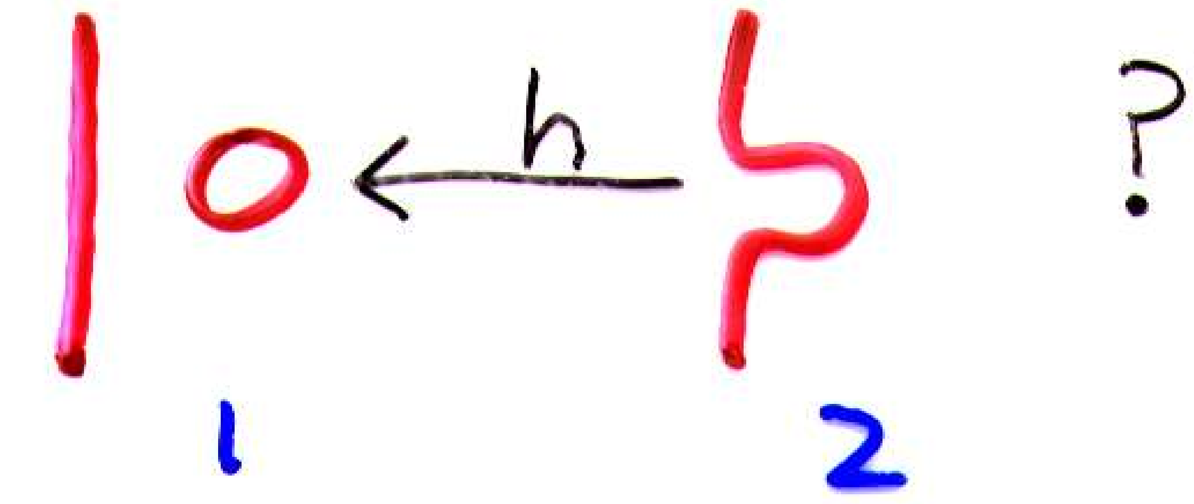


so we better have

Then

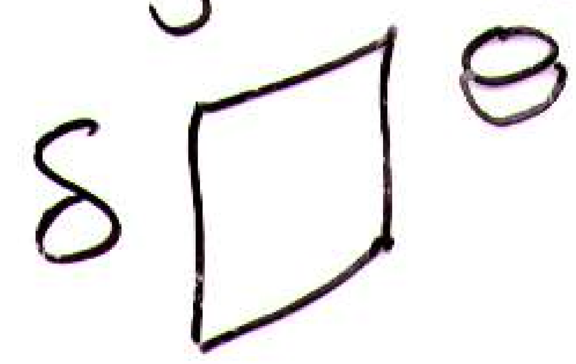


What about



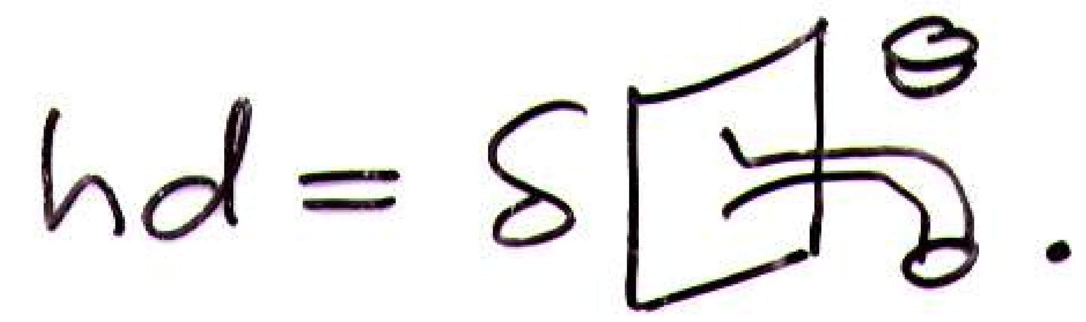
Gradings force $\deg h = 1$, so $\chi(h) = 2$, and so

h must be

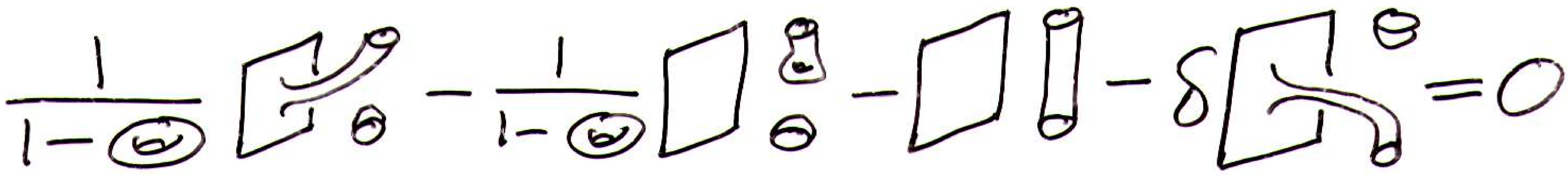


for some $\delta \in R$.

