

Today's overly ambitious goals:

①

- 1) You should want to know the centre of the extended Haagerup fusion categories.
- 2) I can tell you ~~the~~ the fusion ring of the centre, and its modular data.

What are fusion categories?

- A fusion category is a finitely semisimple tensor category.

Examples

- (A) $\text{Rep } G$, for G a finite group
- (B) (the semisimple quotient of)
 $\text{Rep } U_\xi \mathfrak{g}$, for ξ a root of unity
and $\mathfrak{g}$ a semisimple Lie algebra.
- (C) ... other, scarier examples coming from conformal field theory or, even scarier, subfactors.

Some of these are so scary they are not even defined over a cyclotomic field.

Why care about fusion categories?

②

- They are a "noncommutative analogue" of finite groups.

Theorem (Deligne)

A symmetric fusion category ~~with~~ $\mathcal{C}$

~~with integer dimensions or subexponential growth~~

~~is the representation category~~

is $\text{Rep } G$ for some finite group G .

possibly with a slightly more interesting symmetry.

- They are the "symmetries" of $2d$ -dimensional local field theories:

Cobordism hypothesis:

(Lurie) local $n+1$ -dimensional field theories with values in $\mathcal{E}$ $\longleftrightarrow$ fully dualizable objects in $\mathcal{E}$.

(Daglas-Schommer-Pres-Snyder)

when $n=2$, $\mathcal{E} = \text{TensorCat}$, ~~and~~ f.d. objects are nondegenerate fusion categories.

• A fusion category $\mathcal{C}$ has as its braided centre
a modular tensor category $\mathcal{Z}(\mathcal{C})$.

(3) ~~③~~

- The extended Haagerup subfactor gives rise to
a pair of fusion categories, EH1 and EH2 .
(They are Morita equivalent, and hence have the same
centre.)
- They are the only known fusion categories not
known to be constructible from finite groups,
quantum groups, and 'quadratic categories'.
(This probably reflects our ignorance, rather than their
importance. But for now, at least, they are
~~very~~ 'exotic'.)
- We would like to understand them!

$$\text{RT}_{Z(E)}(\Sigma) \stackrel{\text{def}}{=} \sum_i Z(E_i)$$

(4)

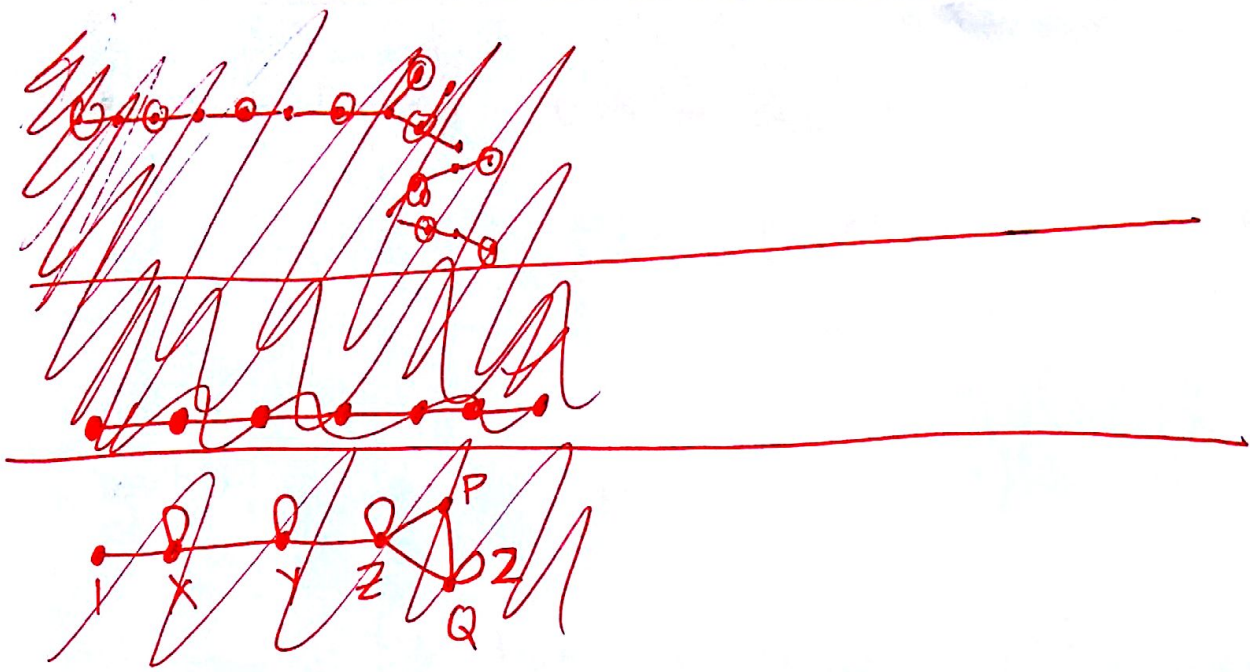
- A very important invariant of a fusion category $\mathcal{C}$ is its (braided) centre $Z(\mathcal{C})$, which is a modular $\otimes$ -category. Often, the centre may give hints as to other objects $\mathcal{C}$ may be related to:

Fusion categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are Morita equivalent if and only if their centres $Z(\mathcal{C}_1)$ and $Z(\mathcal{C}_2)$ are equivalent.

