Generators and relations for $\text{Rep}U_q^{+}\mathfrak{sl}_n$ as a pivotal category.

Scott Morrison, June 2005
UC Berkeley
\[ F S_n = \langle \text{objects: } 1, 2, \ldots, n-1, \quad k^* = n-k, \quad a + b + c = n, \quad a + b + c = n \rangle \text{, free pivotal category} \]

\[ R \]

\[ \text{FundRep}\, U_q \mathfrak{sl}_n = \begin{cases} \text{objects— tensor products of fundamental representations of } U_q \mathfrak{sl}_n \\ \text{morphisms— maps between representations} \end{cases} \]
$$FS_n = \left\{ \begin{array}{l}
\text{objects: } 1, 2, \ldots, n-1 \quad k^* = n-k \\
\text{free pivotal category}
\end{array} \right. $$

$$\xrightarrow{R}$$

$$\xrightarrow{id_{V_a}}$$

$$\xrightarrow{\text{the duality pairing}}$$

$$V_a \otimes V_{n-a} \rightarrow C(q)$$

$$\xrightarrow{\text{the } q\text{-determinant}}$$

$$V_a \otimes V_b \otimes V_c$$

$$\bigwedge_{a+b+c=n} V_i \cong C(q)$$

$$\text{FundRep} U_q \mathfrak{sl}_n = \begin{cases} 
\text{objects} - \\
\text{tensor products of fundamental representations of } U_q \mathfrak{sl}_n \\
\text{morphisms} - \\
\text{maps between representations}
\end{cases}$$
Q: What is the kernel of \( \mathbb{R} \)?

For \( \mathfrak{sl}_2 \) and \( \mathfrak{sl}_3 \), the answer has been known for a while:

\[
\text{FundRep} U_q \mathfrak{sl}_2 \cong \text{the Temperley-Lieb category}
\]

(= the free pivotal category with one object, no extra generating morphisms, modulo \( O = [2] \))

and (Kuperberg, '96)

\[
\text{FundRep} U_q \mathfrak{sl}_3 \cong
\]

\[O = [3] = O\]

\[O = [2]\]

+ \[= \]
$R$ is a full functor

① Kuperberg's argument for the $\mathfrak{sl}_3$ case still works.
   (recognise the image of $FS_n$ under $R$ as a category of representation by a Tannaka-Krein theorem)

② There's also a more direct proof.
   (using the fact that $B_n \rightarrow \text{End}(V_{\otimes^n})$
   via $R$-matrices is surjective, and the centre of $\text{SL}(n)$.)
$FS_n \xrightarrow{\mathcal{R}} \text{FundRep}_{U_q \widehat{sl}_n} \xrightarrow{\text{GT}} \text{Rep}_{U_q \widehat{sl}_{n-1}}$

'forget from $U_q \widehat{sl}_n$ down to $U_q \widehat{sl}_{n-1}$'

$V_a \quad \Rightarrow \quad V_{a-1} \oplus V_a$
\[ \text{FundRep } U_2 \leq \mathfrak{sl}_n \xrightarrow{\text{GT}} \text{Mat}(\text{FundRep } U_2 \leq \mathfrak{sl}_n) \]

\[ \text{Rep } U_2 \leq \mathfrak{sl}_n \xrightarrow{\text{GT}} \text{Rep } U_2 \leq \mathfrak{sl}_{n-1} \]

\[ \text{forget from } U_2 \leq \mathfrak{sl}_n \text{ down to } U_2 \leq \mathfrak{sl}_{n-1} \]

\[ V_a \xrightarrow{} V_{a-1} \oplus V_a \]
Example

A map \( \varphi: V_a \rightarrow V_b \otimes V_c \) becomes a map

\[
\Gamma_T(\varphi): V_{a-1} \oplus V_a \rightarrow V_{b-1} \otimes V_c \oplus V_b \otimes V_c
\]

which we can write as a matrix of maps between the direct summands

\[
\begin{pmatrix}
V_{a-1} & V_a \\
V_{b-1} \otimes V_c & \varphi_{11} & \varphi_{12} \\
V_b \otimes V_{c-1} & \varphi_{21} & \varphi_{22} \\
V_b \otimes V_c & \varphi_{31} & \varphi_{32} \\
V_b \otimes V_c & \varphi_{41} & \varphi_{42}
\end{pmatrix}
\]
\[ \text{Fund Rep} \mathfrak{u}_k \mathfrak{s}_{\mathfrak{l}_n} \xrightarrow{GT} \text{Mat}(\text{Fund Rep} \mathfrak{u}_k \mathfrak{s}_{\mathfrak{l}_n}) \]

*Notice that \( GT \) is faithful.*
\[
\begin{align*}
\text{FS}_n & \quad \text{Mat}(\text{FS}_{n-1}) \\
R_n & \quad \text{Mat}(R_{n-1}) \\
\text{Fund Rep}_k \mathfrak{S}_n & \quad \text{GT} \quad \text{Mat}(\text{Fund Rep}_k \mathfrak{S}_n) \\
\end{align*}
\]

- Notice that $\text{GT}$ is faithful.
- We can extend $R_{n-1}$ to work with matrices of diagrams.
\[ \begin{align*}
\text{Fund Rep } U_{\mathfrak{sl}_n} &\xrightarrow{\text{GT}} \text{Mat}(\text{Fund Rep } U_{\mathfrak{sl}_n}) \\
\text{FS}_n &\xrightarrow{? \text{dGT}} \text{Mat}(\text{FS}_{n-1})
\end{align*} \]

- Notice that GT is faithful.
- We can extend \( R_{n-1} \) to work with matrices of diagrams.

