Local Khovanov homology

(Part 3: "Delooping")

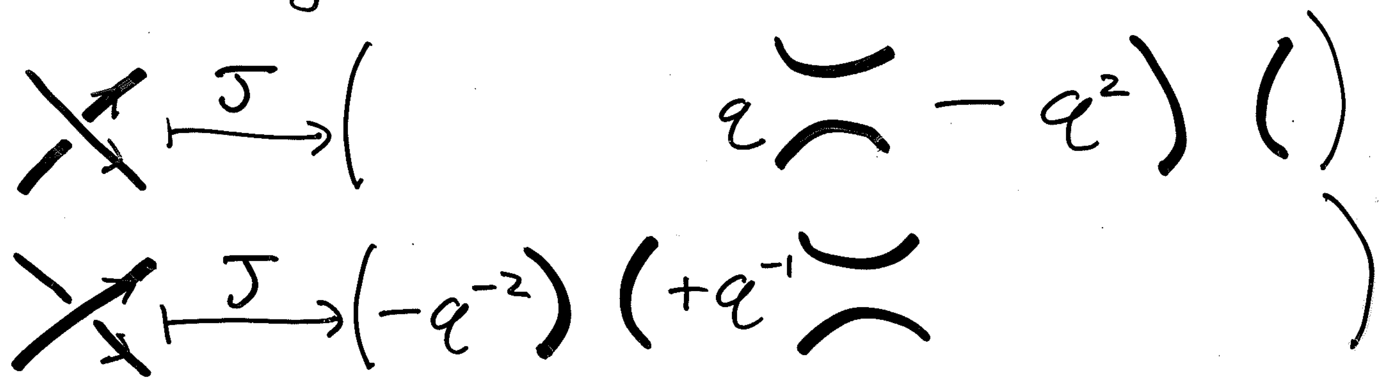
- The Khovanov homology of a knot is the homology of a certain graded complex associated to the knot. The Jones polynomial is the Euler characteristic of this same complex.

- "Local" Khovanov homology uses a different complex, and works for tangles as well as knots. Applying an appropriate functor to the complex of a knot recovers the usual Khovanov homology.

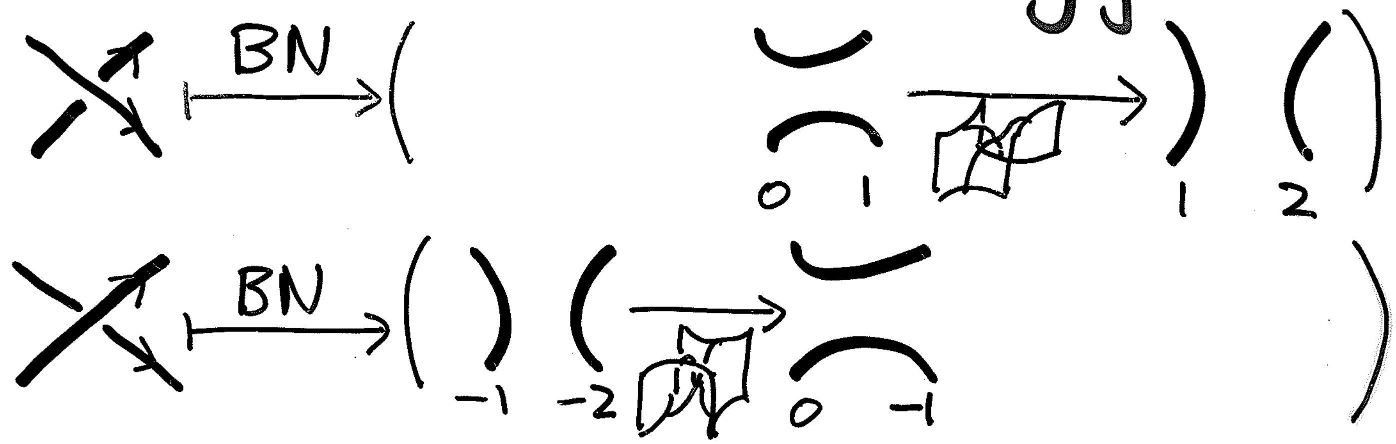
Today:
- definition of the local theory
- 'delooping' allows easy proofs of isotopy invariance
- planar algebra operations allow divide & conquer computations.

