Khovanov homology I (MSRI, January 25 2010)

Khovanov homology is a categorical knot invariant.

It sends knots to objects in a certain category $\mathcal{C}_*$ and isotopies between knots to isomorphisms in that category. (Actually, we get maps for arbitrary cobordisms, too!)

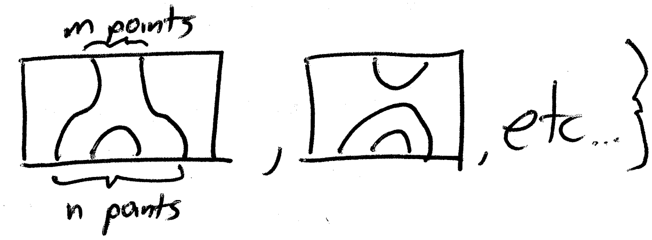
Moreover, the Grothendieck group $K(\mathcal{C}_*)$ is $\mathbb{Z}[q,q^{-1}]$, and the image of a knot there is exactly the Jones polynomial.

$$
\begin{tikzcd}
\overset{\text{a link}}{L} \arrow[r, mapsto, "Kh"] \arrow[d, "\underset{\text{an isotopy}}{\Sigma}"'] & Kh(L) \in \mathcal{C}_* \arrow[r, mapsto] \arrow[d, "Kh(\Sigma)"', "\cong"] & {[Kh(L)] = J(L)} \in K(\mathcal{C}_*) = \mathbb{Z}[q,q^{-1}] \\
L' \arrow[r, mapsto] & Kh(L') \arrow[ur] &
\end{tikzcd}
$$

Many knot invariants come from <u>braided ⊗-categories</u>
The Jones polynomial is an example; it comes from the Temperley-Lieb category.
We'll see that Khovanov homology arises in the same way — we'll sneak up on it by categorifying Temperley-Lieb.

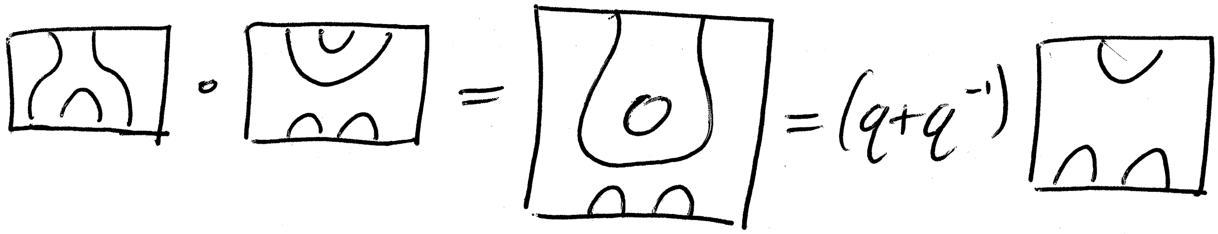
<u>Definition</u> (TL as a ⊗-category)

$\mathrm{Obj}(TL) = \mathbb{N}$

$\mathrm{Hom}^{\circ}_{TL}(n \to m) = \mathbb{Z}[q, q^{-1}]\{$ 

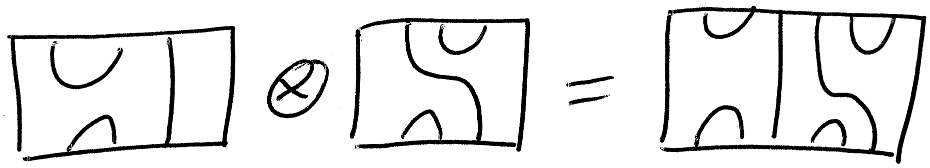
$\}$

Composition is by stacking, and replacing loops with factors of $q+q^{-1}$;

e.g. 

⊗-product is by placing side by side:

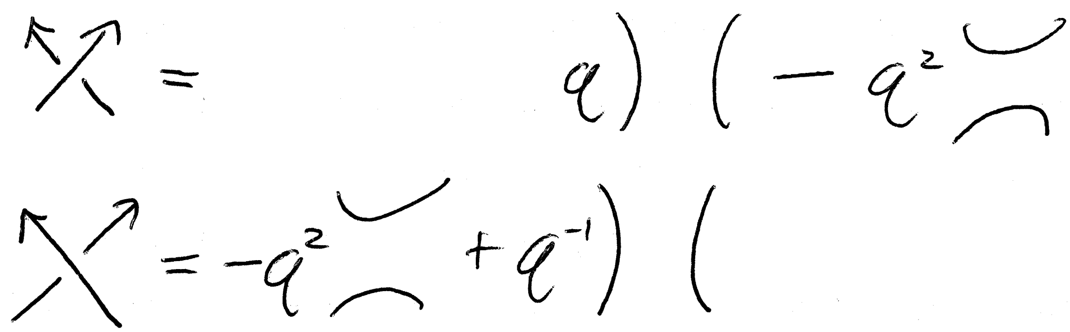
$n \otimes m = n+m$



To say "TL is a braided $\otimes$-category", we'd need a formula for .