**Question** Can we 'lift' the Gelfand-Tsetlin functor to a functor defined (combinatorially) on diagrams?

With such a lift, we can hope to recursively determine the relations:

\[ \ker(R_n) = \text{dGT}^{-1}(\ker(\text{Mat}(R_{n-1}))) \]
The ‘diagrammatic Gel'fand-Tsetlin’ functor is determined by its values on generating morphisms:

\[ d_{\text{GT}}(\begin{array}{c} a \\ \end{array}) = \begin{pmatrix} a-1 & 0 \\ 0 & a \end{pmatrix} \]

\[ d_{\text{GT}}(\begin{array}{c} a \\ b \\ \end{array}) = \begin{pmatrix} 0 & 0 & 0 \\ a-1 & a & 0 \\ a & a & 0 \end{pmatrix} \]

\[ d_{\text{GT}}(\begin{array}{c} a \\ b \\ c \\ \end{array}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a-1 & a & a & 0 \\ a & a & a & 0 \\ a & a & a & 0 \end{pmatrix} \]
It's easy to calculate $\text{dGT}$.

- Arrange the orientations in a diagram so every vertex looks like $\bigtriangledown$ or $\bigtriangleup$.
- For every subset of the boundary, consider all collections of nonoverlapping paths with that boundary.
- Each matrix entry is a sum of "reductions along the paths", with coefficients.

**Example** for $SL_3$.

\[
\text{dGT} \left( \begin{array}{cc}
2 & 1 \\
-1 & 1
\end{array} \right) = \left( \begin{array}{cc}
\frac{1}{2} & 0 \\
0 & 2q + q^{-1}
\end{array} \right)
\]

\[
= \left( \begin{array}{cc}
\frac{1}{2} & 0 \\
0 & 2q + q^{-1}
\end{array} \right)
\]

and we see

\[
R_3 \left( \begin{array}{c}
2 \\
1
\end{array} \right) = [2] R_3 \left( \begin{array}{c}
1
\end{array} \right)
\]
An $s_4$ Example (at $q=1$, for simplicity)

Let's look for a relation of the form

$$\alpha + \beta \gamma + \delta = 0$$

We'll apply $dGT$, then pick out just one matrix entry

$$E = dGT(\alpha + \beta \gamma + \delta) = \alpha 0 + \beta \gamma + \delta$$

For this to be zero, we must have

$$\beta = -\gamma = \delta,$$

by Kuperberg's 'square' relation for $s_3$. 

Looking at another matrix entry, for example $E_{\alpha \beta}$, we find $-\alpha = \beta = -\gamma$.

Thus, for there to be a relation amongst these diagrams, the coefficients must be alternating: $-\alpha = \beta = -\gamma = \delta$.

Checking all the matrix entries, we find

$$R_{n-1}(dGT(\begin{array}{c}
\begin{array}{c}
- & + & - \\
2 & 3 & 3 \\
2 & 2 & 2
\end{array}
\end{array})) = 0$$

and therefore

$$R_n(\begin{array}{c}
\begin{array}{c}
- & + & - \\
2 & 3 & 3 \\
2 & 2 & 2
\end{array}
\end{array}) = 0$$

providing us with a relation in the $SL_4$ representation theory.
\[
\text{Ker } R_n
\]

\[
\begin{align*}
& \quad - \sum_{k} C_{\tilde{a}, \tilde{b}; k, l} \\
& \quad - \frac{2a+1}{2b} \sum_{k=-2b} \sum_{\tilde{d}_{\tilde{a}, \tilde{b}; i, j, k, l}} \\
& \quad \text{for each } j = n-1+\Sigma a, \ldots, \Sigma b \\
\end{align*}
\]

where \( d_{\tilde{a}, \tilde{b}; i, j, k, l} = (-1)^{k+i} \left[ \frac{n-k-j-\max(a)}{n-1-j-\Sigma a} \right] \left[ \frac{k+j+\min(b)}{j+\Sigma b} \right] \)

and \( C_{\tilde{a}, \tilde{b}; i, j, k, l} = \left[ \frac{n-\Sigma a+\Sigma b}{2b+k+l} \right] \) if \( n-\Sigma a+\Sigma b \geq 20 \)

or \( (-1)^{k+l} \left[ \frac{\Sigma a-n+1+k+l}{2b+k+l} \right] \) otherwise.
Applications

• Efficient evaluation of coloured quantum knot invariants for $\mathfrak{su}(n)$.

\[ \begin{array}{c}
\text{a} \quad \text{b}
\end{array} \xrightarrow{R} \sum \left( \begin{array}{c}
\text{c}
\end{array} \right) \]

• The subfactors associated to some fundamental representations of a certain root of unity have intermediate subfactors.

• (Optimistically)
  ‘Foam’ models for $\mathfrak{sl}_n$-Khovanov homology.

(Khovanov’s $\mathfrak{sl}_3$ paper gave categorified versions of Kuperberg’s $\mathfrak{sl}_3$ relations.)