A 4-manifold invariant from Khovanov homology

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Today I'll define an invariant $\mathcal{A}(W^4)$ of smooth 4-manifolds, taking values in doubly-graded vector spaces.

If the manifold has boundary, we can specify a link in the boundary.

It is a generalization of Khovanov homology:

$$\mathcal{A}(B^4; L \subset S^3) \cong \mathrm{Kh}\,(L).$$

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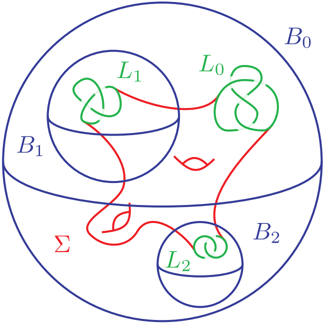
A *lasagna diagram* $(B_0, \{B_i\}, \Sigma)$ consists of

- B_0 a 4-ball
- $\{B_i\}$ a collection of disjoint 4-balls in the interior
- Σ a surface in $B_0 \setminus \bigcup B_i$ meeting the boundaries transversely.

We write $S_i = \partial B_i$ (a boundary sphere) and $L_i = \Sigma \cap S_i$ (a link therein).

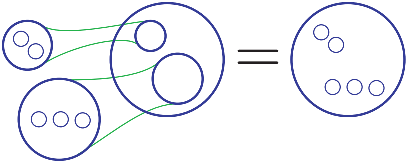


A *lasagna algebra* $\mathcal{A}$ consists of

- $\mathcal{A}(L \subset S)$ a vector space for each link L embedded in a 3-sphere S.
- $\mathcal{A}(\Sigma) : \bigotimes \mathcal{A}(L_i \subset S_i) \to \mathcal{A}(L_0 \subset S_0)$ a linear map, for each lasagna diagram.

such that

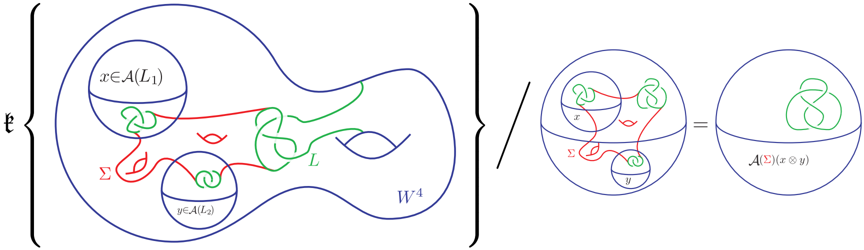
1. the map $\mathcal{A}(\Sigma)$ only depends on the lasaga diagram up to isotopy (rel ∂), and
2. the maps are compatible with gluing lasagna diagrams.



Given a lasagna algebra, we immediately get an invariant of 4-manifolds.

Definition

$\mathcal{A}(W; L \subset \partial W)$ is the linear span of 'labelled lasagna diagrams in W', modulo applying lasagna maps in balls.



This construction is actually a special case of the usual recipe

n-categories with duals $\rightsquigarrow$ invariants on n-manifolds

c.f. my definition of disklike n-categories, and the constructions of our papers *The blob complex* and *Higher categories, colimits and the blob complex* (PNAS May 2011).

A lasagna algebra is a particular kind of 4-category (trivial 0- and 1-morphisms, 2- and 3-morphisms generated by a self-dual 2-morphism).

The TQFT framework ensures that this recipe also associates k-categories to $4-k$-manifolds, and that there are 'gluing formulas' for handle decompositions.

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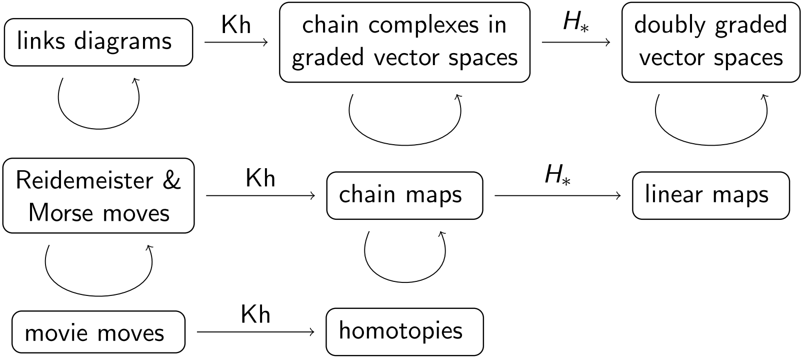
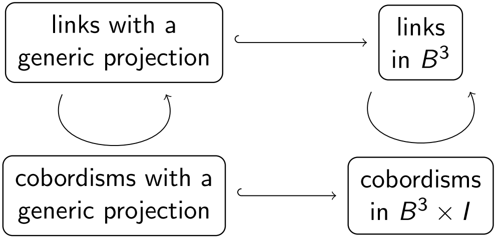
Khovanov homology is defined combinatorially.