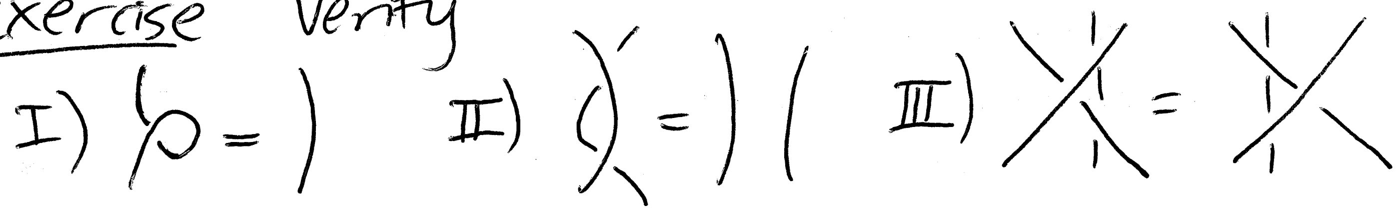
I'll cheat, and write 2 different formulas for oriented crossings.

<u>Definition</u> (TL as a braided $\otimes$-category)



(There's a good explanation for this difficulty: google "tingley a sign that used to annoy me" for a start. The same trick that fixes functoriality-disorientations partially fixes this too.)

<u>Exercise</u> Verify



<u>Remark</u> Really what we did here was define a functor from oriented tangles to TL.

A braided $\otimes$-category $\mathcal{C}$ (or just a $\otimes$-category all with a functor from tangles) immediately gives an invariant of links:

$$L \in \mathrm{Hom}_{\mathcal{C}}(0 \to 0)$$

We'd like to be able to write elements of this Hom space in some nice form.

<u>Example</u> $\mathrm{Hom}^{0}_{TL}(0 \to 0) = \mathbb{Z}[q, q^{-1}]$

Thus the invariant associated to TL is a Laurent polynomial in q - the Jones polynomial.

The Grothendieck group $K(\mathcal{A})$ of a category $\mathcal{A}$ is the $\mathbb{Z}$-module

$$K(\mathcal{A}) = \mathbb{Z}\{[A] \mid A \in \mathrm{Obj}(\mathcal{A})\} \Big/ {\scriptstyle [A]=[B] \text{ if } A \cong B}$$

Temperley-Lieb is a tensor category whose Hom spaces are $\mathbb{Z}$-modules (actually, $\mathbb{Z}[q,q^{-1}]$-modules)

We'll construct $\mathcal{C}$ (and later $\mathcal{C}_*$), a $\otimes$-category whose Hom spaces are themselves categories (this is called a $\otimes$-2-category).

~~Taking the Grothendieck group of each Hom~~

We'll write $K(\mathcal{C})$ to indicate taking the Grothendieck group of each Hom space (exercise for the categorically minded: $\otimes$ and $\circ$ work out)

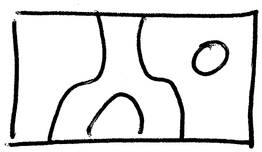
and eventually prove

$$K(\mathcal{C}) \cong TL \qquad \text{(as } \otimes\text{-categories)}$$

$$K(\mathcal{C}_*) \cong TL \qquad \text{(as ``braided'' } \otimes\text{-categories)}$$

<u>Definition</u> the $\otimes$-2-category $\mathcal{C}$:

$Obj(\mathcal{C}) = \mathbb{N}$

$Hom^{\circ}_{\mathcal{C}}(n \to m) = \{ q^k$  $, \ldots \}$

(this is just a set, not a $\mathbb{Z}$-module: no linear combinations)

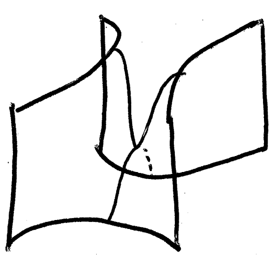
$\circ$-composition is stacking diagrams vertically
(just as before: but no rule for removing loops)

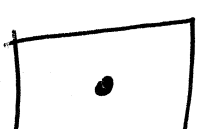
$\otimes$-product is juxtaposing side by side.

This just tells us the <u>objects</u> of the Hom°-space category. What are the morphism? What is the $\bullet$-composition, in the new direction?

