The blob complex (arXiv: 1009.5025 and 1108.5386)

The blob complex is a pairing

$$\mathcal{B}_*(M\,;\mathcal{C}) \longleftarrow \text{a chain complex}$$

a n-manifold   a 'disklike n-category'

which simultaneously generalizes several interesting gadgets

$$\begin{tikzcd}
 & HH_*(\mathcal{C}), \text{ the Hochschild complex} & \\
 & \mathcal{B}_*(M;\mathcal{C}) \arrow[u, "{n=1,\ M=S^1}"'] \arrow[r, "H_0"'] \arrow[dl, "\mathcal{C}=k[t]"] & \text{the TQFT invariant for } \mathcal{C} \text{ on } M \\
C_*(\Sigma^\infty M) & &
\end{tikzcd}$$

The core idea is a disklike n-category which is roughly like a weak n-category, or (∞,n)-category, with duals at all levels, particularly well suited to topological constructions.

A disklike n-category $\mathcal{C}$ consists of:

- a functor $\mathcal{C}_k : \left\{ \begin{matrix} k\text{-balls} \\ \circlearrowleft \\ \text{isomorphism} \end{matrix} \right\} \longrightarrow \mathrm{Set}$ for each $k = 0, \dots, n$.

- for each $Y^{k-1} \subset \partial X^k$, a sub-ball of the boundary of a bigger ball, a <u>restriction map</u>

$$\partial_Y : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(Y)$$

(! many technical details elided here)

- for each $X_1 \cup_Y X_2$, a gluing of two k-balls X_i along a $(k-1)$-ball Y, a <u>gluing map</u>

$$\bullet : \mathcal{C}(X_1) \times_{\partial_Y} \mathcal{C}(X_2) \to \mathcal{C}(X_1 \cup_Y X_2)$$

(• some other minor data)

such that

- the gluing maps are strictly associative
- the subsets $\mathcal{C}_n(X^n; c) \subset \mathcal{C}_n(X)$ with 'fixed boundary data' (i.e., all restrictions agree, determined by c) have the structure of vector spaces
- at level n, if isomorphisms $f, g : X_1^n \to X_2^n$ are <u>isotopic</u>, the maps $\mathcal{C}_n(f) = \mathcal{C}_n(g)$ are equal.

(• many compatibilities)

Examples

- $\mathcal{C}_k(X) = \mathrm{Maps}(X \to T)$ for a fixed target space T.
- $n=2$ $\quad \mathcal{C}_k(X)$ = codimension 1 submanifolds

  linearize $\mathcal{C}_k(X;c)$, and impose the relation $0 = \delta \in \mathbb{C}$.

  This is the "Temperley-Lieb" category
- $n=3$, $\quad$ ~~$\mathcal{C}_{k<n}(X)$ = contact structures on X~~

  $\mathcal{C}_3(X) = \mathbb{C}\{\text{contact structures on } X\}/\text{over twisted discs}$.
- $n=4$, $\quad \mathcal{C}_{0,1,2,3}(X)$ = codimension 2 submanifolds

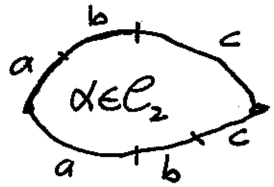
  $\mathcal{C}_4(X; L \subset \partial X) = Kh(L)$, Khovanov homology as a 4-cat.

Although gluing is strictly associative, disklike categories are 'weak'.

- To obtain a composition on $\mathcal{C}_1(I)$, you need a reparametrization $I \cup I \to I$

$$\mathcal{C}_1(I) \times \mathcal{C}_1(I) \to \mathcal{C}_1(I \cup I) \to \mathcal{C}_1(I).$$

- $(a \bullet b) \bullet c \neq a \bullet (b \bullet c)$, but the (omitted) identity axiom ensures there exists

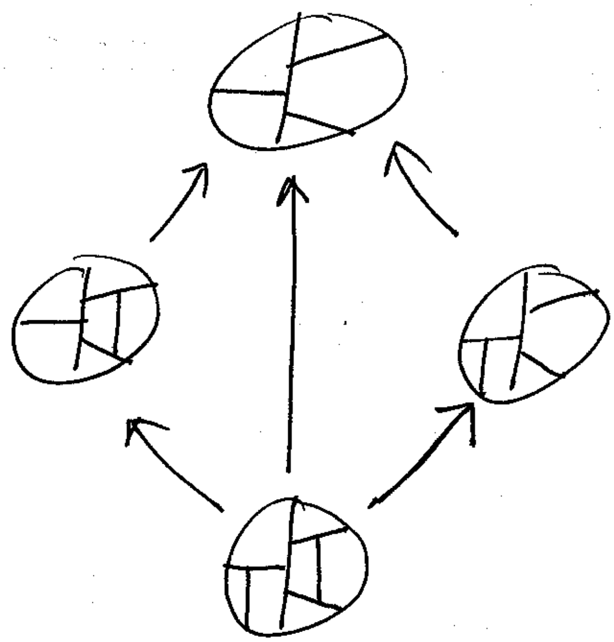


an invertible (up-to higher morphisms) morphism between them.