The Jones polynomial:

$J : \text{Tangles} \longrightarrow \mathbb{Z}[q,q^{-1}]\text{-Temperley-Lieb}$



Bar-Natan's 'local' homology:



Now the right hand sides are up-to-homotopy complexes of cobordisms between 'grading shifted' **TL** diagrams.

The black numbers tell you the homological level.

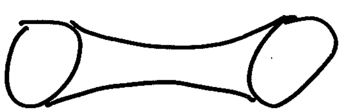
The blue numbers tell you the grading shift.

In fact, the maps in these complexes aren't honest cobordisms, but cobordisms modulo some relations:

"S":  $= 0$ "T":  $= 2$

"G":  $= 0$

and "Neck Cutting":

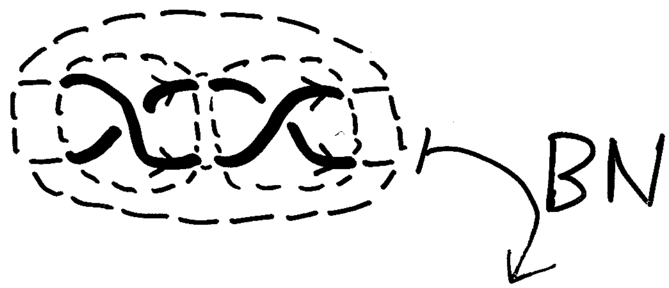
 $= \frac{1}{2}$   $+ \frac{1}{2}$  

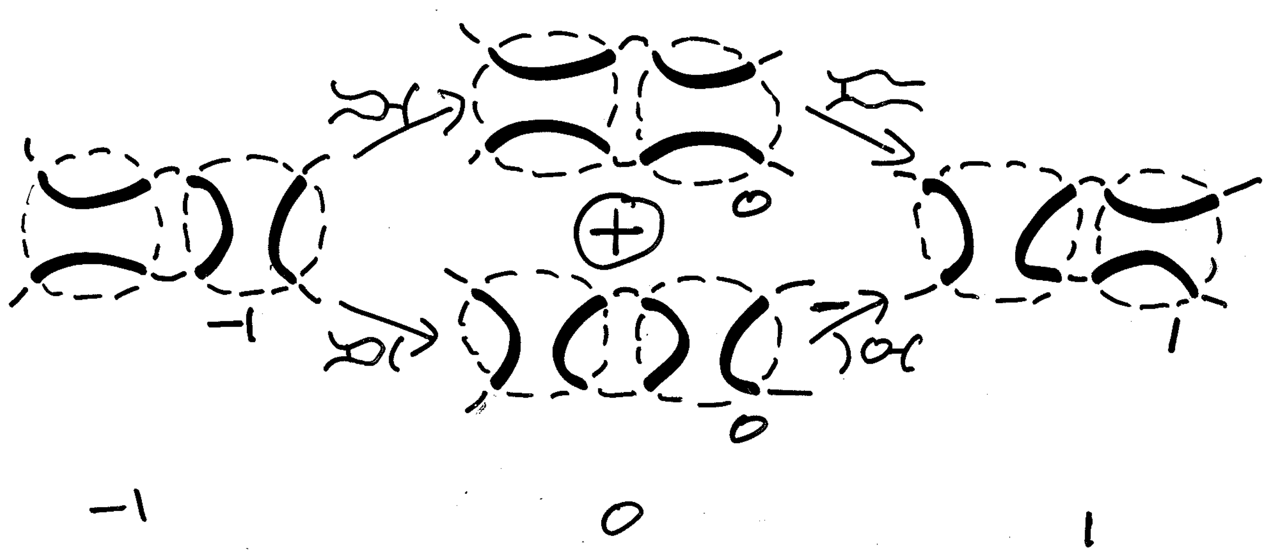
(does it make any difference whether the cobordisms are 'abstract' or 'embedded'?)

