A 'topological' categorification of the representation theory of $U_q \mathfrak{sl}_3$.

University of Oregon, December 4.

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'Khovanov homology without the knot theory'

- For today, I'll take the attitude that the interesting story in Khovanov homology isn't about knot theory, but about representation theory..

- Recently, there's been lots of wonderful work on algebraic models of this sort of categorification (Stroppel, Brundan, Rouquier, ...)

- But I'm not an algebraist, and all that stuff is too hard for me!

- I want to show you some topological models of categorification.

  (repackaging ideas of Khovanov and Bar-Natan and some minor additions by me and Ari Nieh.)

The plan:

- What is categorification?
- Categorifying the representation theory of $SL(2,\mathbb{C})$.
- and of $SL(3,\mathbb{C})$, with applications:
  - Khovanov homology for $SL(3,\mathbb{C})$
  - dual canonical bases.

What does it mean to "categorify"?

"Decategorification" is a functor —
take the Grothendieck group of a category.

$$K(\mathcal{C}) = \mathbb{Z}\cdot \mathrm{Obj}(\mathcal{C}) \Big/ \begin{array}{l}[A]=[B]+[C] \text{ whenever} \\ 0\to B\to A\to C\to 0 \\ \text{is an exact sequence}\end{array}$$

Sometimes the category isn't abelian and we have to settle for the split Grothendieck group

$$K^{\text{split}}(\mathcal{C}) = \mathbb{Z}\cdot \mathrm{Obj}(\mathcal{C}) \Big/ \begin{array}{l}[A]=[B]+[C] \text{ whenever} \\ A \cong B\oplus C\end{array}$$

Examples:

- $K^{\text{split}}(\text{FiniteSets}) \cong \mathbb{Z}$ (shepherds)
- $K(\text{Vect}) \cong \mathbb{Z}$ (rank-nullity theorem)

"Categorification" is an art —
finding 'an ~~interesting~~ section' of decategorification.

More structure

- Often the Grothendieck group is more than just a group:

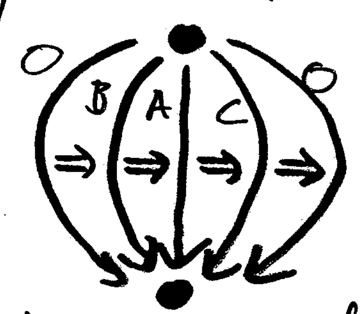
— If $\mathcal{C}$ is a tensor category, $K(\mathcal{C})$ is a ring: $[A][B] = [A \otimes B]$.

— If $\mathcal{C}$ is a 2-category, $K(\mathcal{C})$ is a 1-category

$$\mathrm{Obj}(K(\mathcal{C})) = \mathrm{Obj}(\mathcal{C})$$

$$1\text{-morphisms}(K(\mathcal{C})) = \underline{\mathbb{Z}\cdot(1\text{-morphisms}(\mathcal{C}))}$$

$A = B + C$ if



is an exact sequence.

— If $\mathcal{C}$ is a 'canopolis', $K(\mathcal{C})$ is a planar algebra.

— If $\mathcal{C}$ is a graded category, $K(\mathcal{C})$ is a $\mathbb{Z}[q,q^{-1}]$ module.

- When we want to categorify something, we should reflect as much structure as possible at the category level.

Warmup: $SL(2,\mathbb{C})$.

- I want to categorify $\mathrm{Rep}(SL(2,\mathbb{C}))$, the tensor category of representations of $SL(2,\mathbb{C})$. (a.k.a. 'angular momentum in quantum mechanics)
  - actually, let's do $\mathrm{Rep}(U_q \mathfrak{sl}_2)$, the quantum group version.
- I'm going to show you a <u>diagrammatic presentation</u> of this category, and categorify that.
  - we'll cheat a little first, and restrict our attention to the subcategory $\mathrm{FundRep}(U_q \mathfrak{sl}_2)$

    <u>Objects</u> V, the standard 2-dimensional representation, and $V^{\otimes n}$, tensor powers.

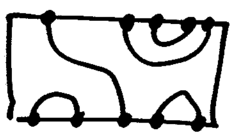
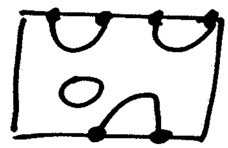
    <u>Morphisms</u>

    $$\mathrm{Hom}(V^{\otimes n}, V^{\otimes m}) = \{U_q \mathfrak{sl}_2 \text{ equivariant linear maps from } V^{\otimes n} \text{ to } V^{\otimes m}\}$$
  - in fact every representation is a subrepresentation of some $V^{\otimes n}$, so this isn't too bad.
- Now — diagrams!

Spider($\mathfrak{sl}_2$), (aka "the Temperley–Lieb category")

Objects points on a line: 

Morphisms $\mathbb{Z}[q,q^{-1}]$ linear combos of diagrams like:

 and 

modulo $\bigcirc = q + q^{-1}$.