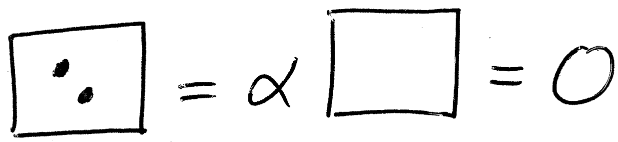
$$Hom^{\bullet}_{\mathcal{C}}(q^k D \to q^{k'} D') =$$

$$\mathbb{C}\left\{ \begin{array}{l} \text{surfaces with dots, with bdy } D \cup D' \\ \text{with Euler characteristic} \\ \chi = k - k' + 2\#\text{dots} + \frac{1}{2}(n+m) \end{array} \right\} \Big/ \begin{array}{l} \text{local} \\ \text{relations} \end{array}$$

e.g.  $\in Hom^{\bullet}_{\mathcal{C}}($  $\longrightarrow q\,)(\,)$

 $\in Hom^{\bullet}_{\mathcal{C}}(\,)(\,\longrightarrow q^2\,)(\,)$

The local relations are:

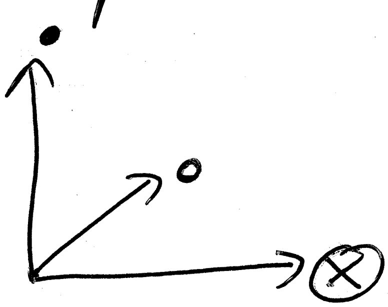


(tomorrow we'll try other values of α)

Note all these are homogeneous in $\chi - 2\#\text{dots}$.

<u>Exercise</u> if D is a closed TL diagram
$\dim \mathrm{Hom}^\bullet_e(q^k \phi \to D) =$ coefficient of q^k in the evaluation of D.

We can compose these surfaces in 3 directions:

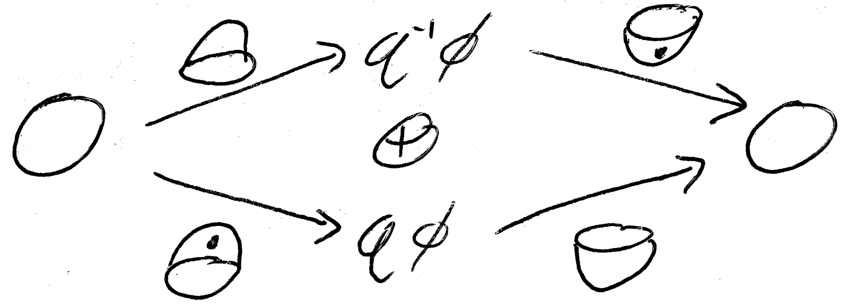


Finally, we really want the matrix version of each of these categories

- objects are formal direct sums (of q-shifted TL diagrams)
- morphisms are matrices of surfaces/relations

Theorem $O \cong q\phi \oplus q^{-1}\phi$

Proof



Exercise (Check the Euler characteristics)

Use the local relations to verify this is an isomorphism followed by its inverse.

Theorem $K(\mathcal{C}) \cong \mathbb{Z}$ as a $\otimes$-category

(Proof: Theorem above says $O = q + q^{-1}$ in $K(\mathcal{C})$. You still need to check there are no more relations: c.f. math.GT/0612754)

But what about the braiding?

The formula $\times = q\,)(\, - q^2 \asymp$ doesn't make much sense in $\mathcal{C}$: what to do about the sign?

(Categorifying $\mathbb{Z}$-modules is harder than categorifying $\mathbb{N}$-modules)

Pass to the <u>homotopy category</u>

$$\mathcal{C}_* = \{\text{formal complexes in } \mathcal{C}\}$$

$$\circlearrowleft$$

$$\{\text{chain maps modulo homotopy}\}$$

To $\circ$-compose or $\otimes$-product complexes, we take the homological tensor product, combining 'chain groups' and differentials using $\circ$ or $\otimes$.

<u>Example</u> (Notation: underline the object in homological height zero.)

$$\left(\underline{A} \xrightarrow{f} B\right) \circ \left(\underline{C} \xrightarrow{g} D\right) = \begin{tikzcd} & A \circ D \arrow[dr, "f \circ 1_C"] & \\ \underline{A \circ C} \arrow[ur, "1_A \circ g"] \arrow[dr, "f \circ 1_C"'] & \oplus & B \circ D \\ & B \circ C \arrow[ur, "-1_B \circ g"'] & \end{tikzcd}$$

Note the sign on one differential.

<u>Rule</u> place a sign wherever the homological height for the first factor is (constant and) <u>odd</u>.