Planar algebra operations

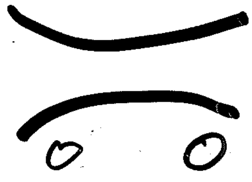
To glue together several complexes by a 'spaghetti and meatballs' diagram, take the tensor product of the complexes, gluing together the constituent TL diagrams and cobordism using the planar diagram.

Thus





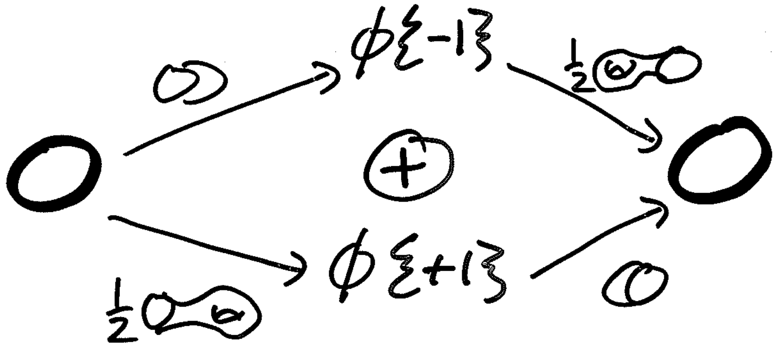
and hopefully this is homotopic to the single step complex



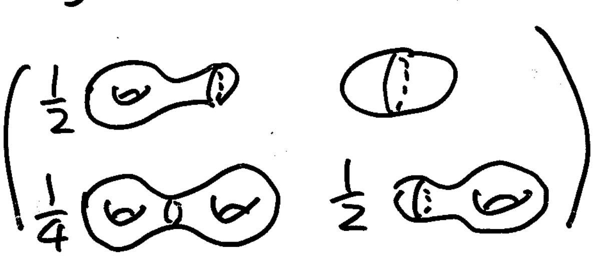
The 'easy' proofs of isotopy invariance.

First, we need a consequence of the 'neck-cutting relation

namely that the object $\bigcirc$ is isomorphic to $\phi\{-1\} \oplus \phi\{+1\}$:



The composition $\bigcirc \longrightarrow \bigcirc$ is the identity by NC, while the composition $\phi\{-1\} \oplus \phi\{+1\} \longrightarrow \phi\{-1\} \oplus \phi\{+1\}$ is



$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

using T, S and G.

Replacing $\bigcirc$ objects in a complex in this manner is called 'delooping'.

Second, we need the <u>reduction lemma</u>

If $\psi: b_1 \to b_2$ is an isomorphism, there's an isomorphism of complexes:

$$[C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\psi & \delta \\ \gamma & \varepsilon\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{(\mu \ \nu)} [F]$$