- As an example, the ^{fusion categories coming from the} 'Asaeda-Haagerup' subfactor formerly looked 'exotic', but have been recognised as Morita equivalent to a quadratic category.

(For the extended Haagerup categories, we know that there is at most one other category in the Morita class, "EH3". If it exists, ~~the~~ today's argument shows it must be Morita equivalent.)

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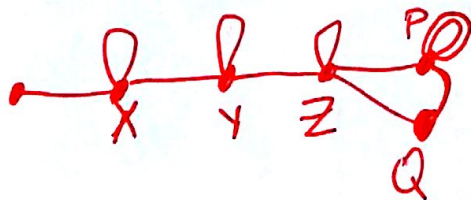


The moral of the story ~~from~~ from here on out is that being a modular tensor category is so ridiculously difficult (overconstrained), that in some cases you can calculate $z(c)$ knowing very little.

At first all we need to know about EHZ is its fusion ring.

This is a ring with a basis (the simple objects), and non-negative integer structure coefficients.

This ring is (easily) uniquely determined by the structure coefficients for just one object, "X":



There is a forgetful functor $R: \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$.

(6)

(Recall $\mathcal{Z}(\mathcal{C}) = \left\{ (X, \beta) \mid \begin{array}{l} X \in \mathcal{C} \\ \beta: X \otimes - \xrightarrow{\beta} - \otimes X \end{array} \right\}$)

There is an adjoint functor $I: \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$,

and $R(I(V)) = \bigoplus_{X \in \text{Irr } \mathcal{C}} X^* \otimes V \otimes X$.

At the level of fusion rings, this means we have

a non-negative integer matrix A so

$$K_0(I): K_0(\mathcal{C}) \rightarrow K_0(\mathcal{Z}(\mathcal{C}))$$

$$v \mapsto Av,$$

and $K_0(R): v \mapsto A^T v$.

The ~~formula~~ formula for $R \cdot I$ above means that if we know the fusion ring of $\mathcal{C}$, we know

$A^T A$. This does not determine A .

However... the dimension of every object in $\mathcal{Z}(\mathcal{C})$

must be:

~~• a cyclotomic integer~~

• a "d-number"

• and divide the global dimension $\dim \mathcal{Z}(\mathcal{C}) = \dim \mathcal{C}^2$ as an algebraic integer,

even though none of these conditions need hold down in $\mathcal{C}$. (7)

Since R and I preserve dimension, this is some interesting "filter" on the allowed rows of A .

Now a miracle occurs, and this completely determines A . (In ~~more~~ less exotic examples, this approach tends to fall flat!)

We learn $Z(EH2)$ has 22 simple objects, and moreover we know the dimensions of each of these objects

We have objects of dimensions

1
177.701
49.396
114.049 (7 times)
176.701 (4 times)
128.304 (4 times)
48.396 (4 times)

These live in a degree 3 extension of $\mathbb{Q}$.
The quotient of the 'big' Galois group acts as indicated

w_0, w_1, w_2

$\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4$

$\sigma_i, \delta_i, \varepsilon_i$

To get further, we need to talk about the S & T matrices for an MTC $\mathcal{D}$ (8)

(some cheating here; some facts are specific to centres of unitary fusion categories)

$$S = \frac{1}{\dim \mathcal{D}} \begin{pmatrix} & & \\ & \textcircled{0} & \\ & & \ddots \end{pmatrix}$$

$$T = \begin{pmatrix} & & 0 \\ & \textcircled{0} & \\ 0 & & \ddots \end{pmatrix}$$

Then: • S is unitary and symmetric,

and ~~some other properties~~

$$\overline{S_{x,y}} = S_{x^*,y}$$

• T is unitary,

$$\frac{S_{x1}}{S_{11}} = \dim x \in \mathbb{R}_{\geq 1}$$

normalized



• $S^2 = (ST)^3 = C$, "charge conjugation", $C^2 = I$
 (So $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto S$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto T$ is a unitary rep of $SL(2, \mathbb{Z})$)

$$N_{xy}^z = \sum_{w \in \text{irr } \mathbb{D}} \frac{S_{xw} S_{yz} S_{zw}}{S_{1w}} \quad (\text{Verlinde's formula}) \quad (9)$$

the multiplicity of

$$\chi_z \text{ in } \chi_x \otimes \chi_y,$$

so in N

The entries of S ~~and T~~ ^{generate some} ~~are~~ some $\mathbb{Q}(\xi_N)$, with $N = \text{ord}(T)$.