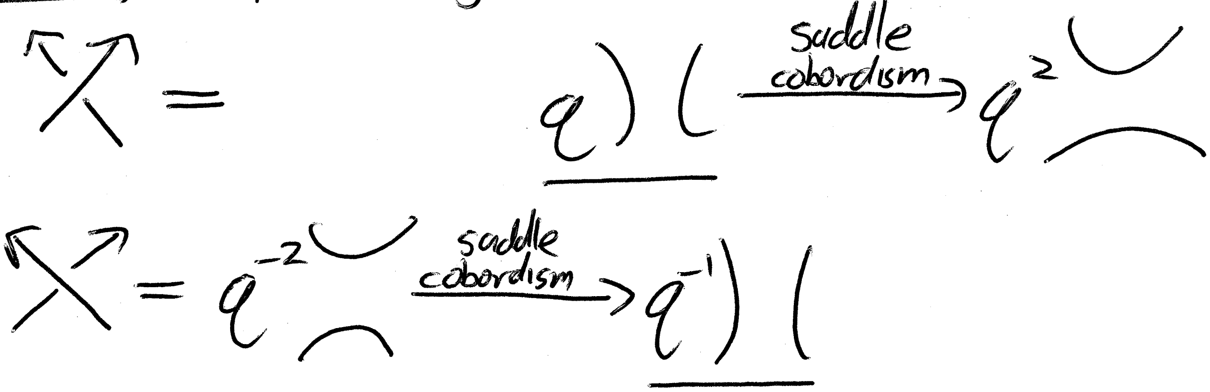
When we take the Grothendieck group of a homotopy category, we add the relation

$$[\cdots \to X_{i-1} \to X_i \to X_{i+1} \to \cdots] = \sum_i (-1)^i [X_i]$$

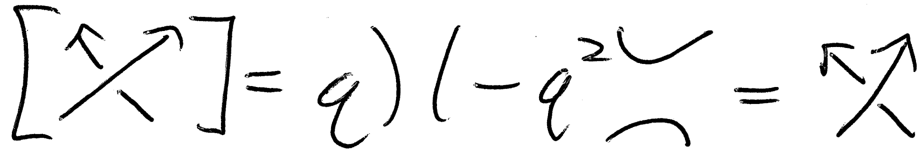
The Grothendieck group doesn't change so

$K(\mathcal{C}_*) = K(\mathcal{C}) = TL$ (still as $\otimes$-categories)

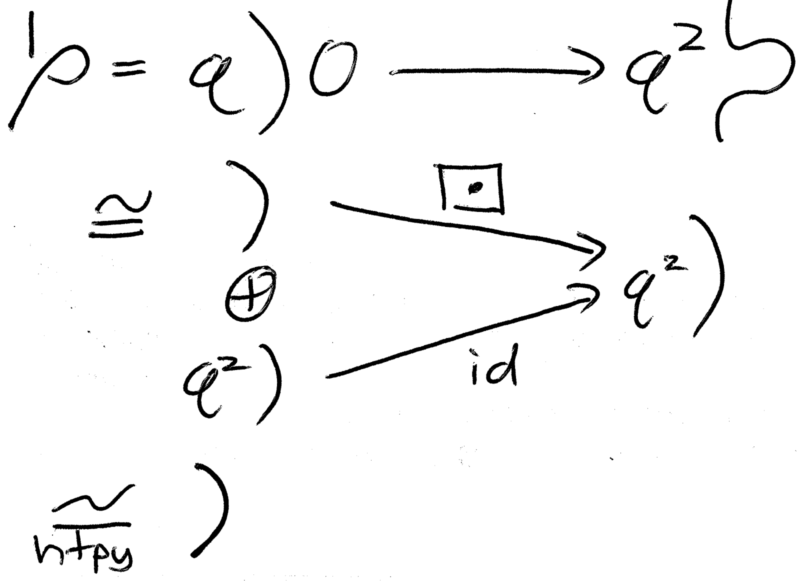
<u>Definition</u> ("braiding" on $\mathcal{C}_*$)

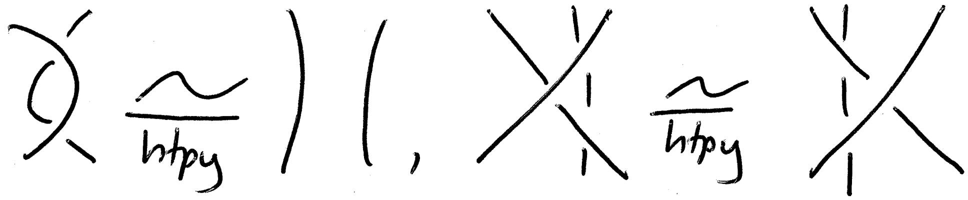


note



<u>Theorem/Exercise</u>



Theorem 

(in all possible orientations!)

Modulo how these homotopy equivalences fit together (tomorrow), $\mathcal{C}_*$ is a "braided" $\otimes$-2-category.