Thus



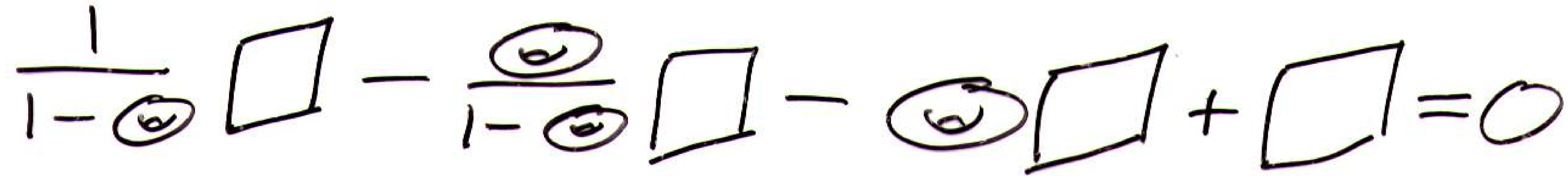
$F^\circ G^\circ - I = hd \implies$



Weird! Fortunately this simplifies somewhat.
Add a 'cup' on the bottom right:

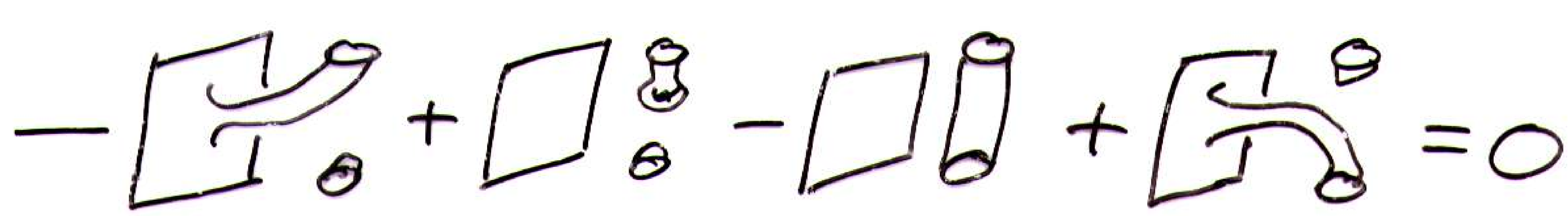
'Thus' $\delta = -1$, and

We can also add a 'tube' connecting the top and bottom circles on the right:



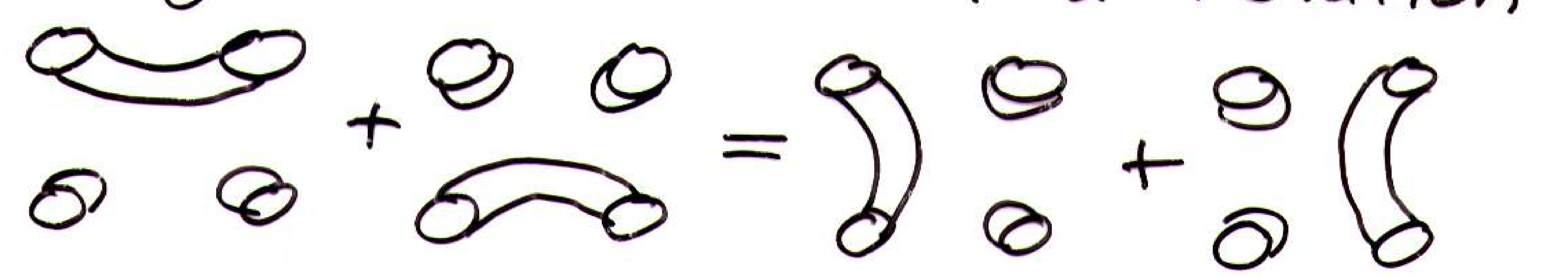
Thus 

Our mysterious cobordism relation becomes



which, drawn more symmetrically, is

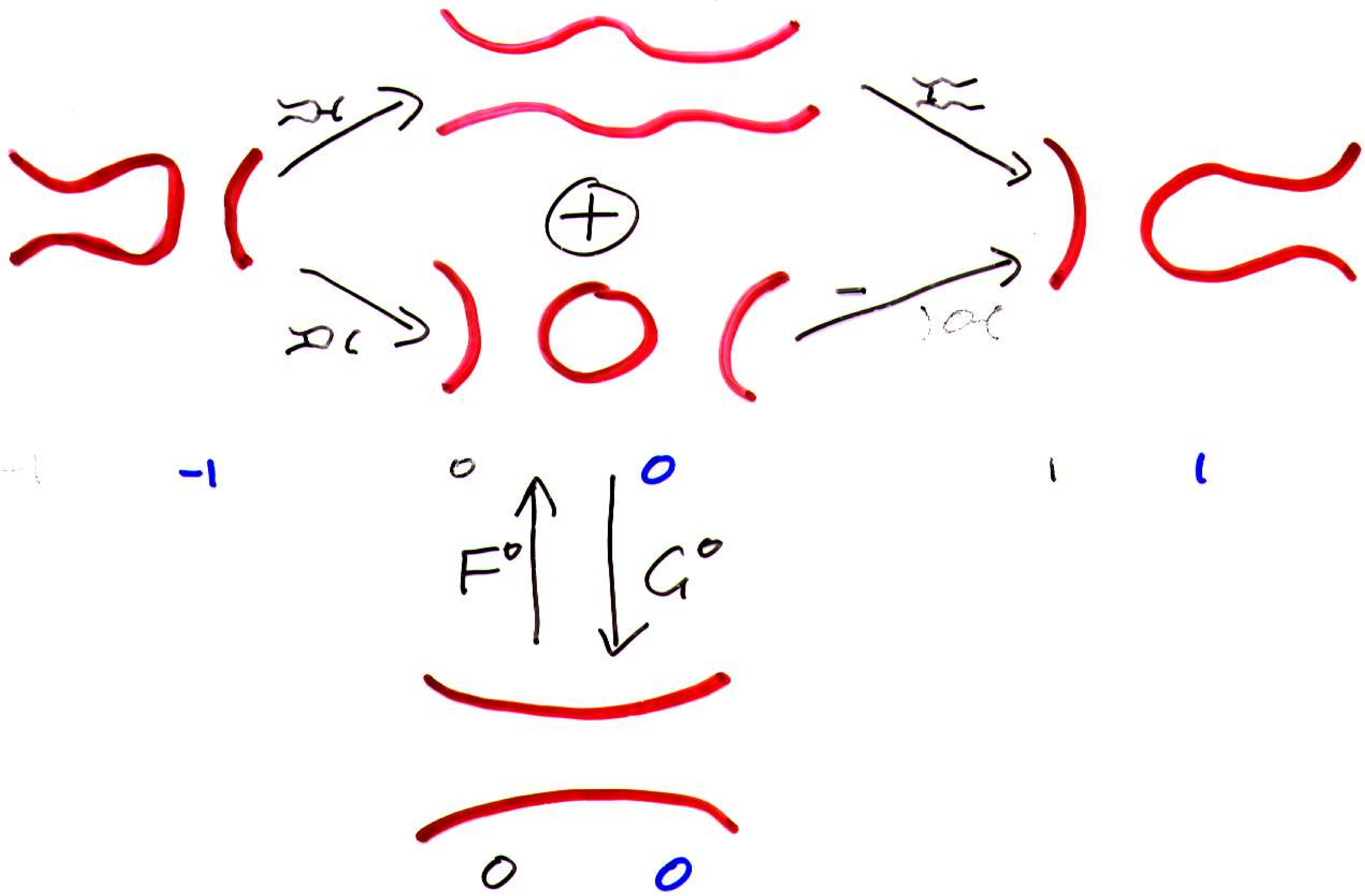
This is what Dror calls the 3S relation. It's equivalent to the 4Tu relation