Since the inclusion



is an equivalence of categories, this lifts to a functorial invariant of links embedded in B^3.

We will define the Khovanov invariant for links in S^3 as the flat sections of a certain vector bundle with parallel transport.

The base space is $S^3 \setminus L$.

The fibre over $x \notin L$ is $\mathrm{Kh}\left(L \subset S^3 \setminus \{x\}\right)$.

The parallel transport along $\gamma : x \to y$ is given by the cobordism $L \times I \subset S^3 \setminus \mathrm{graph}(\gamma)$.

Question

Does this vector bundle have monodromy?

We'll later give conditions that ensure the monodromy is trivial, and for now proceed assuming this.

We define $\mathrm{Kh}\left(L \subset S^3\right)$ as the flat sections of this bundle. Evaluation at any point is an isomorphism.

What about cobordisms? Given $\Sigma \subset S^3 \times I$, choose any path in $S^3 \times I$ connecting the outer and inner boundaries, avoiding Σ. The complement $S^3 \times I \setminus \mathrm{graph}(\gamma)$ is diffeomorphic to $B^3 \times I$.

Definition

The map $\mathrm{Kh}\left(\Sigma \subset S^3 \times I\right)$ *is defined by the composition*

$$\begin{aligned}\mathrm{Kh}\left(L_1 \subset S^3\right) &\xrightarrow{\cong} \mathrm{Kh}\left(L_1 \subset S^3 \setminus \{\gamma(1)\}\right)\\ &\xrightarrow{\mathrm{Kh}\,(\Sigma)} \mathrm{Kh}\left(L_0 \subset S^3 \setminus \{\gamma(0)\}\right)\\ &\xrightarrow{\cong} \mathrm{Kh}\left(L_0 \subset S^3\right).\end{aligned}$$

Question

Is this definition independent of the choice of γ*?*

Associating a linear map to a cobordism in $S^3 \times I$ was a warm-up to the general definition of a lasagna algebra.

Given a lasagna diagram $(B, \{B_i\}, \Sigma)$, choose arcs γ_i connecting $x_i \in S_i$ to $x_0 \in S_0$, avoiding Σ. In the complement of the arcs, we just have a cobordism in $B^3 \times I$, from $\bigsqcup L_i$ to L_0.

Definition

The map $\mathcal{A}(\Sigma)$ is given by the composition

$$\begin{aligned}
\bigotimes \mathrm{Kh}\left(L_i \subset S_i\right) &\xrightarrow{\cong} \bigotimes \mathrm{Kh}\left(L_i \subset S_i \setminus \{x_i\}\right) \\
&\longrightarrow \mathrm{Kh}\left(\bigsqcup L_i \subset B^3 \times \{0\}\right) \\
&\xrightarrow{\mathrm{Kh}\,(\Sigma)} \mathrm{Kh}\left(L_0 \subset B^3 \times \{1\}\right) \\
&\xrightarrow{\cong} \mathrm{Kh}\left(L_0 \subset S_0\right).
\end{aligned}$$

Here we're using the map from the tensor product of Khovanov homologies of distant links to the Khovanov homology of the disjoint union.

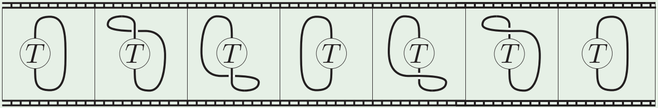
Question

Is this independent of the choice of γ*?*

All three of our questions (monodromy, independence of γ for a single ball, and for multiple balls) can be answered at once.

Theorem

The vector bundle is flat and our constructions above are well-defined exactly if

$\mathrm{Kh}\Bigg($ 

$\Bigg) = \mathbf{1}$

for all tangles T.

