Khovanov homology over $\mathbb{C}[[\varepsilon]]$

- Recall in Bar-Natan's cobordism model of Khovanov homology, we imposed relations:

  \[
  \begin{align*}
  \varepsilon \cdot 0 &= 0 \\
  \varepsilon \cdot \varepsilon &= 2 \\
  \varepsilon \cdot \varepsilon + \frac{1}{2} 0 &= \frac{1}{2} \varepsilon \cdot 0 + \frac{1}{2} \varepsilon \\
  \text{and} & \quad \varepsilon \cdot \varepsilon = 0
  \end{align*}
  \]

- The last one is actually unnecessary: it's just there to make Hom spaces finite dimensional over $\mathbb{C}$.

  But it's a bad tradeoff—instead, we should write $\alpha = \varepsilon \cdot 0$ and absorb it into the coefficient ring.
• First a little calculation from yesterday:

\[
\frac{1}{2} + \frac{1}{2} = 2
\]

So \[
= 0.
\]

• Then

\[
\frac{1}{2} \frac{1}{2} + \frac{1}{2} \frac{1}{2} = \frac{\alpha}{2}
\]

• We can run this both ways, eliminating even numbers of handles for \(\alpha\), or ‘reattaching’ \(\alpha\) to a sheet. Thus we have two descriptions of \(\text{Hom}(, , )\):

\[
\text{Hom}(, , ) = \mathbb{C}[[\alpha]] = \mathbb{C}[\alpha][\square, \square]
\]
Cutting open the knot

• Before calculating Khovanov homology, cut open the link at a chosen point, producing a “1-1 tangle”.

• Each resolution of a 1-1 tangle looks like \( \text{\includegraphics[width=0.2\textwidth]{tangle.png}} \); a single “through strand” with circles on either side.

• We can take the complex for a 1-1 tangle, e.g.

\[
\text{Kh}_{\text{cut}}(\text{\includegraphics[width=0.2\textwidth]{tangle.png}}) = \text{Kh}\left(\frac{1}{\text{\includegraphics[width=0.2\textwidth]{tangle.png}}}\right)
\]

\[
\cong a^{-2} \oplus \left(\begin{array}{c}
\vdots
\end{array}\right) \oplus \left(\begin{array}{c}
\vdots
\end{array}\right) \oplus \left(\begin{array}{c}
\vdots
\end{array}\right)
\]

and “deloop” it, so every object is a direct sum of strands:

\[
\cong a^{-2} \oplus \left(\begin{array}{c}
\vdots
\end{array}\right) \oplus \left(\begin{array}{c}
\vdots
\end{array}\right) \oplus \left(\begin{array}{c}
\vdots
\end{array}\right)
\]
Gaussian elimination over $C[t]$:

- We saw yesterday how to simplify the complex for a tangle by “Gaussian elimination,” successively stripping off invertible matrix entries, turning them into contractible direct summands.

- Today, this won’t get us as far: $\mathbf{E}$ is not invertible. Nevertheless, we can strip off the lowest power of $\mathbf{E}$.

If $k \leq l, m \in \mathbb{N}$, and $\varphi$ is an isomorphism, there’s an isomorphism of complexes:

\[
\begin{align*}
A & \xrightarrow{(0)} B \oplus \left( \begin{array}{cc} t^k \varphi & t^k \lambda \\ \mu & \nu \end{array} \right) \oplus (\delta \varepsilon) \rightarrow C \oplus E \rightarrow F \\
\end{align*}
\]

- Since our matrix entries are homogeneous elements of $C[\mathbf{E}]$, they’re all either $0$ or $\mathbf{E}^k \varphi$ for some $k \in \mathbb{N}$, $\varphi \in \mathbb{C}_x$. 