We've crept up on the definition of Khovanov homology

$$Kh(L) = L \in \mathrm{Hom}^{\circ}_{\mathcal{C}_*}(0 \to 0)$$

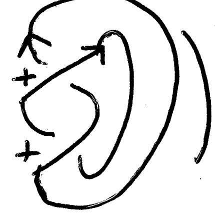
Let's make this much more explicit.

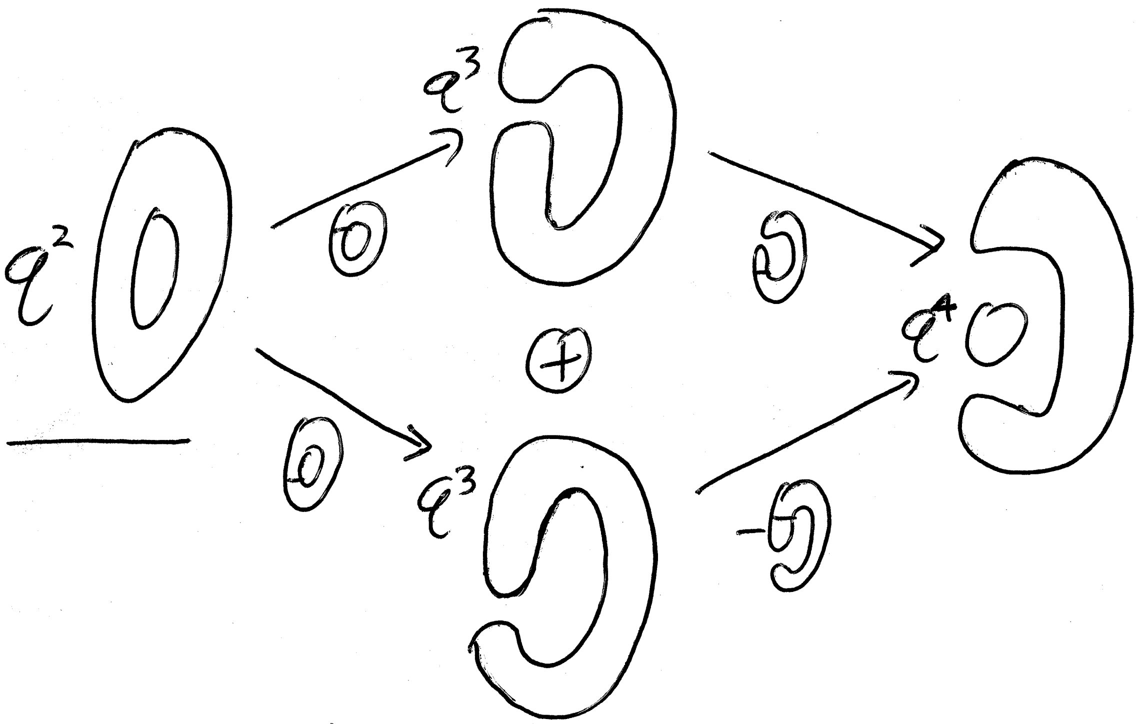
Build a big cube, with

- vertices "resolve each crossing $)_0($ or $\underset{1}{\smile\frown}$"
  with coefficient $q^{\#(+,0)+2\#(+,1)-\underbrace{\#(-,0)}-2\#(-,1)}$
  and homological height $\#(+,1)-\#(-,1)$ ← number of negative crossings resolved as 0
- edges saddle cobordisms changing a single crossing (from 0 to 1 for +ve crossings, 1 to 0 for -ve crossings)
  with a sign
  $(-1)^{\#\text{ of } 1 \text{ resolutions in 'later' crossings}}$
  ↳ this requires an ordering on crossings

This is a chain complex!
It's the 'usual' definition, c.f. math.GT/0410495.

Example

Kh() =



Applying $\mathrm{Hom}^\bullet_{\mathcal{C}}(\phi, -)$:

q^6					$\mathbb{C}$
q^4	$\mathbb{C}$	$\xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}}$	$\mathbb{C}\oplus\mathbb{C}$	$\xrightarrow{(1\ \ -1)}$	$\mathbb{C}\oplus\mathbb{C}$
q^2	$\mathbb{C}\oplus\mathbb{C}$	$\xrightarrow{\begin{pmatrix}1&1\\1&1\end{pmatrix}}$	$\mathbb{C}\oplus\mathbb{C}$	$\xrightarrow{(1\ \ -1)}$	$\mathbb{C}$
q^0	$\mathbb{C}$				

Taking homology

$$H_0 = q^0\mathbb{C} \oplus q^2\mathbb{C} \qquad H_1 = 0 \qquad H_2 = q^4\mathbb{C} \oplus q^6\mathbb{C}$$

<u>Exercise</u> Understand this!