$$\cong$$

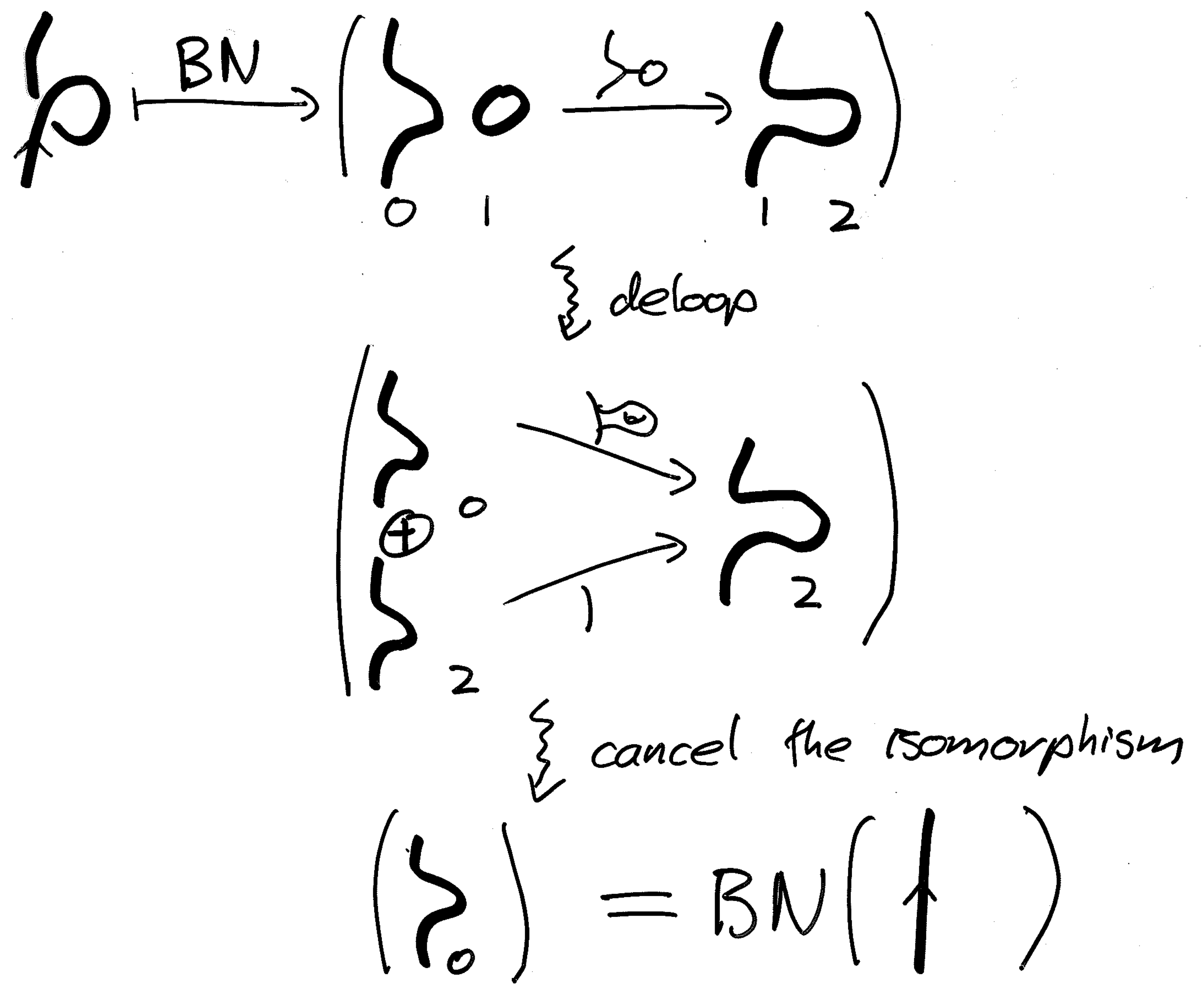
$$[C] \xrightarrow[\begin{pmatrix}0\\ \beta\end{pmatrix}]{} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow[\begin{pmatrix}\psi & 0 \\ 0 & \varepsilon-\gamma\psi^{-1}\delta\end{pmatrix}]{} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow[(0 \ \nu)]{} [F]$$

This second complex has a contractible direct summand $[b_1] \xrightarrow{\psi} [b_2]$ and so up to homotopy it's just

$$[C] \xrightarrow{(\beta)} [D] \xrightarrow{(\varepsilon-\gamma\psi^{-1}\delta)} [E] \xrightarrow{(\nu)} [F]$$