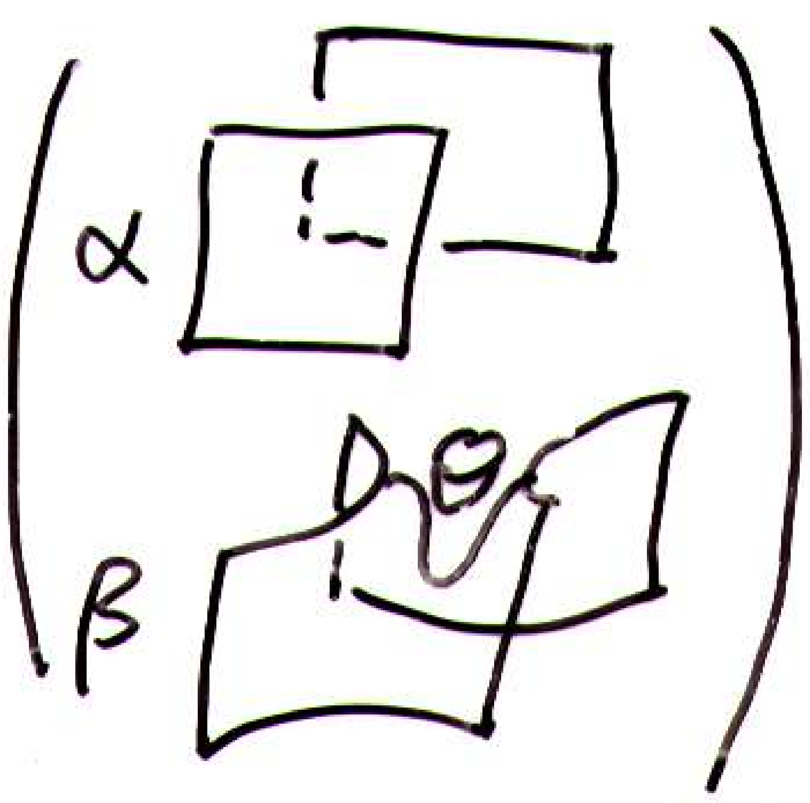
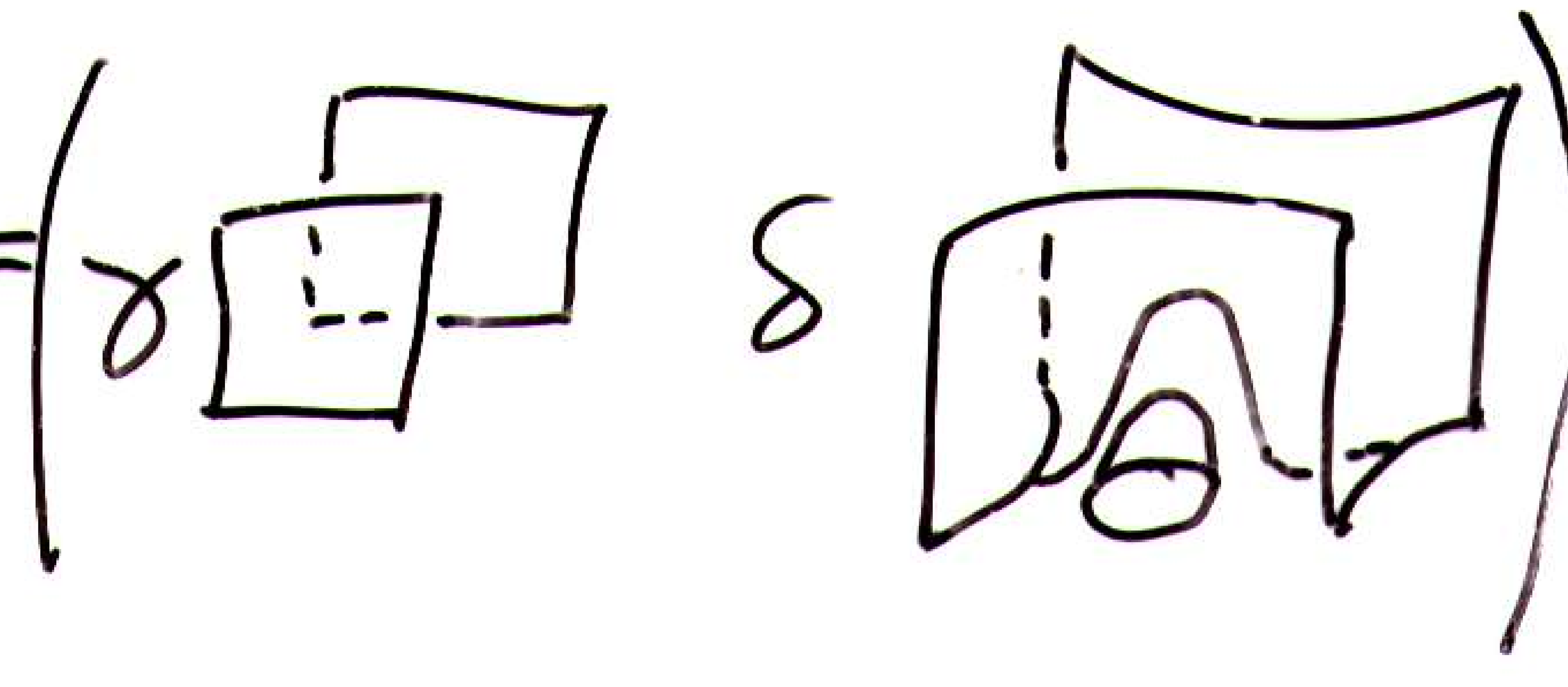
and implies the 'neck cutting relation'