We'd like to have a 'standard form' for elements of $\mathrm{Hom}^{\circ}_{\mathcal{C}_*}(0\to 0)$ (ie nice homotopy representatives) along the lines of $\mathrm{Hom}^{\circ}_{TL}(0\to 0)\cong \mathbb{Z}[q,q^{-1}]$

<u>Theorem</u> $\mathrm{Hom}^{\circ}_{\mathcal{C}_*}(0\to 0)\cong \{\text{bigraded vector spaces}\}$

<u>Proof</u> We'll describe this isomorphism two ways. The second time, we'll actually show it's an equivalence.

① There's a functor

$$\bigoplus_s \mathrm{Hom}^{\bullet}_{\mathcal{C}}(q^s\phi, -)$$

from $\mathrm{Hom}^{\circ}_{\mathcal{C}}(0\to 0)$ to graded vector spaces.

Given X, a complex in $\mathrm{Hom}^{\circ}_{\mathcal{C}_*}(0\to 0)$, we can apply this functor, then take homology. This gives a bigraded vector space.

② Objects in $\mathrm{Hom}^{\circ}_{\mathcal{C}}(0\to 0)$ are — up to isomorphism — just direct sums of (grading shifts of) empty diagrams.

Thus every complex $X\in \mathrm{Hom}^{\circ}_{\mathcal{C}_*}(0\to 0)$ has an isomorphism representative with 'chain groups' of this form.

What are the possible differentials?

$$\mathrm{Hom}^{\bullet}_{\mathcal{C}}(\phi, q^k \phi) = \begin{cases} \mathbb{C} & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases}$$

Thus an matrix entry in a differential is either <u>zero</u> or <u>invertible</u>.

<u>Lemma</u> any such complex has a homotopy representative with all differentials zero.

<u>Proof</u> Repeated application of the following homotopy:

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[ddr] & B \arrow[r, "\beta"] \arrow[ddr, "\lambda"] & C \arrow[r, "\gamma"] & D \\
 & \oplus & \oplus & \\
 & E \arrow[uur, "\mu"] \arrow[r, "\psi \text{ invertible}"'] & F \arrow[uur] &
\end{tikzcd}$$

IS htpy

$$A \xrightarrow{\alpha} B \xrightarrow{\beta - \mu\psi^{-1}\lambda} C \xrightarrow{\gamma} D$$

(This also gives an algorithm – essential for computations.)

~~Alternatively, the subcategory of $\mathcal{C}_*$ with all objects ϕ is isomorphic to grVec_* (complexes of graded vector spaces) and all complexes there are homotopic to their homology.~~

You can think of this as an explicit form of the fact that complexes of vector spaces are homotopic to their homology.

We can apply $\bigoplus_s \mathrm{Hom}^\bullet_e(q^s\phi, -)$ to one of these complexes without differentials. It just converts each $q^s\phi$ to a copy of $\mathbb{C}$ in grading s.
There's no need to take homology now.

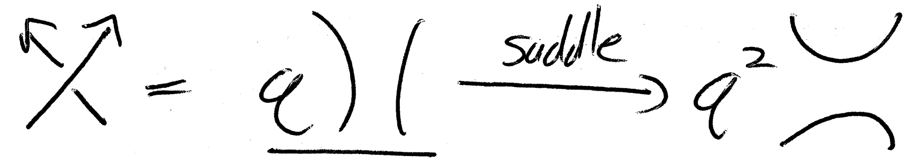
If $Kh(L)_i$ denotes the i-th homological height and for a graded vector space $V = \bigoplus_j n_j \mathbb{C}$ we write $q\dim V = \sum_j n_j q^j$, the Poincaré polynomial of $Kh(L)$ is

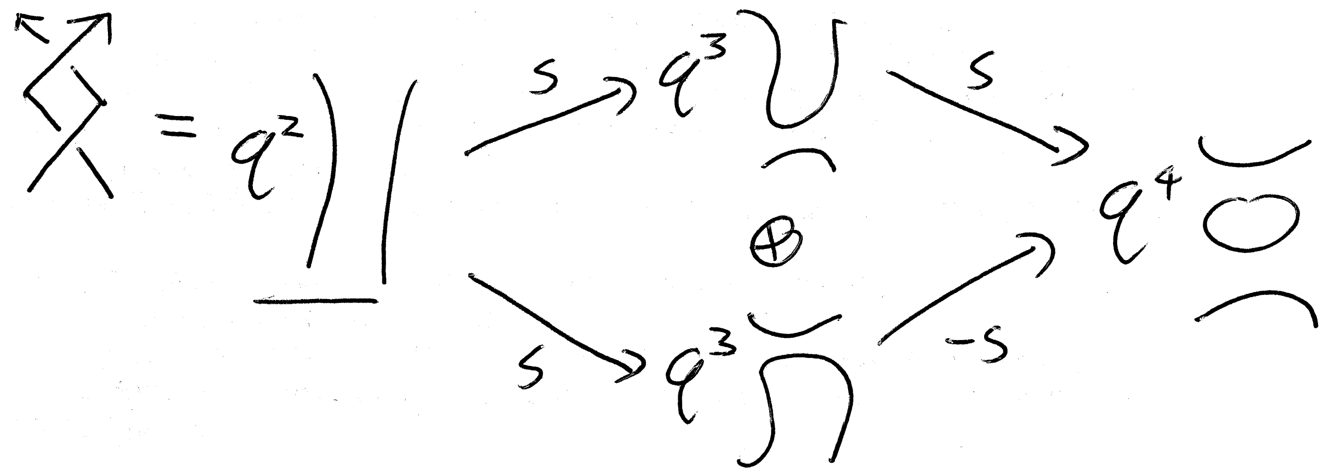
$$Kh(L)(q,t) = \sum_i t^i q\dim Kh(L)_i.$$