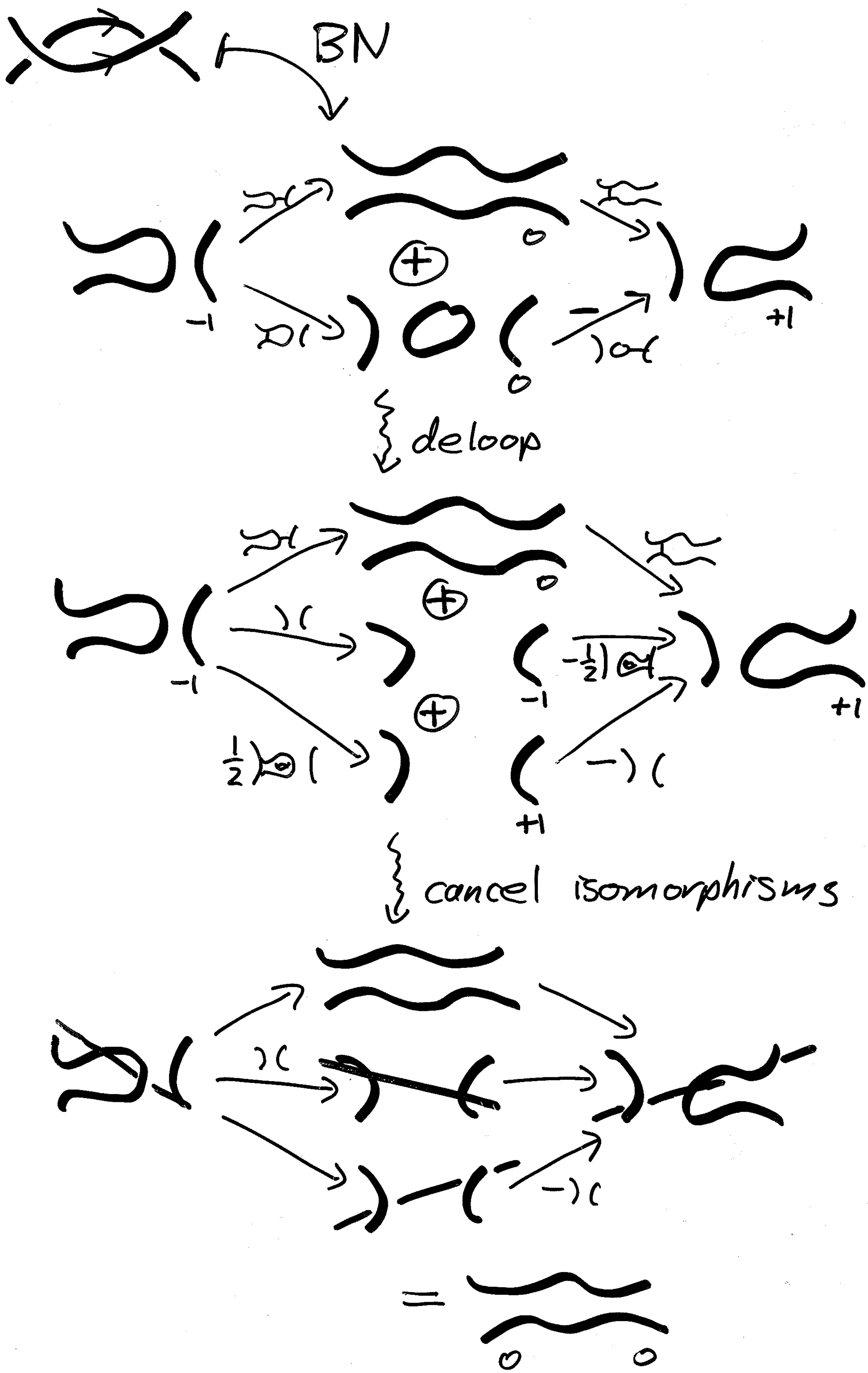
This trick will make our up-to-homotopy complexes almost as manageable as the Jones polynomial itself!

Okay — now we're ready for the 'modern' isotopy invariance proofs.

