The blob complex (arXiv:1009.5025 arXiv:1108.5386)

The blob complex is a [redacted] gadget

$$\mathcal{B}_*(\underset{\substack{\uparrow \\ \text{a } n\text{-manifold}}}{M} ; \underset{\substack{\nwarrow \\ \text{a 'disklike'} \\ n\text{-category}}}{\mathcal{C}})$$

taking an n-manifold and an n-category and producing a chain complex.

It generalizes several nice ideas:

(1) When $M = S^1$, and $\mathcal{C}$ is an associative algebra,

$$\mathcal{B}_*(S^1; \mathcal{C}) \underset{q.i.}{\simeq} HC_*(\mathcal{C})$$

the Hochschild complex.

(2) The zeroth homology $H\mathcal{B}_0(M; \mathcal{C})$ is the "skein module" for M coming from the topological quantum field theory associated to $\mathcal{C}$.

(3) With $\mathcal{C} = k[t]$ ~~thought of~~ as an n-category

$$\mathcal{B}_*(M; k[t]) = C_*(\Sigma^\infty M)$$

singular chains on the infinite configuration space of M.

Let's start with Hochschild homology, and see how to generalize it.

$$HC_*(A) = \bigoplus_{k=1}^{\infty} A^{\otimes k}$$

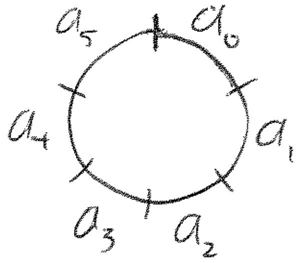
$$d(a_0 \otimes a_1 \otimes a_2) = a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 + a_2 a_0 \otimes a_1$$

The differential is the signed sum over ways of multiplying two cyclically adjacent factors.

Note $HH_0 = A / \{ab - ba\}$ = "the coinvariants of A"

(in fact, HH is the 'derived functor of coinvariants'; the complex given above is a nice presentation, but we can also think about Hochschild homology via its universal properties)

Let's sprinkle some geometry:

$HC_k =$ 

- subdivide the circle into k standard intervals
- label each interval with an element of A.

The differential glues two adjacent intervals together (and then moves them all slightly)

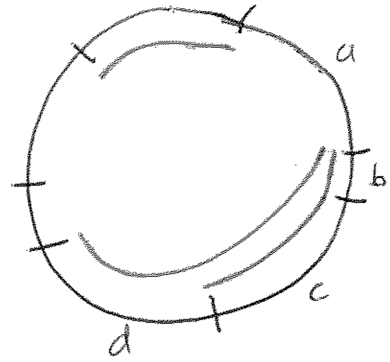
If you're used to thinking about the Hochschild complex, you might think of the bar complex definition above as 'unnecessarily large': for specific algebras you can usually find much smaller free resolutions and use those for computations.

We're going to go in the opposite direction, replacing the bar complex with something much larger.

We're trying to formulate Hochschild homology so we can generalize it to higher dimensions.

- We need to make sure our approach works on 'some 1-manifold' not just the circle
- We want to rely on dimension 1 as little as possible.

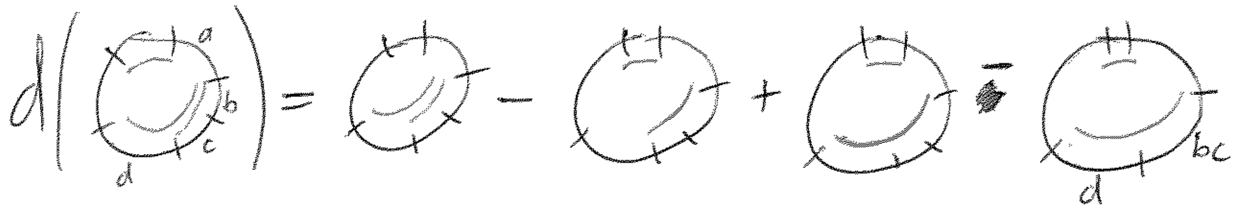
$B_k(S^1; A) =$



- a subdivision of the circle into <u>any</u> number of intervals
- labels from A on each.
- a collection of k "blobs", each an interval made up of intervals from the subdivision, which are pairwise disjoint or nested.

The differential is the signed sum over ways to

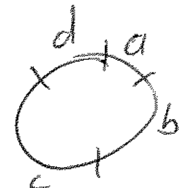
- forget a blob, or
- forget an 'innermost blob' and glue up all its intervals, multiplying their labels.

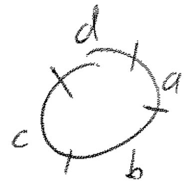


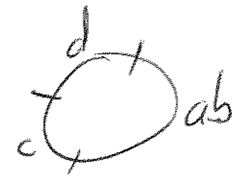
It's not particularly obvious that this gives the same answers as the usual bar complex!

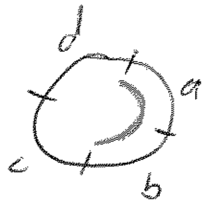
(See our paper for the low degrees of an explicit map, and a check that this satisfies the universal properties of Hochschild homology.)