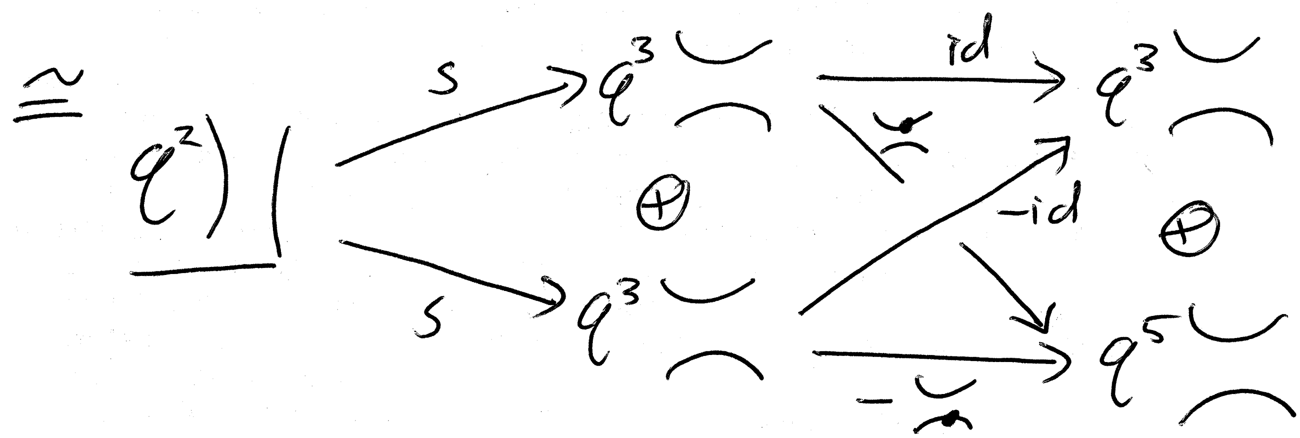
Today we worked over $\mathbb{C}$. Over $\mathbb{Z}$, complexes aren't homotopic to their homology, and $\mathbb{Z}$-torsion appears.

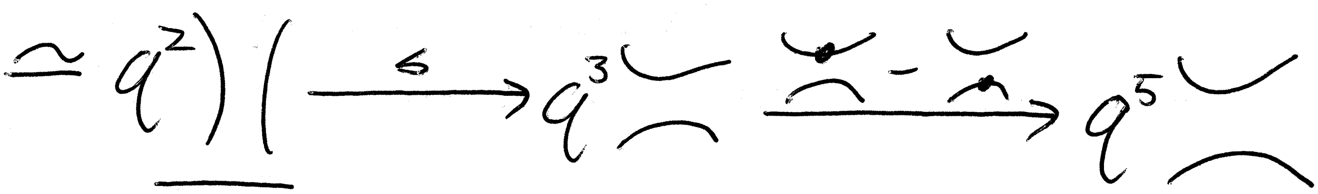
Tomorrow: functoriality, working over $\mathbb{C}[\alpha]$, the s-invariant.