There's a tensor functor

$$\text{Spider}(\mathfrak{sl}_2) \longrightarrow \text{FundRep}(U_q\mathfrak{sl}_2)$$

 $\longmapsto V$

 $\longmapsto \mathrm{id}_V$

 $\longmapsto$ the duality pairing $V \otimes V \to A$

 $\longmapsto$ the copairing $A \to V \otimes V$

Theorem this is an equivalence of categories.

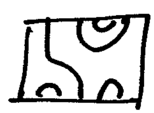
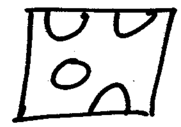
(signs, oh dear!)

- For the next while, you can forget about any representation theory, and just think about this diagrammatic category.

(It's actually more than a category; it's a planar algebra

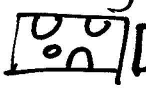
$Cob(\mathfrak{sl}_2)$

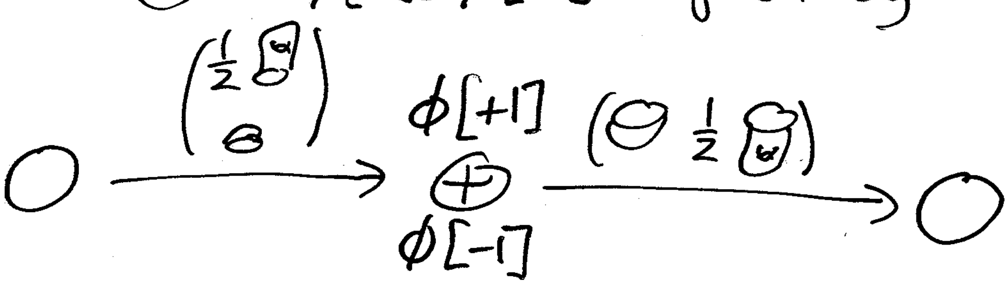
- We want to define $Cob(\mathfrak{sl}_2)$ so
$$K^{split}(Cob(\mathfrak{sl}_2)) \cong Spider(\mathfrak{sl}_2) \cong FundRep(U_q\mathfrak{sl}_2)$$

<u>Objects</u> diagrams like  and 

(but no linear combinations, and no relations)

<u>Morphisms</u> $\mathbb{Z}/2$ linear combinations of cobordisms between diagrams, modulo

- We can put a grading on this category, by allowing 'formal grading shifts' on objects (e.g.  $[2]$) and defining
$$\deg(C : D_1[m_1] \to D_2[m_2]) = \chi(C) - \tfrac{1}{2}\#\partial + m_2 - m_1$$
- What is the Grothendieck group of this category? There's an isomorphism (which we saw on Friday)
$$\bigcirc \cong \phi[+1] \oplus \phi[-1] \quad \text{given by}$$



- Thus in $K(Cob(\mathfrak{sl}_2))$ we have $\bigcirc = q + q^{-1}$.

But could there be further isomorphisms?

Nondegenerate pairings

For two diagrams D_1 & D_2 with common boundary

$$\mathrm{qdim}\,\mathrm{Hom}_{Cob(\mathfrak{sl}_2)}(D_1, D_2) = q^{-\#\partial/2}\langle D_1, D_2\rangle_{\text{spider}(\mathfrak{sl}_2)}$$

spider($\mathfrak{sl}_2$) pairing

Example

$\mathrm{Hom}(/,/) = \mathbb{Z}/2\{$  $\}$

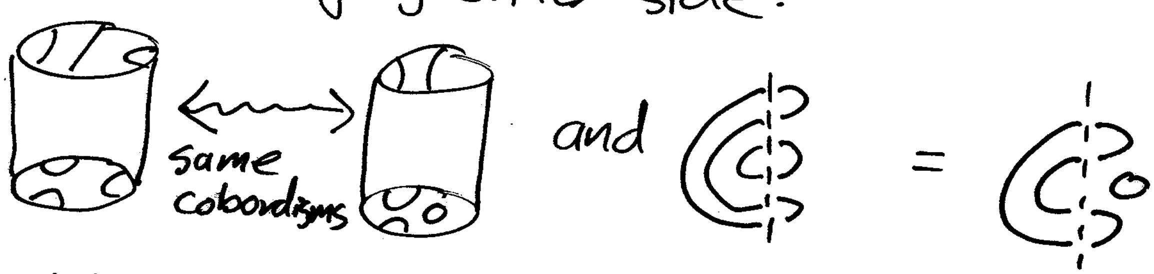
so

$$\mathrm{qdim}\,\mathrm{Hom}(/,/) = 1 + q^{-2}$$

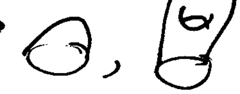
and

$$q^{-\#\partial/2}\langle /,/\rangle = q^{-1}\bigcirc = 1 + q^{-2}$$

Proof • You can slide an arc from D_1 to D_2 without changing either side:



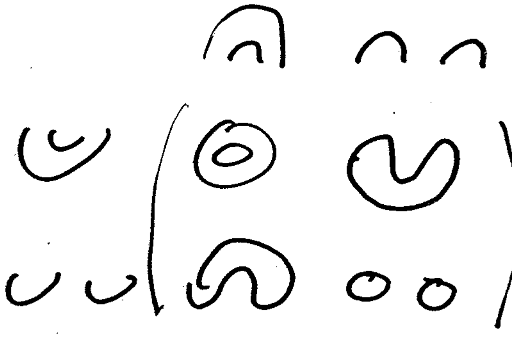
• You can remove a loop from D_2, dividing both sides by $q+q^{-1}$, since

$$\mathrm{qdim}\,\mathrm{Hom}(\bigcirc, \phi) = \mathrm{qdim}\,\mathbb{Z}/2\{$$  $$\}$$

$$= q + q^{-1}$$

The Spider($\mathfrak{sl}_2$) pairing matrix is nondegenerate on diagrams without loops

- Consider the 'pairing matrix':



$$= \begin{pmatrix} (q+q^{-1})^2 & q+q^{-1} \\ q+q^{-1} & (q+q^{-1})^2 \end{pmatrix} = \begin{pmatrix} q^2+\cdots & q+\cdots \\ q+\cdots & q^2+\cdots \end{pmatrix}$$

- The greatest numbers of loops, and hence the greatest powers of q, appear on the diagonal
- Thus the 'diagonal' term in the determinant cannot be cancelled, so $\det\langle -,- \rangle \neq 0$.

- Now we know $\operatorname{qdim} \operatorname{Hom}_{Cob(\mathfrak{sl}_2)}(-,-)$ is nondegenerate too, ruling out any isomorphisms amongst diagrams without loops:

  If $\bigoplus D \cong \bigoplus D'$ then

  $$\operatorname{Hom}(\bigoplus D, A) \cong \operatorname{Hom}(\bigoplus D', A) \text{ for all } A,$$

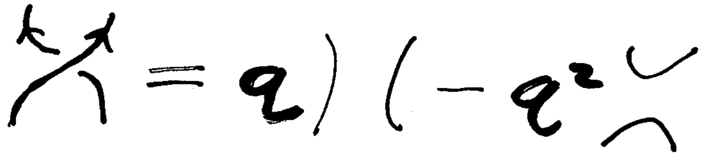
  so $\langle \Sigma D - \Sigma D', A \rangle = 0$ for all A.

Thus $K(Cob(\mathfrak{sl}_2)) \cong \text{Spider}(\mathfrak{sl}_2)$.

This isomorphism is compatible with planar gluings of diagrams, so it's actually an isomorphism of tensor categories not just groups

Braidings

Spider($\mathfrak{sl}_2$) isn't just a tensor category — it's a <u>braided tensor category</u>:

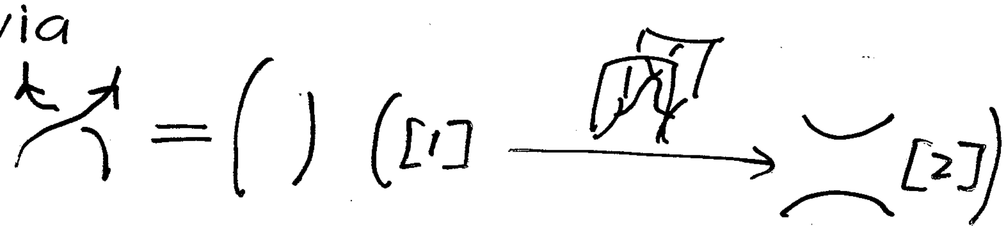


(⇝ the Jones polynomial)

Is this structure reflected in Cob($\mathfrak{sl}_2$)?

Not quite! But it is in $\mathrm{Kom}_{/\mathrm{htpy}}(\mathrm{Cob}(\mathfrak{sl}_2))$ (which, loosely speaking, is the derived category)

via



- That was the content of the Basic Notions talk on Friday!
- Note that $K(\mathrm{Kom}_{/\mathrm{htpy}}(\mathrm{Cob}(\mathfrak{sl}_2)) \cong K(\mathrm{Cob}(\mathfrak{sl}_2)$ $\cong \mathrm{Spider}(\mathfrak{sl}_2)$

  so the derived category is also a categorification of the representation theory, and a "better one", because it reflects the braiding.

Now for $U_q\mathfrak{sl}_3$

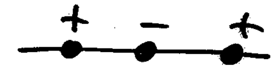
FundRep($U_q\mathfrak{sl}_3$) has objects

V, the standard 3-dimensional representation

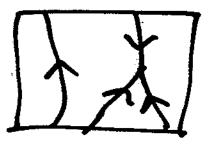
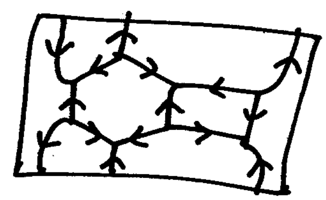
V^* its nonisomorphic dual, $\cong \Lambda^2 V$.

and tensor products of these

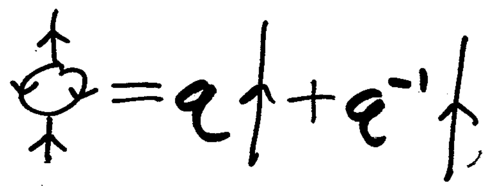
Spider($\mathfrak{sl}_3$) has

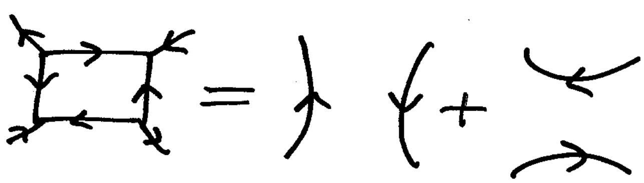
Objects oriented points on a line 

Morphisms oriented trivalent graphs with two types of vertices  and .