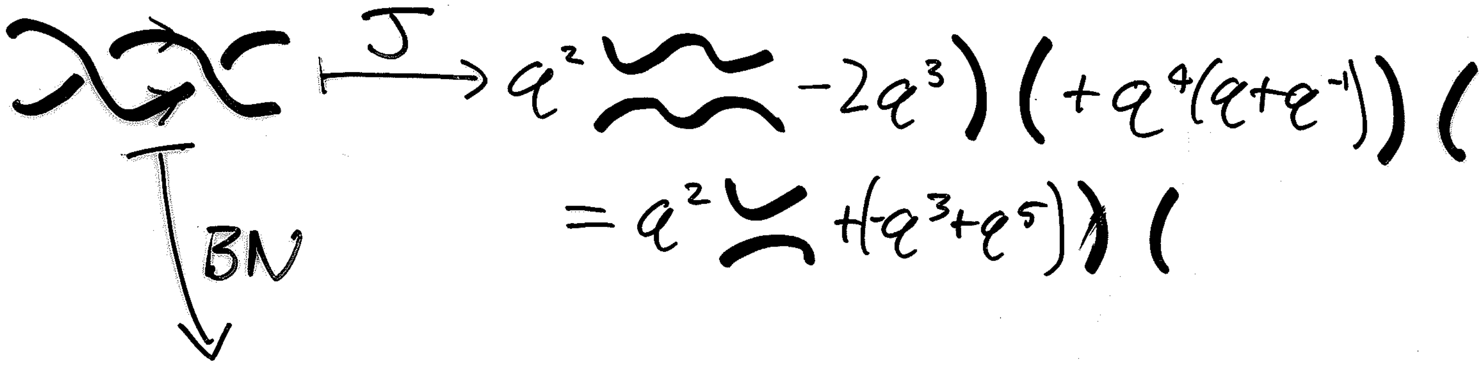


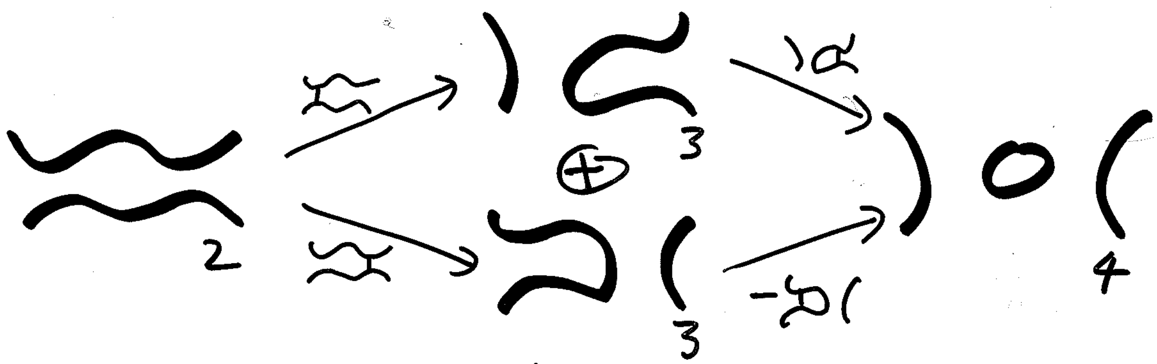
Neat, huh?

Make sure you understand the morphisms in the complex in the middle line!

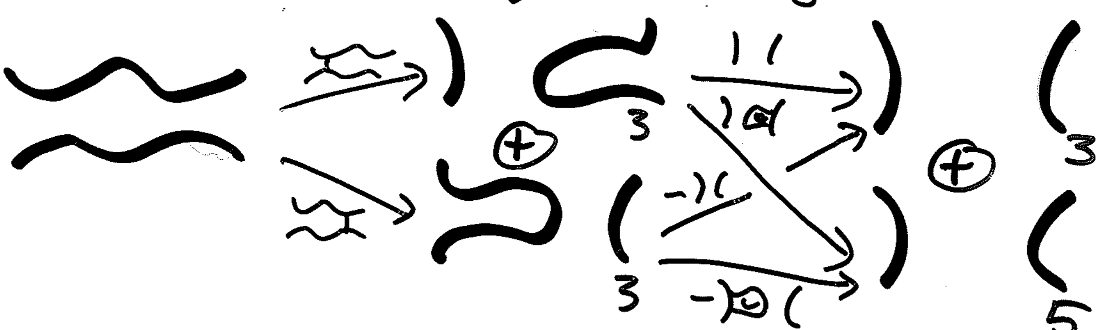

BN
deloop
cancel isomorphisms

And something more interesting:





delooping



reduction