- At level n equations hold on the nose, by the isotopy axiom.

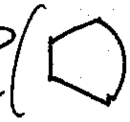
The blob complex is now defined as a certain homotopy colimit.

The 'ball-decomposition poset' $\mathbb{D}(M^n)$ of an n-manifold:



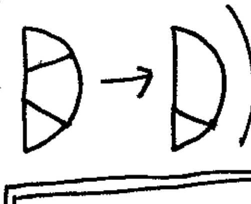
A disklike n-category $\mathcal{C}$ defines a functor $\mathbb{D}(M) \to Vec$
(if M has boundary we need to specify boundary data)

$\mathcal{C}($  $) =$

$\bigoplus \quad \bigotimes \quad \mathcal{C}($  $; c)$

boundary data on the k-1 skeleton

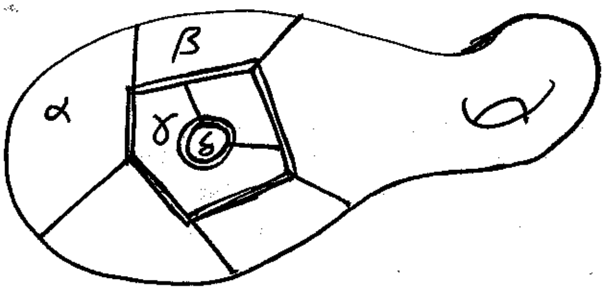
pieces 

$\mathcal{C}($  $)$ is just the relevant gluing maps.

Then: $\boxed{B_*(M; \mathcal{C}) = \operatorname{hocolim}_{D(W)} \mathcal{C}}$

Let's make this much more explicit!

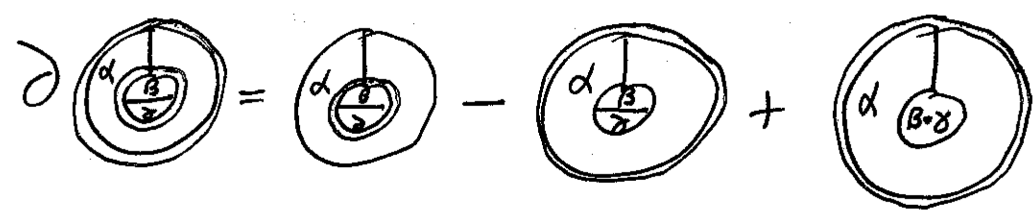
$B_{*k}(M;\mathcal{C})$ is spanned by diagrams



- a ball decomposition of M
- labels from $\mathcal{C}$
- k 'blobs', pairwise nested or disjoint formed from a union of balls.

The differential $\partial : B_k(M;\mathcal{C}) \to B_{k-1}(M;\mathcal{C})$ is the signed sum of ways to

- forget a blob, or
- forget an 'innermost' blob, and glue up its interior.



Observe $H_0 = B_0 / \partial B_1 =$ "pictures from $\mathcal{C}$ drawn on M, modulo local relations"

i.e. the usual TQFT invariant.

<u>Theorem</u> (Families of diffeomorphisms act)

There are chain maps

$$\mathcal{C}_*(\mathrm{Diff}(M)) \otimes B_*(M;\mathcal{C}) \longrightarrow B_*(M;\mathcal{C})$$

- so $C_0 \mathrm{Diff}(M) \otimes B_*(M;\mathcal{C}) \to B_*(M;\mathcal{C})$ is the obvious action,
- compatible with gluing (up to homotopy)
- and in fact uniquely (up-to-homotopy) determined by these conditions.

<u>Examples</u>

- $B_*(S^1;\mathcal{C})$ is the Hochschild complex; rotation around S^1 gives the cyclic differential
- rotation along rational slopes on $\mathbb{T}_2$ giving a degree-raising map $HB_*(\mathbb{T}^2;\mathcal{C}) \to HB_{*+1}(\mathbb{T}^2;\mathcal{C})$

<u>Sketch</u> define BT_*, total complex of $BT_{ij} = C_j$(i-blob diagrams)

BT_* has an obvious action of $C_*\mathrm{Diff}$

the inclusion $B_* = BT_{*0} \subset BT_*$ is a homotopy equivalence

<u>Sketch</u> $B_* = B_*^{\mathcal{U}}$ (blobs smaller than an open cover $\mathcal{U}$)

$BT_* = BT_*^{\mathcal{U}}$

$BT_*(B^n)$ is contractible (acyclic in positive degrees)

To state the next theorems, we first need the notion of

'A_∞ disklike n-categories'

as before, but

- $\mathcal{C}_n(X;c)$ is a chain complex, not a vector space
- the action of diffeomorphisms of balls lifts to

$$\mathcal{C}_*(\mathrm{Diff}(X^n)) \otimes \mathcal{C}_n(X) \to \mathcal{C}_n(X)$$

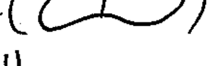
<u>Theorem</u> With M a k-manifold, $\mathcal{C}$ a disklike n-category, the association $X^{n-k} \longmapsto B_*(M \times X; \mathcal{C})$ defines an A_∞ disklike $(n-k)$ category, which we call $B_*(M)$ (or $\underset{\to}{\mathcal{C}}(M)$, depending on what we want to emphasize)

<u>Theorem</u> If $M^{n-1} \subset \partial N^n$, $B_*(N)$ is a module over $B_*(M)$.

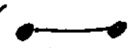
<u>Theorem</u> If $N = N_1 \underset{M}{\cup} N_2$

$$B_*(N) \underset{q.i.}{=} B_*(N_1) \overset{A_\infty}{\underset{B_*(M)}{\otimes}} B_*(N_2)$$

Sketch: prove a much more general fibre product formula

$B_*($  $)$

$\parallel$

$B_*($  $)$

k-cells labelled by order $(n-k)$ modules

Prove this using 'small blobs', acyclic models, and a somewhat technical argument!

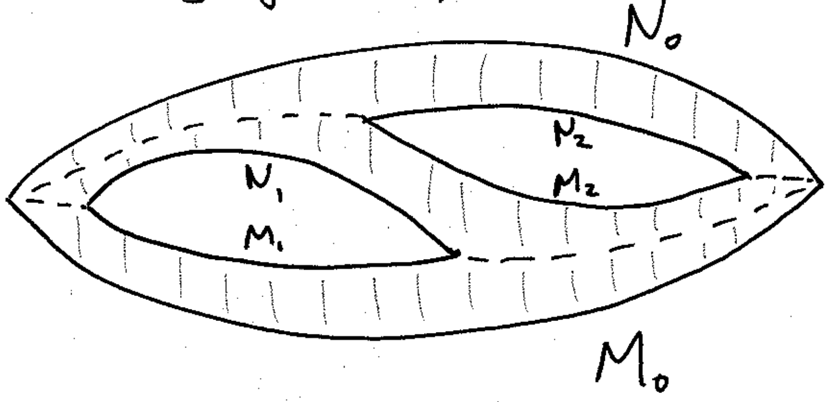
<u>Theorem</u> Define the fundamental (∞,n)-groupoid of T: $\pi_\infty^{\leq n}(T)(X^n) = C_*(\mathrm{Maps}(X \to T))$

Then $B_*(S; \pi_\infty^{\leq n}(T)) \underset{q.i.}{=} C_*\mathrm{Maps}(S \to T)$.

<u>Corollary</u> $\mathrm{Hoch}_*(C_*\Omega T) \simeq \mathrm{Hoch}_*(\pi_{\leq 1}^{\infty}(T)) \simeq B_*(S^1; \pi_{\leq}^{\infty}(T)) \simeq C_*(LT)$

Generalized Deligne conjecture

The 'surgery cylinder operad': $SC_{M,N}$



- $\partial M_i = \partial N_i = E_i$
- mapping cylinders between.
- $SC_{M,N}$ has a natural topology.

Write $\hom_i = \mathrm{Hom}_{B_*(E_i)}\big(B_*(M_i) \to B_*(N_i)\big)$.

Theorem There are a collection of maps

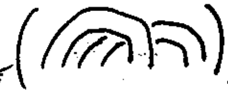
$$C_*(SC_{M,N}) \otimes \bigotimes_i \hom_i \longrightarrow \hom_0$$

giving an action of the operad up to coherent homotopy.

Specialising to $n=1$, $N_i = M_i = I$, this gives the Deligne conjecture: the little discs operad acts on Hochschild cochains.

(Even at $n=1$ there more here:

$(S^1 \to$  $\to S^1)$ gives a map

$\mathrm{Hom}_{B_*(\cdots\cdots\cdot)}\big(B_*(\smile \smile \smile \smile) \to B_*($$)\big)$

$\downarrow$

$\mathrm{End}(HC_*)$)