<u>Examples</u>  and 

modulo the relations

 $= q^2 + 1 + q^{-2}$, 



<u>Theorem</u> (Kuperberg 96)

There is an equivalence of categories

$$\text{Spider}(\mathfrak{sl}_3) \longrightarrow \text{FundRep}(U_q\mathfrak{sl}_3)$$

 $\longmapsto$ the determinant map

$$V \otimes V \otimes V \longrightarrow A$$

Let's invent $\mathrm{Cob}(\mathfrak{sl}_3)$.

<u>Objects</u> trivalent graphs built from  and  without relations

<u>Morphisms</u> cobordisms with trivalent seams modulo some relations to be determined.

- We want $\operatorname{qdim} \operatorname{Hom}(\phi, \circlearrowright) = \langle \circlearrowright \rangle_{\mathrm{Spider}(\mathfrak{sl}_3)}$

$$= q^2 + 1 + q^{-2}$$

- Let's guess $\deg(C) = 2\chi(C) - \frac{1}{2}\#\partial + \#\curlywedge$

  and $\operatorname{Hom}(\phi, \circlearrowright) = \mathbb{Q}\{$  $\}$

- Declare  $= 3$,  $= 0$.
- Further let's assume

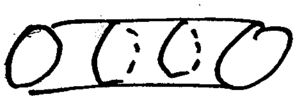
$$\operatorname{Hom}(\circlearrowright, \circlearrowright) \cong \operatorname{Hom}(\phi, \circlearrowright\circlearrowleft)$$

$$\cong \operatorname{Hom}(\phi, \circlearrowright) \otimes \operatorname{Hom}(\phi, \circlearrowleft)$$

(with $\operatorname{qdim} = q^{-4} + 2q^{-2} + 3 + 2q^2 + q^4$)

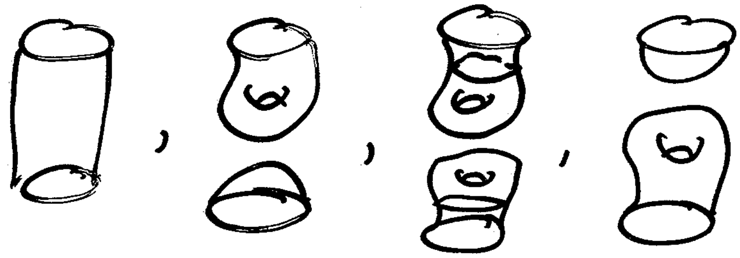
since this is what happened in $\mathrm{Cob}(\mathfrak{sl}_2)$.

- Looking at the degree 4 part, which is 1-dimensional, we must have

 $= \lambda$  ("bamboo")

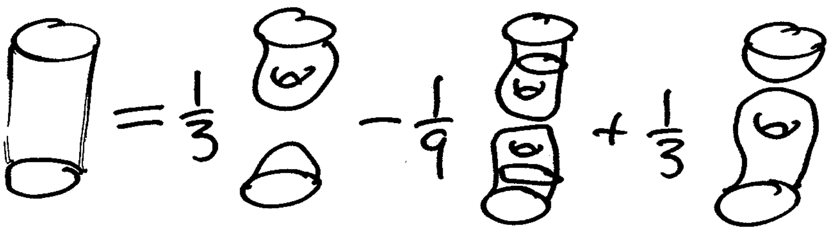
for some λ; lets take $\lambda = -1$.

- Now look at the degree 0 piece; it's 3-dimensional, so there must be a relation between



- By considering different ways to 'cap off' these cobordisms (and the assumptions $\mathrm{qdim}\,\mathrm{Hom}(\phi,\phi)$ and $\mathrm{qdim}\,\mathrm{Hom}(\uparrow,\uparrow)$ are in $\mathbb{N}[q^{-1}]$ not just $\mathbb{N}[q,q^{-1}]$.)

we find



("neck cutting")

- There are some more relations 'in the local kernel' of the ones we've found so far — that is, cobordisms with boundary, all of whose closures give zero.