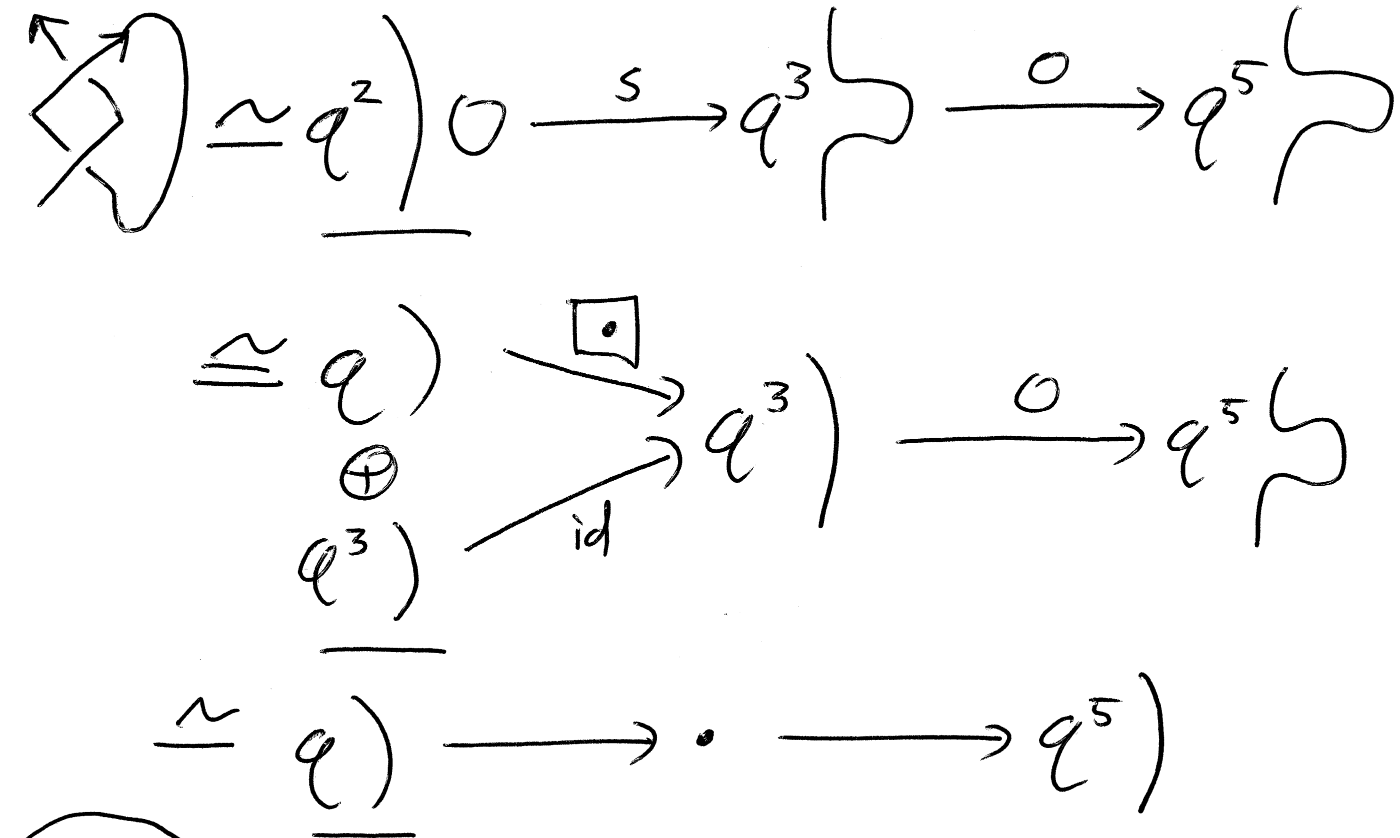
Example











$$\simeq \underline{\begin{matrix}\phi\\ \oplus\\ q^2\phi\end{matrix}} \longrightarrow \bullet \longrightarrow \begin{matrix}q^4\phi\\ \oplus\\ q^6\phi\end{matrix}$$

The Poincaré polynomial is

$$t^0(1+q^2)+t^2(q^4+q^6)$$

<u>Exercise</u> verify that at $t=-1$ this is the Jones polynomial

<u>Exercise</u> Understand all the steps in this calculation

<u>Exercise</u> Compute the Khovanov homology of the trefoil.
(hint: reuse the previous page)

<u>Addendum ~~on~~ on tangles</u>

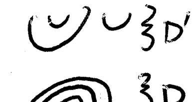
You can recover Khovanov's "complex of H_n-bimodules" invariant of tangles from this setup.

<u>Claim</u> $H_n \cong \mathrm{Hom}^{\bullet}_{\mathcal{C}}\Big(\bigoplus_{\substack{\text{TL diagrams}\\ D \text{ with } n\\ \text{bdy pts}}} D, \bigoplus_{\text{''}} D\Big)$

Given a complex [redacted]

$X \in \mathrm{Hom}^{\circ}_{\mathcal{C}_*}(n \longrightarrow m)$, we can form

e.g.



$$\bigoplus_{D \in TL_n} \bigoplus_{D' \in TL_m} \bigoplus_k \mathrm{Hom}^{\bullet}_{\mathcal{C}_*}\left(D' \circ D, q^k X\right)$$

which is a complex of graded vector spaces.

In fact, it's an H_n-H_m bimodule - exactly Khovanov's invariant.

You can 'compose tangles' by taking tensor products over H_n, but you <u>can't</u> compose in arbitrary directions. This makes it less useful for large computations.