Let's just check that $HB_0(A) = HH_0(A) = A/_{ab-ba}$.

$B_0(S^1; A) =$ 

but 

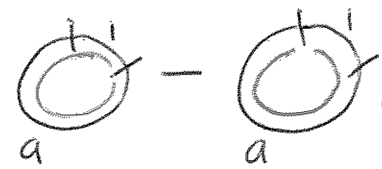
$\sim$ 

via 

so certainly $HB_0(A)$ is some quotient of A

and 

$\sim$ 

via 

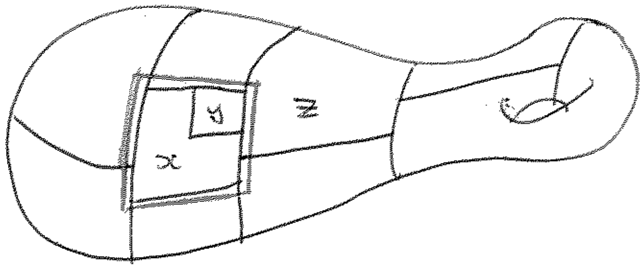
It's not hard to convince yourself that if 

there's a factorization $a = xy$, $b = yx$.

Time to raise the dimension!

Let's not worry for a moment what "$\mathcal{C}$, a disklike n-category" means, and just define the blob complex.

$\mathcal{B}_k(M^n; \mathcal{C}) =$

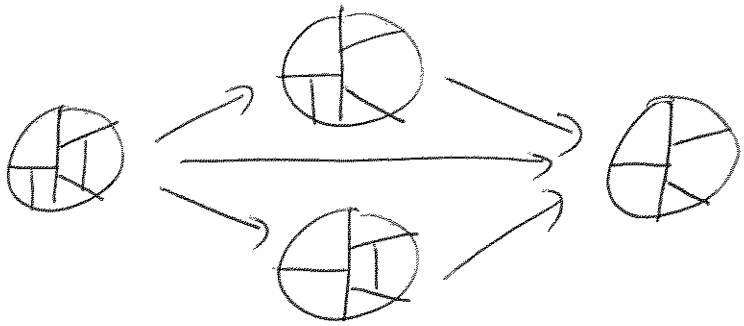


- a decomposition of M into n-balls,
- labelled by morphisms from the n-category
- a collection of k blobs
  (each an n-ball obtained by gluing balls from the decomposition together)
  pairwise disjoint or nested.

The differential is again a sum over ways to
- forget a blob, or
- forget an 'innermost blob' and glue up its contents.

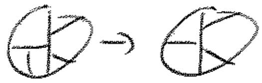
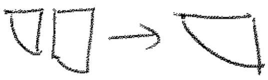
More fancyly:

$D(M)$ is the 'ball decomposition poset' of M:



$\mathcal{C}$ gives a functor $D(M) \to \mathrm{Vec}$

$\mathcal{C}\Big($  $\Big)$ = tensor product of n-morphism spaces of shapes 

$\mathcal{C}\Big($  $\Big)$ = composition map 

then $B_*(M;\mathcal{C}) = \operatorname{hocolim}_{D(M)} \mathcal{C}$.

So what is a disklike n-category?

- a functor $\mathcal{C}_k : \left\{\begin{matrix}k\text{-balls} \\ \text{\& isomorphism}\end{matrix}\right\} \longrightarrow \text{Set}$ for $k=0,\dots,n$.

- for each $Y^{k-1} \subset \partial X^k$, a sub-ball of the boundary of a bigger ball, a <u>restriction map</u>

  $\partial_Y : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(Y)$

- for each $X_1 \cup_Y X_2$ a gluing on two k-balls X_i along a (k-1)-ball, a <u>gluing map</u>

$$\mathcal{C}(X_1) \times_{\partial_Y} \mathcal{C}(X_2) \longrightarrow \mathcal{C}(X_1 \cup_Y X_2)$$

(+ other minor data)

such that:

- the gluing maps are strictly associative.
- the subsets $\mathcal{C}_n(X^n; c) \subset \mathcal{C}_n(X)$ with 'fixed boundary data' have the structure of vector spaces.
- at level n, if isomorphisms $f, g : X_1^n \to X_2^n$ are isotopic, the maps $\mathcal{C}_n(f) = \mathcal{C}_n(g)$ are equal.

(+ many compatibilities)

Examples

- $\mathcal{C}_k(X) = \mathrm{Maps}(X \to T)$ for a fixed target space T

  (we can prove a nice result here; recovering $C_*(\mathrm{Maps}(M \to T))$.)

  This works in any dimension.

- $n=2$ any pivotal category
  (e.g. $\mathrm{Rep}\,G$, G a finite group; HB is concentrated in degree 0.)

- $n=3$ - any ribbon category
  (e.g. $\mathrm{Rep}\,U_q\mathfrak{g}$)

  - tight contact structures

- $n=4$ $\mathcal{C}_{0,1,2,3}(X)$ = codimension 2 submanifolds
  $\mathcal{C}_4(X; L \subset \partial X) = Kh(L)$, the Khovanov homology.