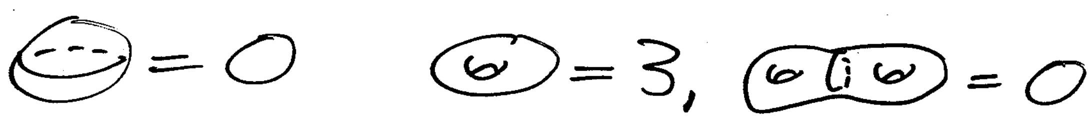
— "tube"

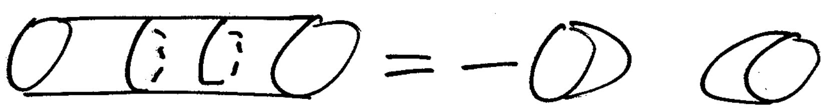
— "rocket"

Cob($\mathfrak{sl}_3$)

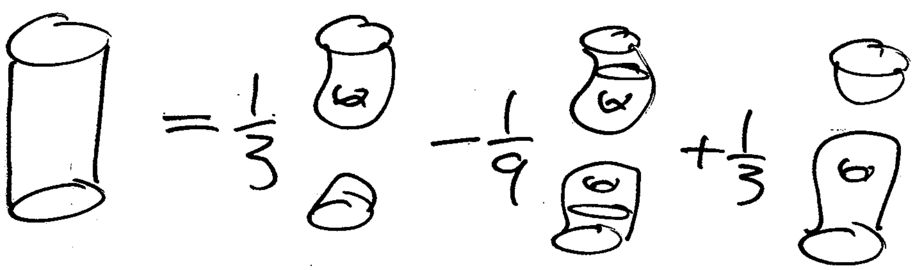
Objects diagrams from Spider($\mathfrak{sl}_3$), no coefficients, no relations.

Morphisms cobordisms with trivalent seams modulo the relations

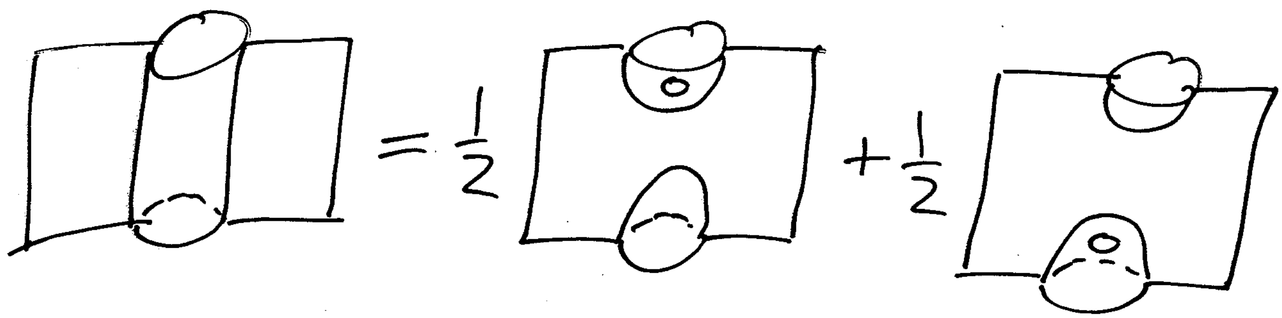




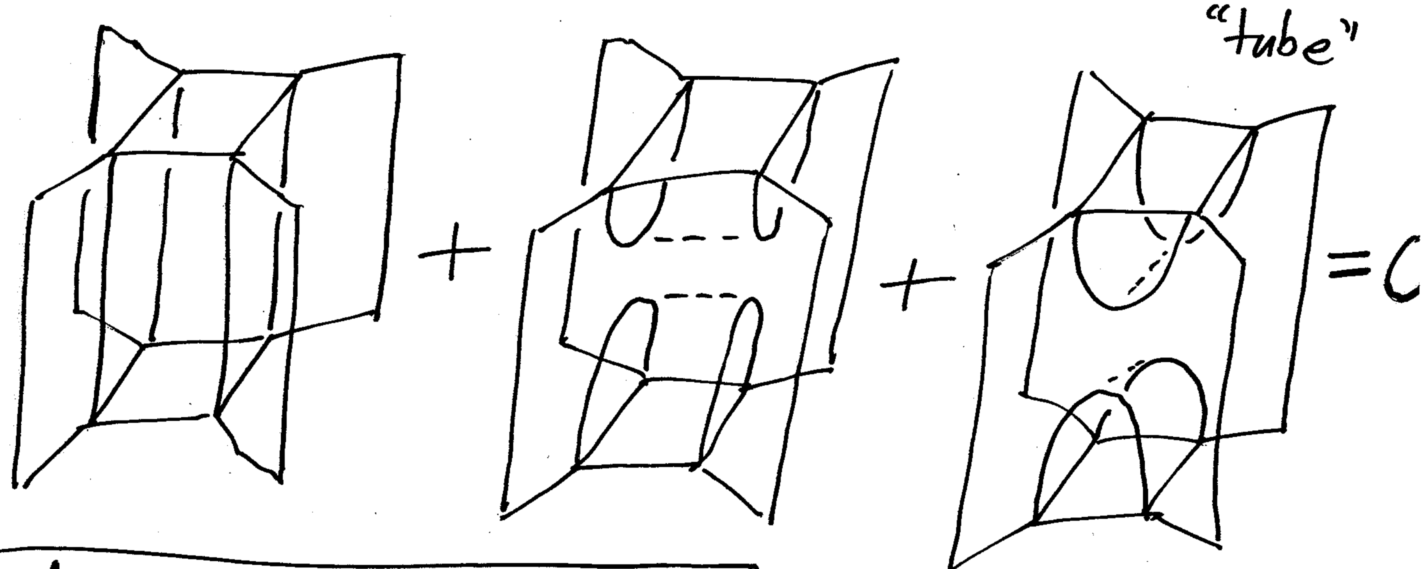
"bamboo"



"neck cutting"



"tube"



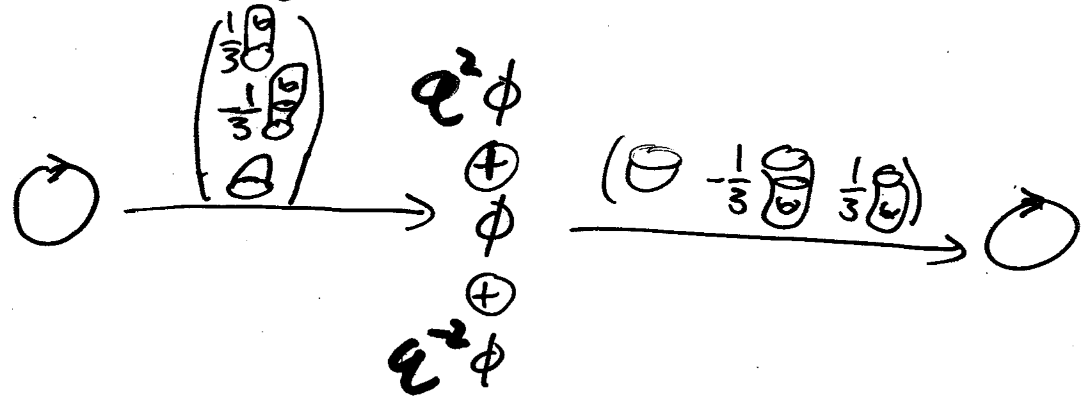
"3 rockets"

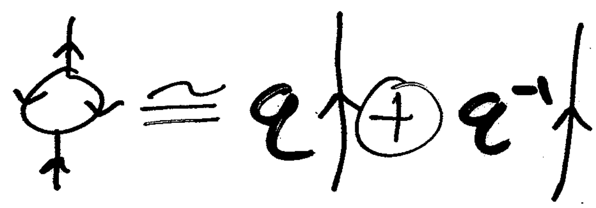
Grading:

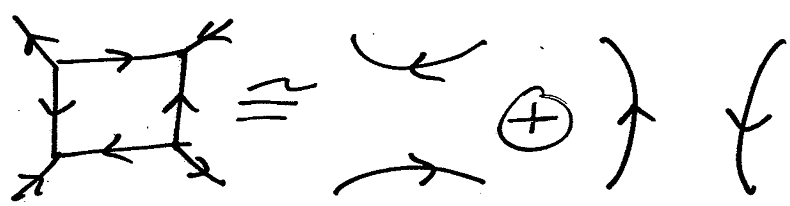
$$\deg(C) = 2\chi(C) - \tfrac{1}{2}\#\partial + \#\text{⅄}$$

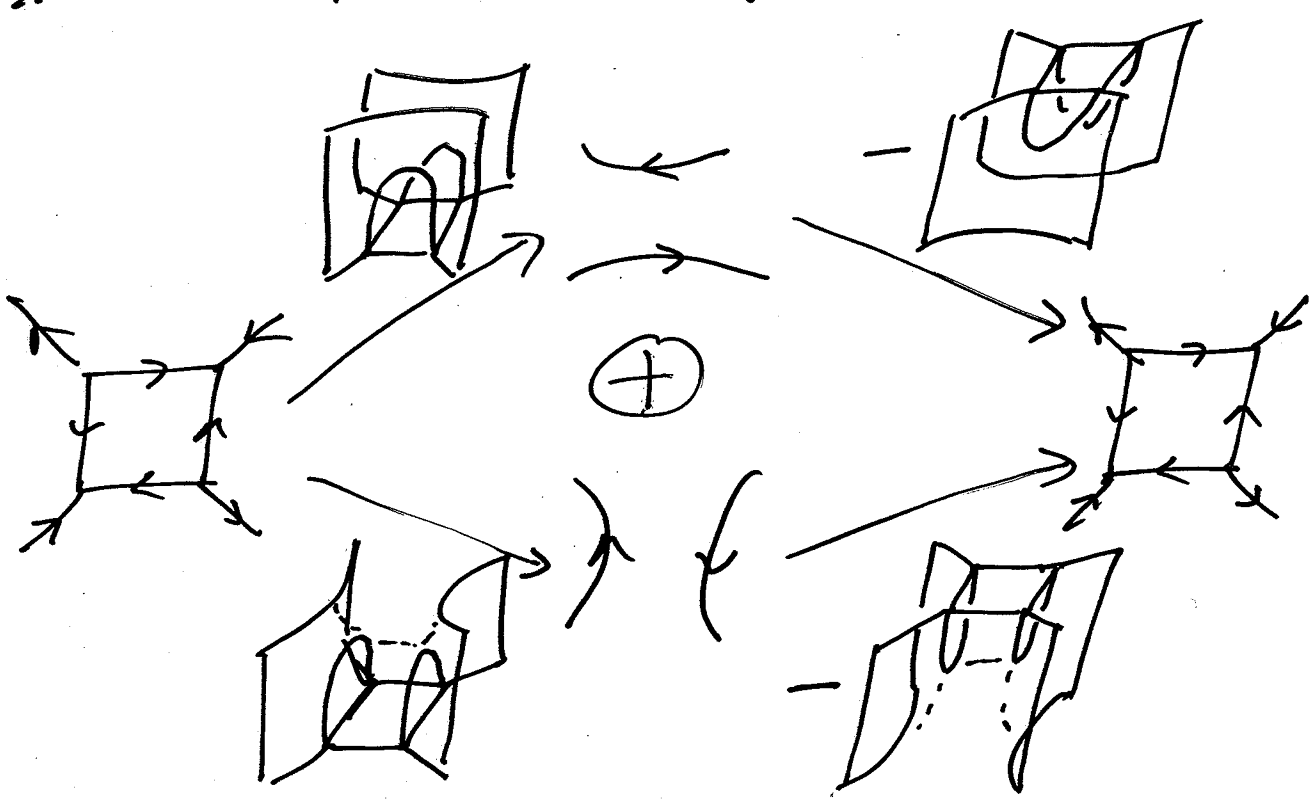
Relations amongst cobordisms imply isomorphisms amongst objects