(actually equivalent if $\frac{1}{2} \in R$)

Reidemeister 2 (is easy after that!)



Grading considerations force $(\chi(F^\circ)=\chi(G^\circ)=2)$

$F^\circ =$  $G^\circ =$ 

The chain map conditions $dF^\circ=0$, $G^\circ d=0$ force $\alpha=\beta$, $\gamma=-\delta$

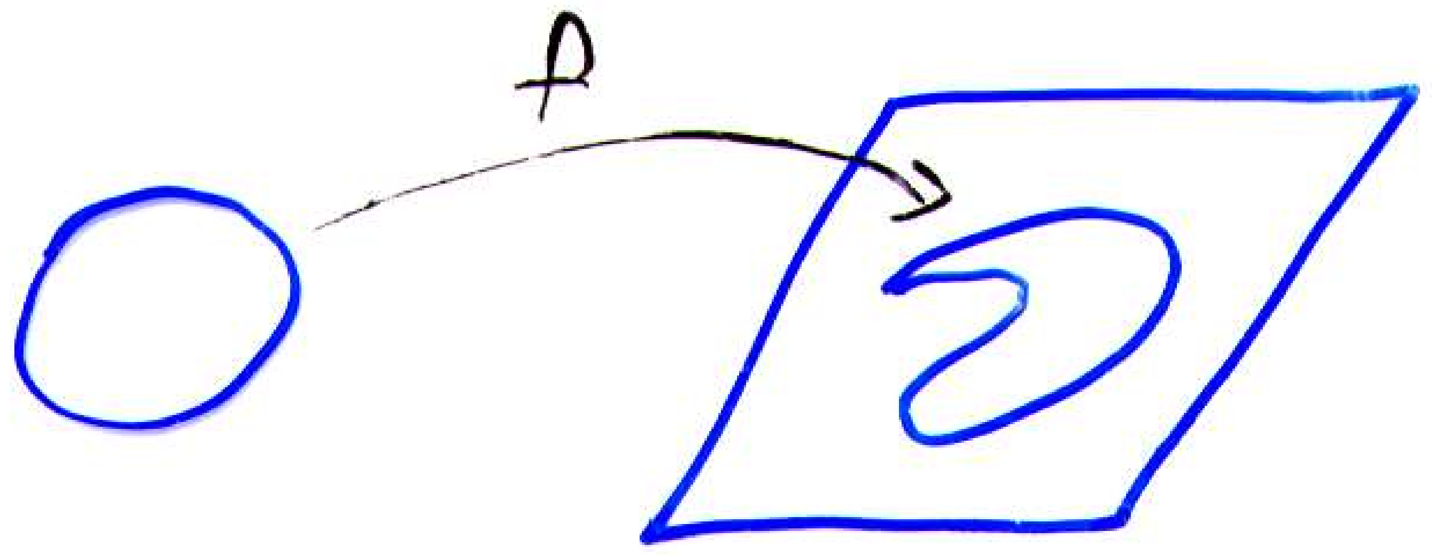
With these restrictions, $F^\circ G^\circ \sim I$ and $G^\circ F^\circ = I$, just using  $=0$,  $=2$, and the 3S relation.

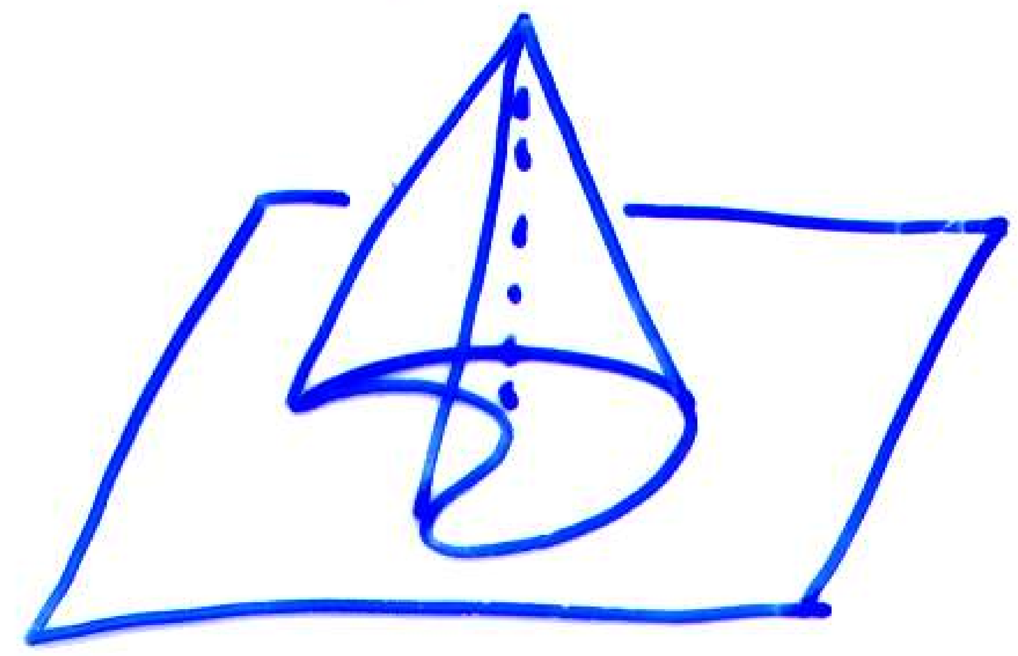
Mapping cones and retracts.