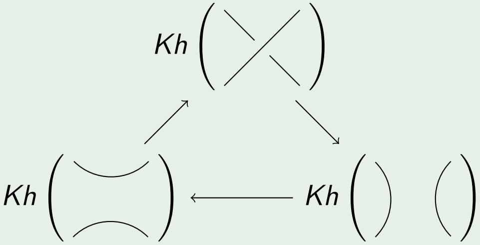
This is a “non-local movie move”. This surface is isotopic to a cylinder in S^3 (by ‘inflating through ∞’) but nontrivial in B^3.

It's hard to prove!

Earlier with David Clark and Kevin Walker I constructed a variant of Khovanov homology over the integers which is functorial in B^3.

We can't prove this identity in that setting. In order to construct the lasagna algebra structure, we've had to settle for working in characteristic two.

Working mod 2, the usual version of Khovanov homology (for unoriented links) has an exact triangle

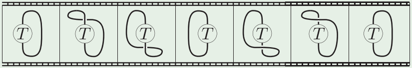


with the maps induced by the obvious cobordisms.

Since these maps are induced by cobordisms, the exact triangle is ‘natural’: it commutes with cobordisms outside the ball containing the crossing and its resolutions.

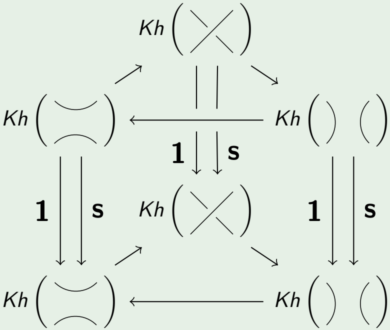
Theorem

In this variant of Khovanov homology,

Kh() $= \mathbf{1}$.

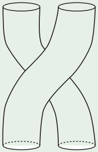
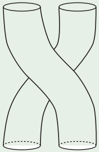
Proof.

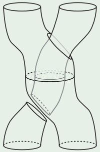
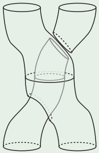
Induct on the number of crossing, using the five lemma.



Proof (continued).

On unlinks, the result is easy to prove, using the fact that

Kh () = Kh () =

= Kh () = Kh ().

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We can extend the invariant obtaining ‘disklike k-categories’ for codimension k-manifolds.

Definition

Given a finite subset $c \subset \partial M^3$, we define a $*$-category $\mathcal{A}(M^3, c)$ with objects tangles in M, and $\mathrm{Hom}(T_1, T_2) = \mathcal{A}(M \times I, T_1 \cup T_2)$.

Theorem

If $M \subset \partial W^4$, then the collection of vector spaces

$$\{\mathcal{A}(W, T_1 \cup T_2)\}_{T_2}$$

(here T_1 is a tangle in $\partial W \setminus M$, and T_2 is a tangle in M, both with boundary $c \subset \partial M$) forms a module over the category $\mathcal{A}(M, c)$.

Theorem

If $M \sqcup M^{op} \subset \partial W^4$, then $\mathcal{A}(W, -)$ forms a bimodule over $\mathcal{A}(M)$, and

$$\mathcal{A}(W \bigcup_{M} \circlearrowleft) \cong \mathcal{A}(W) \bigotimes_{\mathcal{A}(M)} \circlearrowleft .$$

Example

We can express $B^3 \times S^1$ as $B^4 \bigcup_{B^3} \circlearrowleft$. To compute $\mathcal{A}(B^3 \times S^1, L)$, where L wraps around S^1 twice, we need to understand the category $\mathcal{A}(B^3, 2\text{pts})$.

We have a tentative answer.

Usually, Khovanov homology is calculated recursively via the long exact sequence.

The long exact sequence appears to fail when $W \neq B^4$!

This is unsurprising: we construct $\mathcal{A}$ by taking a big quotient, so expect to lose exactness.

The blob complex allows us to define a chain complex $\mathcal{A}_\bullet(W, L)$, with $H_0 = \mathcal{A}(W, L)$. The long exact sequence survives here, giving a spectral sequence converging to zero. This may allow computation of $\mathcal{A}(W, L)$, if we can compute all of $\mathcal{A}_\bullet(W, L')$ for some simpler links L'.