and the Galois group $\mathbb{Z}_N^\times$ acts by

$$\sigma_\ell(\xi_N) = \xi_N^\ell.$$

In fact, there is an ~~signed~~ action of the Galois group by signed permutations on $\text{Irr } \mathbb{D}$: $\sigma(S_{xy}) = \varepsilon_\sigma(x) S_{x^\sigma y} = \varepsilon_\sigma(y) S_{xy^\sigma}$

$\frac{S_{xy}}{S_{1y}}$ is an eigenvalue of $N_x = (N_{xa})_{a,b}$

so is an algebraic integer.

Thus $\dim(1^\sigma)$ is an algebraic unit, since

$$\dim(1^\sigma) = \frac{S_{11}}{S_{11}}$$

$$\text{and } \left(\frac{S_{11}}{S_{11}} \right)^{-1} = \frac{\varepsilon_\sigma(1)}{\varepsilon_\sigma(1)} \sigma \left(\frac{S_{11}}{S_{11}} \right)$$

Moreover, the action of $SL(2, \mathbb{Z})$ factors through
the ~~principal congruence subgroup~~

$$\Gamma(N) = \{A \in SL(2, \mathbb{Z}) \mid A \equiv I \pmod{N}\}$$

finite group $SL(2, \mathbb{Z}/N\mathbb{Z})$.

Cauchy's Theorem for modular tensor categories

The primes dividing N are the same primes
dividing the norm of ~~its~~ ~~of~~ ~~the~~ $\dim \mathcal{D}$.

$$\text{so: } N = 5^a 13^b.$$

$$T_{x^2, x^2} = T_{xx}^{x^2}$$

This tells us that if a or b are large,
there must be many distinct eigenvalues for T .

Since we know $Z(EH1)$ has exactly 22
simple objects $N = 65$.

We can also work out how the Galois group
acts (e.g. to start there are exactly three
objects whose dimension is a unit,
so these are permuted)

($S_{x_3} \neq 0 \Rightarrow t_{x_i}, t_{y_i} \in \mathcal{I}(P_i)$ for some i) (first, no tensor product irreps)

$$\dots \rho \cong \rho_{14}^{(13)} \oplus \rho_{(5)}, \quad \text{where } \rho_{(5)} \text{ factors through } \mathrm{SL}(2, \mathbb{Z}/5).$$

$\S$
the Kth irrep of $\mathrm{SL}(2, \mathbb{Z}/13)$ in GAP's table

$$\rho_5 \cong \rho_6^{(5)} \oplus 1 \oplus 1 \oplus 1$$

End game

From these decompositions of ρ , we know ~~the~~ the eigenvalues of T . There are 6 ones, and the rest are distinct.

We have explicit ~~these~~ matrices S' and T' for ρ , in some stupid basis, and we know

$$\textcircled{1} QT = T'Q, \quad \textcircled{2} QS = S'Q$$

for some change of basis Q .

From $\textcircled{1}$ Q is (a permutation^{*} and) a 6×6 block, and otherwise the identity.

Then:- i) we solve the linear equations in

(2)

$$QS = S'Q. \quad (\dots \text{and the modular invariants})$$

ii) $\det Q$ has one of its components as a factor,

$$\text{so } Q_{1,w_0} \neq 0.$$

iii) some of the linear equations are now

$$\chi Q_{2,w_0} = 0.$$

So all such χ are zero.

iv) Some quadratics are now $X^2 = 0$, so solve those.

v) Again $\det Q$ factors ~~as~~ ^{as} $Q_{15,w_0} \cdot W$,
~~reps~~ as above.

vi) Finally, treat $QS = S'Q$ as quadratics jointly in S_{xy} and Q_{15} ,
and find they're solvable, with
4 out of the 5 solutions making
 $\det Q = 0$.

As a consistency check, Verlinde produces
non-negative integers.

What else?

(13)

- Constructing $EH3$.
- Finding the remaining data for $Z(EH_i)$ as an MTC.
- Does this modular data lie in a family?

ρ_5 is the Steinberg representation of $SL(2, \mathbb{Z}/5)$

ρ_3 is in the principal series of $SL(2, \mathbb{Z}/13)$.

If we had $n \perp \rho_r + \rho_p$,

$n = r - 2$, and much of S is determined.

The undetermined part looks rather simple...

- Find ~~the~~ compatible character vectors (vumt), which would have to be available if

$$Z(EH) = \text{Rep } V.$$

(Done, some constraints on the central charge of V .)

- One way or another, explain the existence of EH .