Topology (think this!)

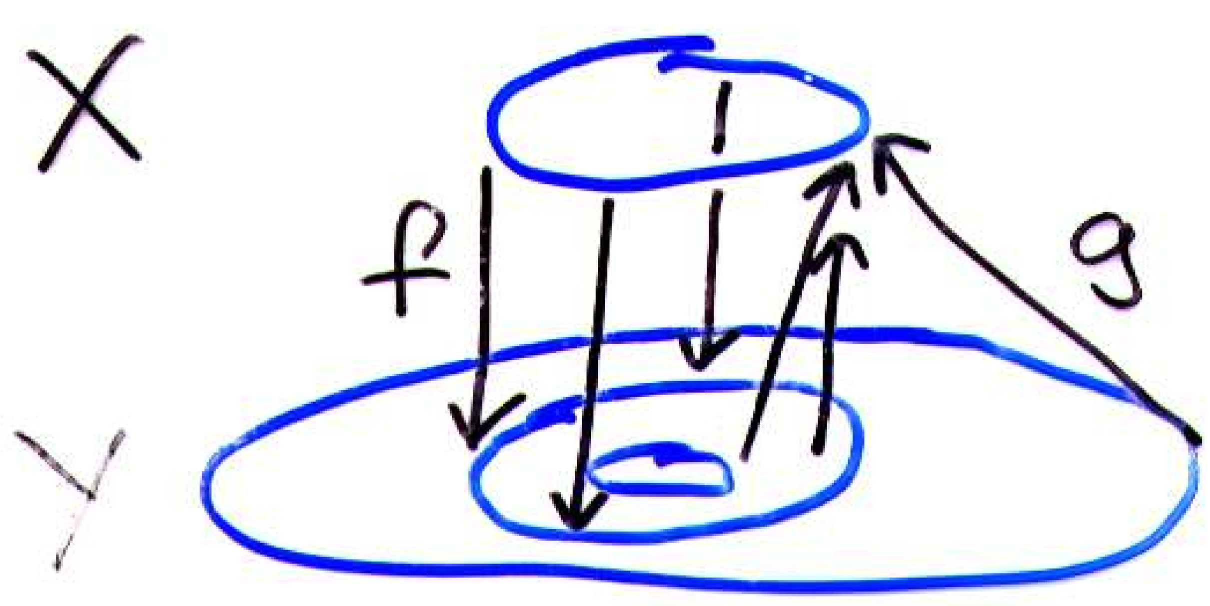
$f : X \to Y$

$Cf = CX \coprod_f Y$





Strong deformation retracts:



$gf = id_X$

$fg \underset{htpy}{\simeq} id_Y$ via a homotopy fixing f

g is a strong deformation retract with inclusion f.

Homotopy (but do this)

$\psi : (\Omega^*_0, d_0) \longrightarrow (\Omega^*_1, d_1)$ a chain map

$$C\psi = \begin{tikzcd}
\cdots \arrow[r] & \Omega_0^{r-1} \arrow[r, "d_0"] \arrow[d, "\mp\psi"] \arrow[dl, dashed, "\oplus"'] & \Omega_0^{r} \arrow[r, "d_0"] \arrow[d, "\pm\psi"] \arrow[dl, dashed, "\oplus"'] & \Omega_0^{r+1} \arrow[r] \arrow[d, "\pm\psi"] \arrow[dl, dashed, "\oplus"'] & \cdots \\
\cdots \arrow[r] & \Omega_1^{r-1} \arrow[r, "d_1"] \arrow[d, dashed, "\oplus"] & \Omega_1^{r} \arrow[r, "d_1"] \arrow[d, dashed, "\oplus"] & \Omega_1^{r+1} \arrow[r] \arrow[d, dashed, "\oplus"] & \cdots \\
\cdots \arrow[r] & (C\psi)^{r-1} \arrow[r, "d"] & (C\psi)^{r} \arrow[r, "d"] & (C\psi)^{r+1} \arrow[r] & \cdots
\end{tikzcd}$$

Say we have chain maps $\Omega_a \underset{G}{\overset{F}{\rightleftarrows}} \Omega_b$

and a homotopy $\Omega_a \circlearrowleft h$ satisfying

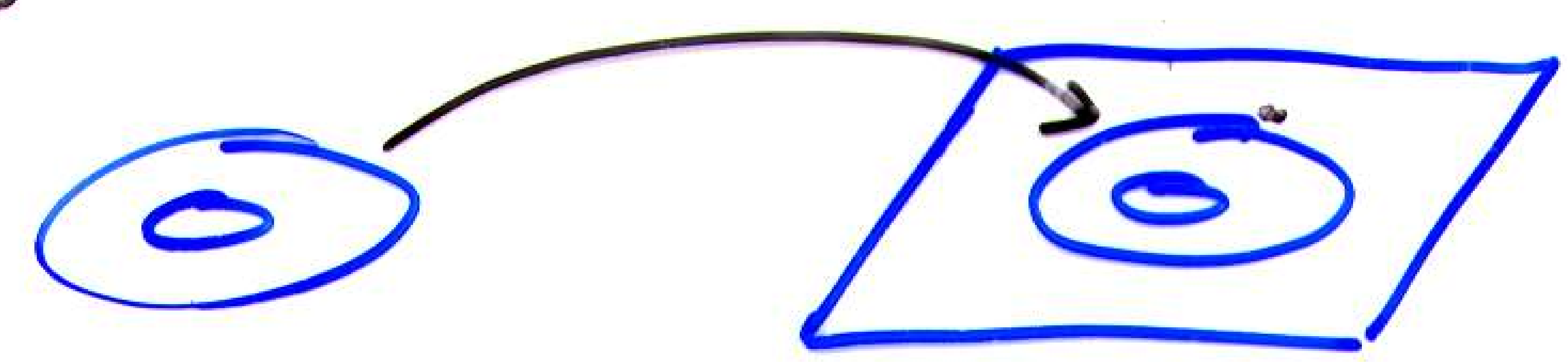
$$GF = I, \quad FG - I = dh + hd \text{ and } hF = 0.$$

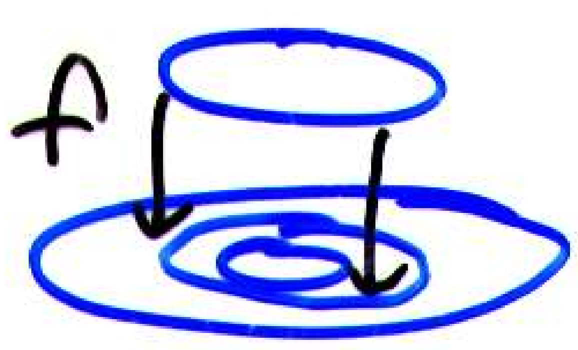
We call F the <u>inclusion</u> in the strong deformation retract G.

Lemma: $\Omega_a \underset{G}{\overset{F}{\rightleftarrows}} \Omega_b \xrightarrow{\psi} \Omega_c$

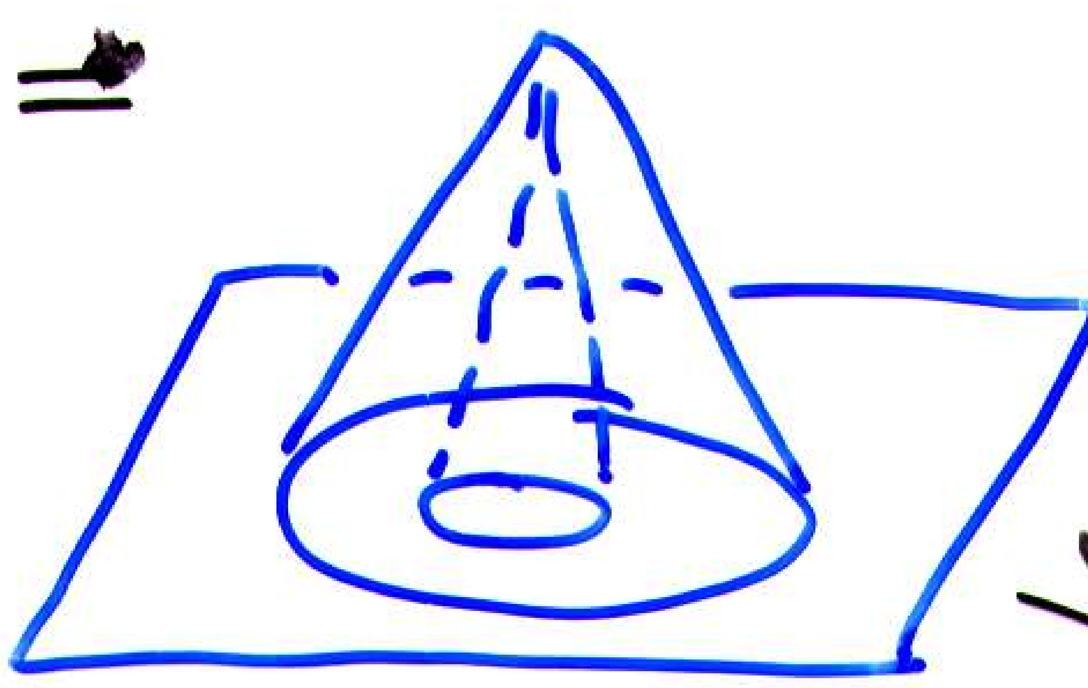
Say F is an inclusion in the strong deformation retract G. Then

$$C(\psi F) \underset{\text{htpy}}{\simeq} C(\psi)$$

'Proof' Say ψ : 