- $\bigcirc \cong q^{2}\phi \oplus \phi \oplus q^{-2}\phi$ via



-  via the "tube" relation

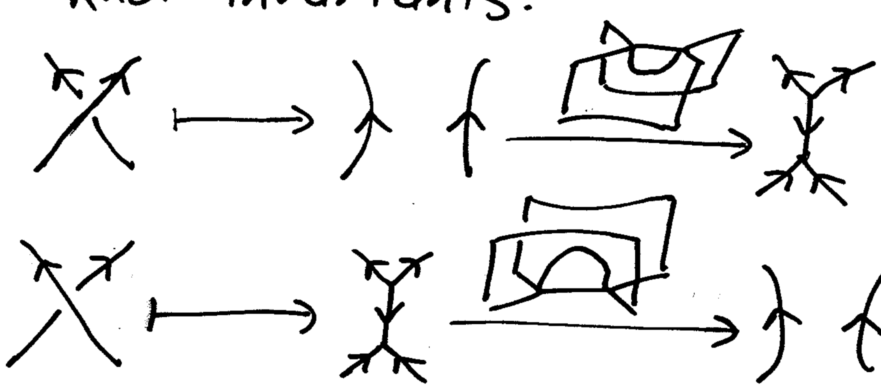
-  via



$Cob(\mathfrak{sl}_3)$

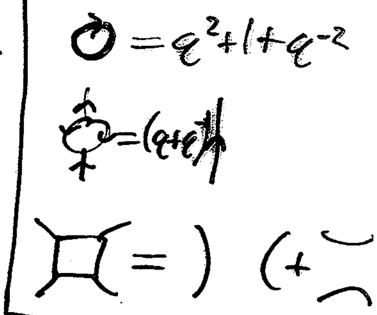
- by the same argument as for $Cob(\mathfrak{sl}_2)$,
  $$K(Cob(\mathfrak{sl}_3)) \cong Spider(\mathfrak{sl}_3)$$
  (although now we actually use some representation theory — I don't know a 'diagrammatic' proof that $\langle -, - \rangle_{Spider(\mathfrak{sl}_3)}$ is nondegenerate)

- $Cob(\mathfrak{sl}_3)$ isn't <u>braided</u>, but $Kom_{/htpy}(Cob(\mathfrak{sl}_3))$ is. This lets us define a link homology theory categorifying the quantum $SU(3)$ knot invariants:



- $Cob(\mathfrak{sl}_3)$ can actually explain (modulo a conjecture!) some representation theory: the difference between Kuperberg's 'non-elliptic diagram' basis for $\mathfrak{sl}_3$, and Lusztig's dual canonical basis.

Recall:

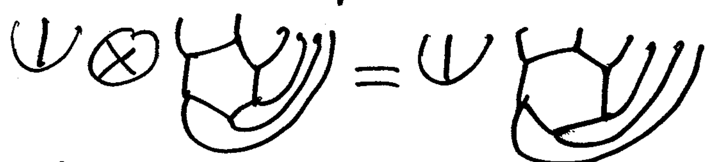


Spider($\mathfrak{sl}_3$) has a natural basis

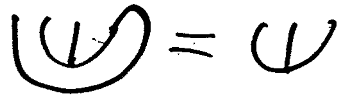
consisting of diagrams without loops, bigons or squares

This basis shares many properties with the dual canonical basis for $\mathrm{Inv}(\bigotimes V^{(*)}) = \mathrm{Hom}(A, \bigotimes V^{(*)})$.