Diagram:

\[
\begin{align*}
A & \xrightarrow{(0)} B \oplus (0) \rightarrow C \oplus D \oplus (0) \rightarrow E \oplus (0) \rightarrow F \\
\end{align*}
\]
The Gaussian elimination lemma decomposes our complex into direct sums of the following short complexes:

\[ E = \) \]

\[ C_n = q^{-2n} \rightarrow \) \]

each with grading a homological height shifts attached. Thus at the level of complexes,

\[ \text{Kh}^{\text{cut}}(L) = \sum_{x,r} a_{x,r} q^{x} t^{-r} E + \sum_{x,r,n} b_{x,r,n} q^{x} t^{-r} C_n \]

- Taking homology (implicitly replacing ) with \( \text{Hom}(\), \))

\[ H^\bullet() = C[t] \quad (\text{in height } \bullet = 0) \]

\[ H^\bullet(q^{-2n}) \rightarrow \) \]

There's one generator in homology for each indecomposable complex, and it's either "free" or "t-torsion".
The fancy version of all this would say

- "$C[t]" has homological dimension 1"
- "\(\text{Kom}(C[t]\text{-modules})\) is Krull-Schmidt,"
  i.e. has unique decomposition into indecomposables

Observe \(E\) and \(C_n\) are not simple.
(exercise, write down all the chain maps between them, which ones are homotopically trivial?)

Be careful with "torsion" here: everything is over \(C\), so this is orthogonal to the also-interesting notion of \(\mathbb{Z}\)-torsion in Khovanov homology. Nothing I'm saying today really works over \(\mathbb{Z}\).
Recovering the usual invariant:

- We need to close the cut open invariant, and set $\alpha = 0$.

\[ E = \begin{array}{c}
\text{close} \\
\text{delooping}
\end{array} \xrightarrow{\phi} q^{2} \phi \oplus q^{-2} \phi
\]

\[ \text{take Hom} (q^{2} \phi \oplus q^{-2} \phi)
\]

\[ C_{1} = q^{-2} \rightarrow \begin{array}{c}
\text{close} \\
\text{delooping}
\end{array} q^{-2} \phi \oplus (0 1) q^{-1} \phi
\]

\[ \text{htry} q^{-2} \phi \xrightarrow{\alpha = 0} q^{2} \phi
\]

\[ \text{take Hom} q^{-2} \phi \oplus (q^{-1} \phi \oplus q^{2} \phi)
\]

(without setting $\alpha = 0$, we would have just got $q \mathbb{C}[\alpha]/\alpha = 0$.)
Thus in classical Khovanov homology

E contributes

\[ \begin{array}{c}
2c \\
\rightarrow\end{array} \]

\[ \begin{array}{c}
c \\
\rightarrow\end{array} \]

often called an “exceptional pair”

\[ \begin{array}{c}
C_1 \text{ contributes} \\
2c \\
\rightarrow\end{array} \]

\[ \begin{array}{c}
c \\
\rightarrow\end{array} \]

called a “knight’s move”

C_2 contributes

\[ \begin{array}{c}
C \\
\rightarrow\end{array} \]

\[ \begin{array}{c}
C \\
\rightarrow\end{array} \]

indistinguishable from a pair of knight’s moves.

C_3 contributes

\[ \begin{array}{c}
C \\
\rightarrow\end{array} \]

\[ \begin{array}{c}
C \\
\rightarrow\end{array} \]

and so on.

(However, as far as I know C_3 hasn’t yet been seen in the wild.)
Some “phenomenology”

• In the homology of a knot, there's a single copy of $E$, in homological height 0. Its $q$-grading is the “Rasmussen s-invariant.”

• In alternating knots, only $C_1$ shows up, and they're all “on diagonal”:

$$Kh_{\infty}(K_{\text{alternating}}) = q^5 E + \sum b_r q^{s+2r} t^r C_1$$

• $C_2$ first appears in $S_{19}$, and is eventually common in non-alternating knots, especially closures of positive braids and torus knots.

• I haven't seen a $C_3$ yet, but there's no reason not to expect them?
Lee homology

- We can recover Lee homology by setting
  \[ \tilde{Z} = \mathbb{C}^\times. \]
  First, notice
  - \( \tilde{Z} \) is now invertible:
    \[ \tilde{Z} \times \frac{2}{\mathbb{Z}} \tilde{Z} = \frac{2}{\mathbb{Z}} \mathbb{Z} = \mathbb{Z} \]
    \[ \square = \frac{1}{2} \square + \frac{1}{\sqrt{2} \mathbb{Z}} \]
    \[ \square = \frac{1}{2} \square - \frac{1}{\sqrt{2} \mathbb{Z}} \]
  - \( \square \) splits into orthogonal idempotents:
  - What happens?
    \( \mathbf{E} \) survives
    each \( \mathbf{C}_n \) becomes contractible.
  - People used to say "Lee homology is boring,"
    (2 \# components - 1 copies of \( \mathbf{E} \))
    "but boring in an interesting way."
    There's a spectral sequence
    \( E_2 = \mathbf{C} \mathbf{K} \mathbf{h}, \ E_2 = \mathbf{K} \mathbf{h}, \ldots \ E_\infty = \mathbf{K} \mathbf{h}_{\text{Lee}} \)
  - This is easy to understand using \( \mathbf{K} \mathbf{h} \).
After delooping the complex for a 1-h tangle, each matrix entry appearing in a differential is an element of