and f 

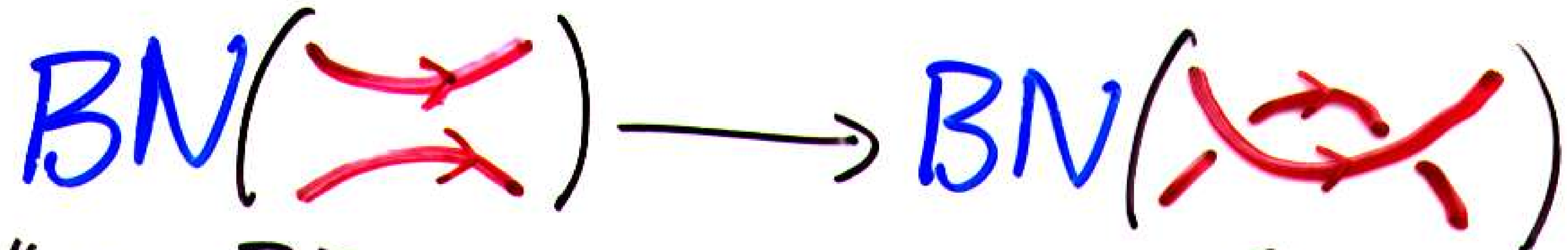
Then $C\psi =$



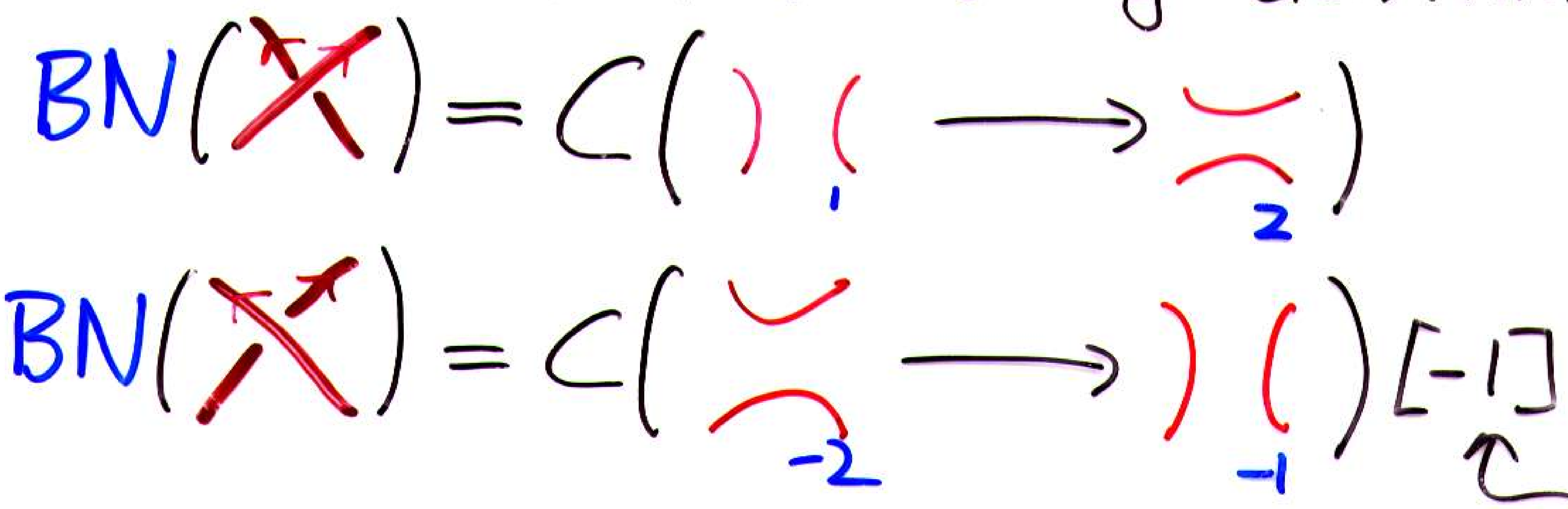
htpy!

and $C(\psi f) =$

What was that for?

① The morphism F: 

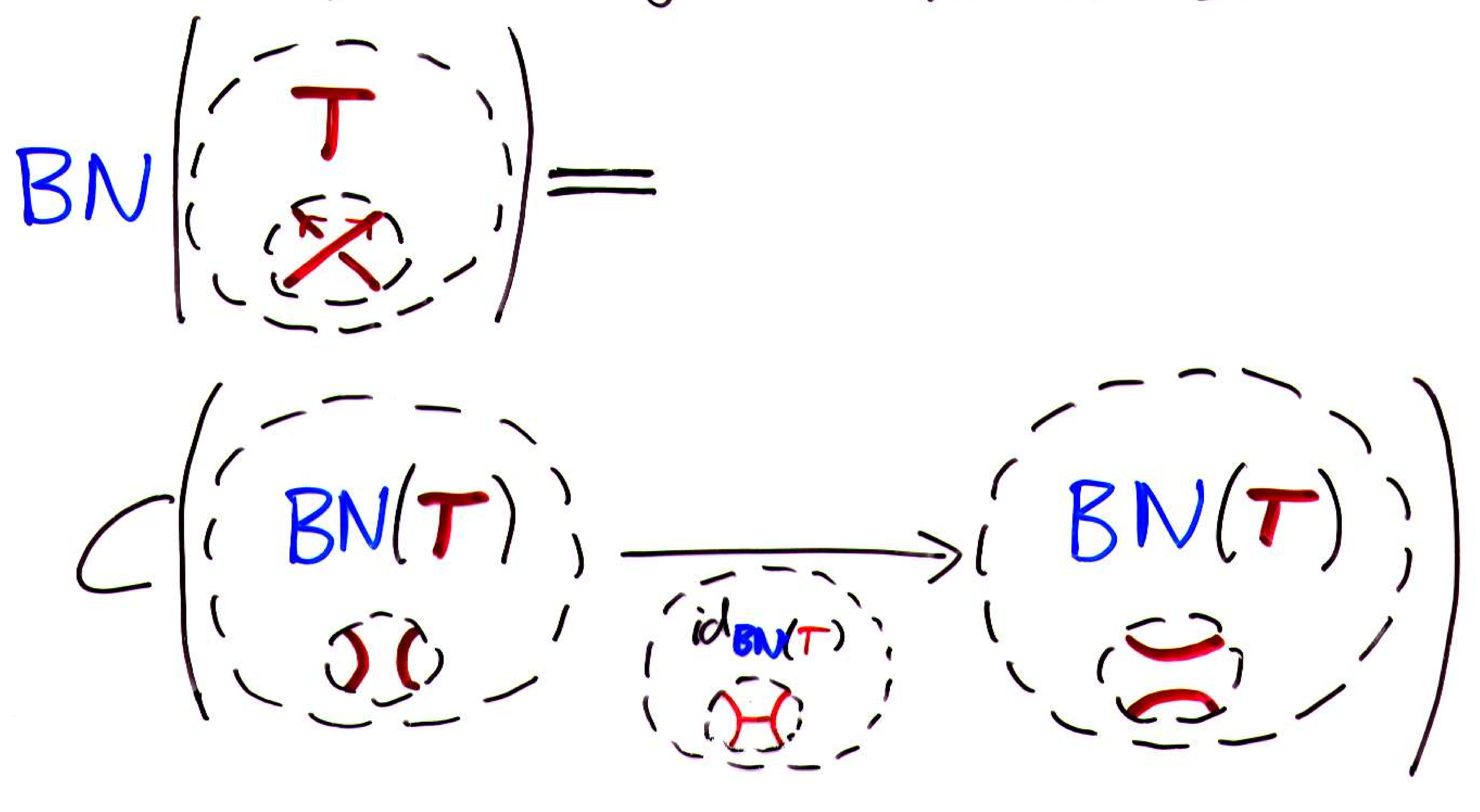
constructed in the R2 invariance proof is an inclusion in a strong deformation retract.

② 

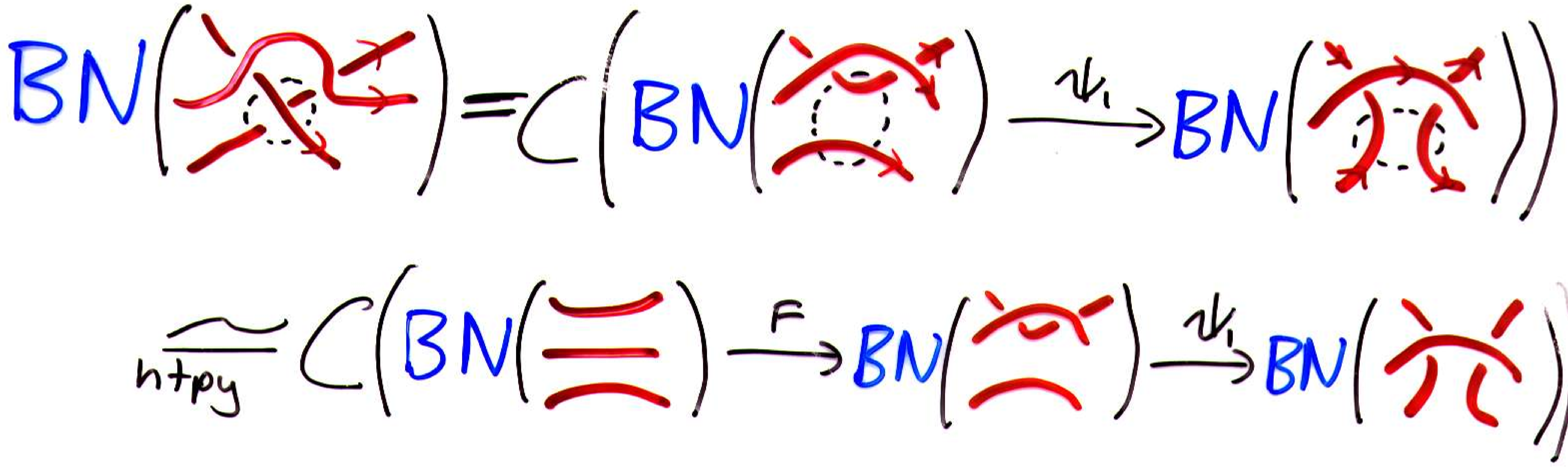
level shift in the complex

These facts are trivial to check; we're just taking the cone of a morphism between single step complexes (no differentials!)

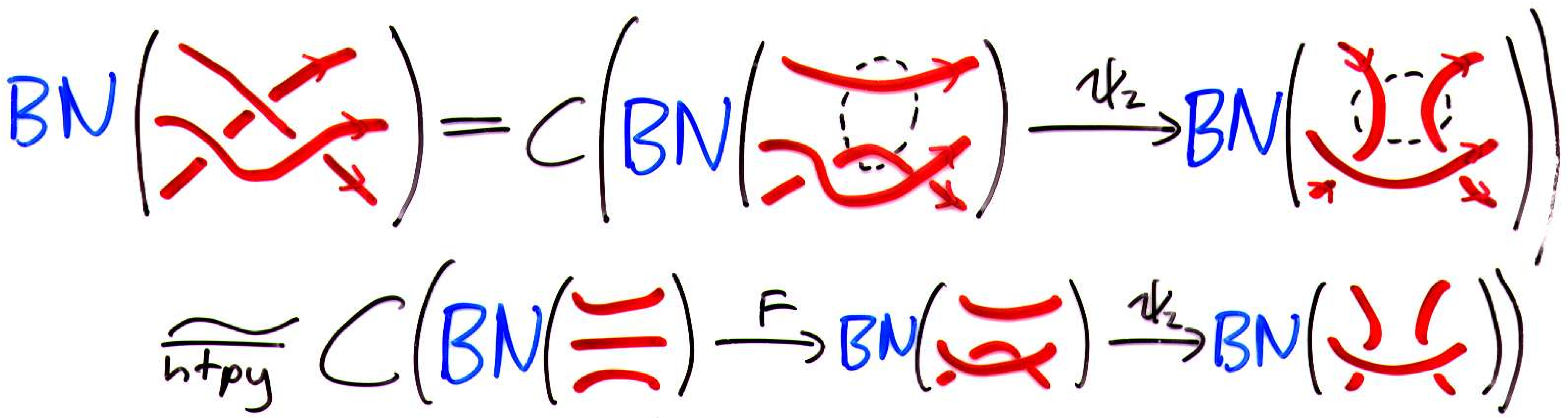
Harder to get your head around is the interaction between the cone construction and the planar algebra operations:



Putting these facts together, we get an easy proof of R3 invariance.



and



We're taking the cone of the same morphism in each case: $F\psi_1 = F\psi_2$

A brief history of homologification