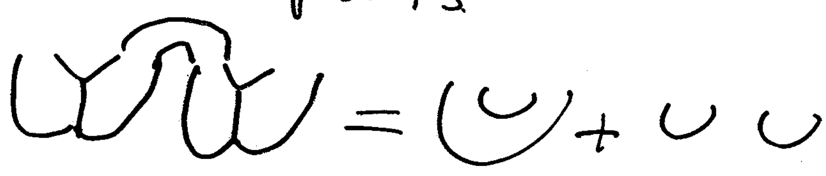
- closed under tensor product



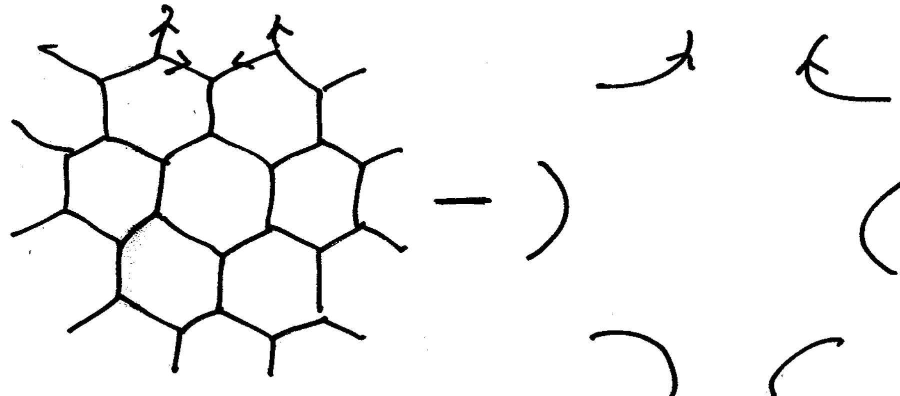
- closed under cyclic permutation of tensor factors:



- if you 'stitch' two basis diagrams together, you get a $\mathbb{N}[q, q^{-1}]$ linear combination of basis diagrams



But they're not the same! They coincide for a long time, until



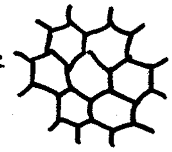
(Kuperberg & Khovanov, '96)

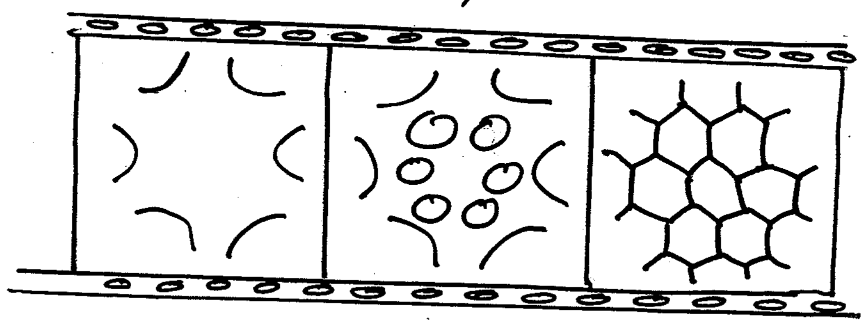
(which is dual canonical, while the first term isn't)

- We can 'enlarge' $Cob(\mathfrak{sl}_3)$ by adding idempotents as extra objects. This is called the 'Karoubi envelope'.

<u>Theorem</u> $K(Kar(Cob(\mathfrak{sl}_3))) \cong K(Cob(\mathfrak{sl}_3))$

<u>Conjecture</u> the minimal idempotents in $Cob(\mathfrak{sl}_3)$ give the dual canonical basis in $Spider(\mathfrak{sl}_3)$.

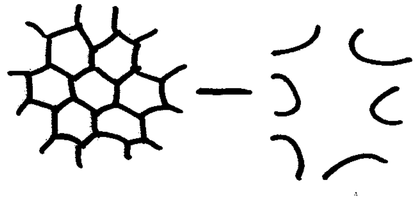
<u>Example</u> The identity cobordism on $H =$ 
is the sum of two minimal idempotents.
Consider the cobordism

$x =$ .

Then $p = x^* x$ is a projection, and
$id_H = p + (1-p)$.

The idempotent p factors through  so the correspondence in the conjecture is

$p \longleftrightarrow$ 

$1-p \longleftrightarrow$ 

Sketch of the theorem

We want to set up a 1-1 correspondence between non-elliptic diagrams (the basis for $K(Cob(\mathfrak{sl}_3))$) and minimal idempotents up to isomorphism (the basis for $K(Kar(Cob(\mathfrak{sl}_3)))$.)

Order the nonelliptic diagrams 'by complexity', $\prec$.

Write $id_D = \sum_\alpha p_\alpha$, as a sum of orthogonal minimal idempotents.

Claim 1 Only one idempotent contains an 'identity cobordism' term. Call this the leading idempotent.

Claim 2 We've seen all the other idempotents, up to isomorphism, as leading idempotents of less complex diagrams.

Proofs something about 'standard forms' of cobordisms between non-elliptic diagrams.