$$\text{Hom}(\), () = \mathbb{C}[\alpha]\{\square, [\text{red}]\}$$

We can write the differential as

$$d = d_0 + \alpha d_4 + \alpha^2 d_8 + \ldots$$

Associated to this is a spectral sequence

$$(\text{CKh}_1, d_0) \Rightarrow (H^*(\mathbb{C}, d_0), d_4^*) \Rightarrow (H^*(H^*(\mathbb{C}, d_0), d_4^*), d_8^*)$$

The second page is normal Khovanov homology:

at $\alpha = 0$, $d_0 = d$.

The $\infty$-page is $H^*(\text{CKh}_1, d_0 + d_4 + d_8 + \ldots)$, which is Lee homology; at $\alpha = 1$, $d = d_0 + d_4 + d_8 + \ldots$.
Don’t be scared, each step of this spectral sequence really is a complex!

First, write

\[ d^2 = d_0^2 + \alpha (d_4 d_0 + d_0 d_4) + \alpha^2 (d_8 d_0 + d_4 d_4 + d_0 d_8) + \ldots = 0. \]

Next, observe \( d_4 : \ker d_0 \to \). \( d_0 x = 0 \Rightarrow d_0 d_4 x = -d_4 d_0 x = 0. \)

On \( \ker d_0 / \text{im} d_0 \), \( d_4^2 = 0 \):

\[ d_4^2 x = -d_8 d_0 x - d_0 d_8 x = -d_0 d_8 x \in \text{im} d_0. \]

and so on for higher levels....
Genus bounds from the $s$-invariant.

- Write $E(L)$ for the "$E$ part" of $\text{Kh}_{2h}(L)$.
- A cobordism $\Sigma : L_1 \rightarrow L_2$ gives a map of grading $\chi(\Sigma)$; $\text{Kh}(\Sigma) : \text{Kh}(L_1) \rightarrow \text{Kh}(L_2)$.

If $\Sigma$ is connected, this map is nonzero when restricted to $E(L_1) \rightarrow E(L_2)$.

(there's some work to do there!)

- For $L_1 = 0$ and $L_2 = K$, a knot,

$$\text{Kh}(\Sigma) : E \rightarrow q^{s(K)} E$$

Calculating gradings, $\chi(\Sigma) = s(K) - 2k$, so $\chi(\Sigma) \leq s(K)$.

The same argument with the mirror image says $\chi(\Sigma) \leq -s(K)$, so $\chi(\Sigma) \leq -|s(K)|$.

- For a cobordism $\Sigma' : \phi \rightarrow K$ we have

$\chi(\Sigma') \leq -|s(K)| + 1$. 
Genus bounds for links, too!

• First switch to the ‘framing grading’ $t \mapsto \tau^2 q^{-3}$

• To obtain a genus bound for surfaces inducing an orientation differing in writhe from the original by $\Delta$, look at all copies of $E$ in height $\tau^{-\Delta}$:

$$\chi(\Sigma) \leq -\min_{\tau^{-\Delta}q \cdot E}(r) + 1$$

and

$$\chi(\Sigma) \leq \max_{\tau^{-\Delta}q \cdot E}(r) + 1$$

• Unfortunately, this can be very weak. For example, if $\sum r^3 = 3 - 1, 13$ we only learn $\chi(\Sigma) \leq 2$. 
An Example.

\[ \text{Kh}_{\text{L4al}} \left( \begin{array} \{ \text{\text{\text{}}} \end{array} \right) = q^3 E + L^6 C + q^{-1} L^8 E \]

• Thus \( \chi \leq -3 + 1 = -2 \) for surfaces inducing the given orientation, which is sharp:

• The other orientation differs in writhe by \(-8\), so we look at the \( q^{-1} L^8 E \) term, obtaining \( \chi \leq 0 \), also sharp: