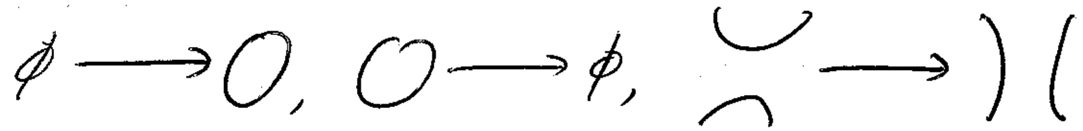
Functoriality

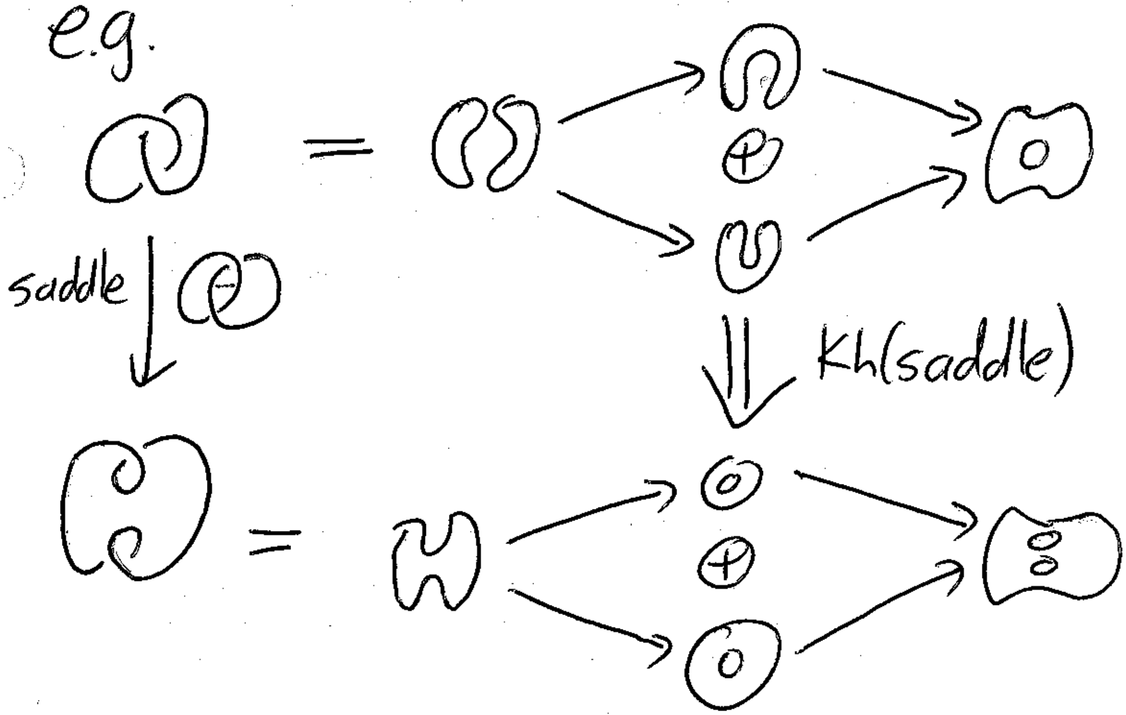
We saw there are homotopy equivalences corresponding to Reidemeister moves

e.g.

We also get chain maps for 'Morse moves'



e.g.

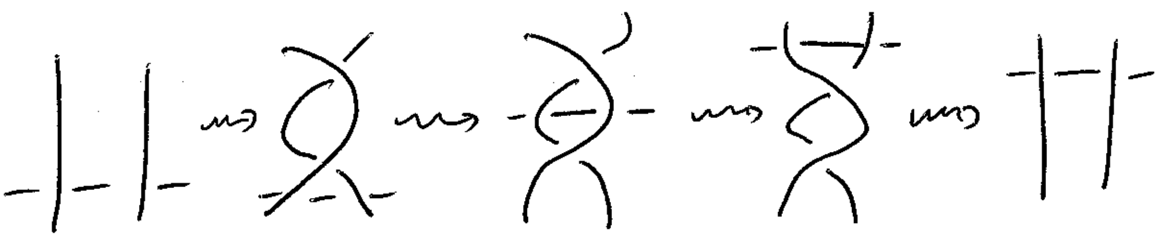


Kh(saddle) is 'diagonal': apply the same Morse move everywhere.

We'd like to associate a chain maps to every link cobordism, by writing it as a composition of these elementary moves.

Unfortunately, different ways of decomposing a cobordism give different answers!

<u>Example</u>



is just the identity, but however we pick our homotopy equivalences for Reidemeister moves, this composition is assigned $-\mathrm{id}$.

There's a fix involving 'disorientations', but that would take us too far afield.

It's important to get functoriality right if you want to think about Khovanov homology for links in other 3-manifolds!

Deformations

Recall we had a relation $\boxed{\,\bullet\!\!\bullet\,} = 0$.

Let's work over $\mathbb{C}[\alpha]$, rather than $\mathbb{C}$ (warning: not a field!), and replace this relation with $\boxed{\,\bullet\!\!\bullet\,} = \alpha \boxed{}$.

(We'll also tweak the rule about euler characteristics, by giving α a 'formal euler characteristic' of -4.)

Before, every complex in $\mathrm{Hom}^\circ_{\mathcal{C}_*}(0 \to 0)$ decomposed as a direct sum of copies of $q^k \emptyset$.

This is no longer true. It's convenient to now think about complexes in $\mathrm{Hom}^\circ_{\mathcal{C}_*}(1 \to 1)$.

<u>Theorem</u> every complex in $\mathrm{Hom}^\circ_{\mathcal{C}_*}(1 \to 1)$ decomposes as a direct sum of complexes

$$E = \big)$$

$$C_n = \big) \xrightarrow{\boxed{\bullet^n}} q^{2n}\big)$$

(each sum can come with an overall factor of q^k, and in any homological height.)

Example/Exercise

$\mathrm{Kh}\Big($  $\Big) = \underline{q^{2}\,\big)} \longrightarrow \bullet \longrightarrow q^{6}\big) \xrightarrow{\boxed{\cdot}} q^{8}\big)$

$$= q^{2} E \oplus q^{6} t^{2} C_1$$

Theorem (Lee) For a k-component link, exactly 2^{k-1} copies of E appear

Definition the s-invariant of a knot K is the q-grading of the unique copy of E in $\mathrm{Kh}(K^{cut})$

Theorem (Rasmussen) $|s(K)| \leq |g_{slice}(K)|$

Conjecture Only E, C_1 & C_2 appear in the invariants of links.

Corollary $$s(K) = \frac{\mathrm{Kh}(K)(q, t = -q^{-4})}{q + q^{-1}}$$