On Khovanov's cobordism theory for $\mathfrak{su}_3$ knot homology.

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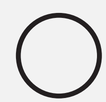
What is the $\mathfrak{su}_3$ spider?

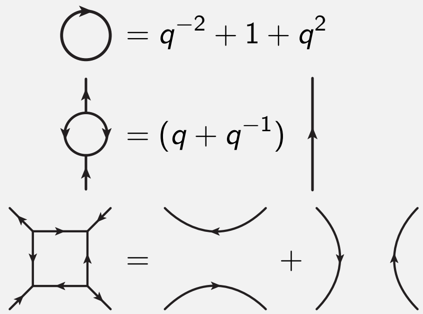
- To begin, we'll recall how to calculate quantum knot invariants using "spiders". For $\mathfrak{su}_2$ this is just another way of talking about the Kauffman bracket. The spider for $\mathfrak{su}_3$ was introduced by Kuperberg.
- In both cases, the spider is a type of *planar algebra*. It consists of $\mathbb{Z}[q, q^{-1}]$-linear combinations of certain diagrams drawn in discs.

In the $\mathfrak{su}_2$ spider, the diagrams consist simply of unoriented arcs, with no crossings (like in the Temperley-Lieb algebra or the Kauffman skein module). There's just one relation:

 $= q + q^{-1}$

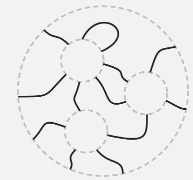
- In the $\mathfrak{su}_3$ spider, diagrams have oriented edges, and two types of vertices:

- Now there are three relations:



What is a "planar algebra"?

Definition
A 'planar tangle' is a diagram like this one:



Planar tangles can be composed in obvious ways, by gluing smaller planar tangles into the internal discs of a larger planar tangle. Tangle composition is associative.

Definition
A planar algebra $\mathcal{P}$

- associates a set $\mathcal{P}_k$ to each disc with k marked points on the boundary,
- associates to a planar tangle T with internal discs $D_{k_1}, D_{k_2}, \ldots, D_{k_n}$ and k_0 external marked points a map $\mathcal{P}_T : \mathcal{P}_{k_1} \times \mathcal{P}_{k_2} \cdots \times \mathcal{P}_{k_n} \to \mathcal{P}_{k_0}$,
- and this association is compatible with tangle composition.

If $\mathcal{C}$ is some monoidal category, a "planar algebra over $\mathcal{C}$" associates an object of $\mathcal{C}$ to each disc, instead of just a set, and associates an appropriate morphism in $\mathcal{C}$ to each planar tangle.

Tangles and webs form planar algebras

Tangles form a prototypical example of a planar algebra.

- Crossings and Reidemeister moves provide a presentation by generators and relations.

$\mathfrak{su}_3$ webs form a planar algebra.

- Discs have oriented marked points on their boundaries.
- Instead of just sets we have $\mathbb{Z}[q, q^{-1}]$ modules.

Other examples...

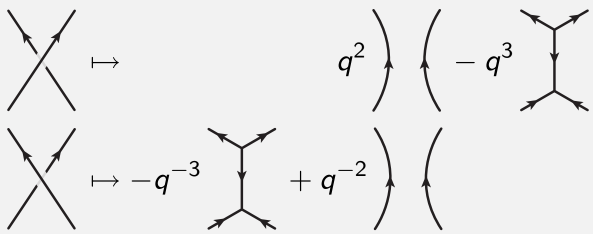
- Planar contraction of tensors.
- Subfactor planar algebras.

Calculating knot invariants

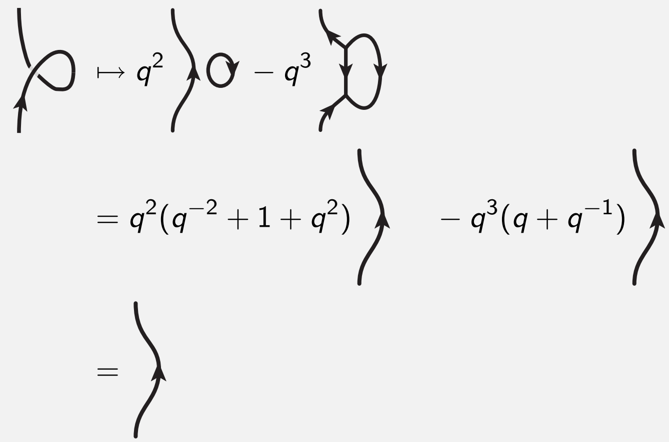
The $\mathfrak{su}_3$ quantum knot invariant is a map of planar algebras

$$\textbf{Tangles} \to \textbf{Spider}(\mathfrak{su}_3)$$

defined by

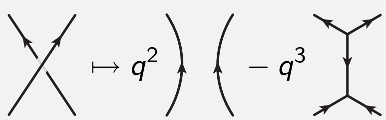


Example (Reidemeister I invariance)

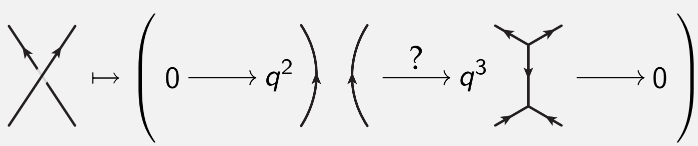


A recipe for categorification

- Positive integers are secretly dimensions of vector spaces.
- Integers are secretly Euler characteristics of complexes.
- This recipe tells us we should try to categorify quantum knot invariants by associating *complexes* to tangles, instead of linear combinations. We want to replace



with some complex



What is a "seamed cobordism"?

Definition

Given two $\mathfrak{su}_3$ webs, D_1 and D_2, drawn in a disc D both with boundary points ∂, a seamed cobordism from D_1 to D_2 is an oriented 2-complex F (the 'foam') embedded in a cylinder $D \times [0,1]$, such that:

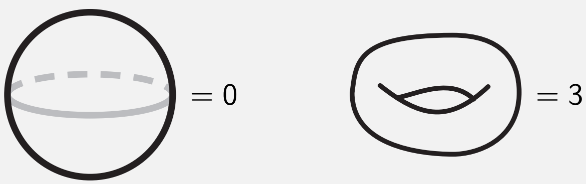
- The foam intersects the cylinder in $D_1 \times \{0\} \cup D_2 \times \{1\} \cup \partial \times [0,1]$.
- Inside the cylinder, the foam is locally homeomorphic to either
  - an oriented disc
  - or three half-planes meeting at an edge.

Seamed cobordisms form a category, which we'll call **Foam** ($\mathfrak{su}_3$). The objects of this category are the $\mathfrak{su}_3$ web diagrams. We compose seamed cobordisms simply by stacking them one atop the other.

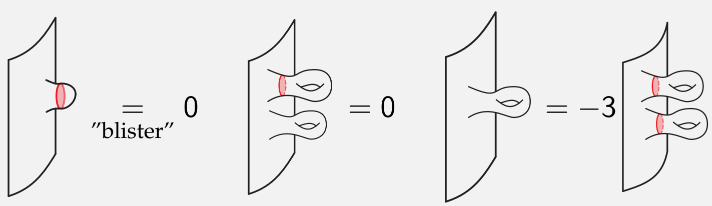
Local relations

- Just as we needed to add relations amongst $\mathfrak{su}_3$ webs to obtain the quantum knot invariant, we need to introduce relations amongst $\mathfrak{su}_3$ foams to obtain the categorified invariant.
- Using these relations, we'll discover certain isomorphisms amongst the objects in the $\mathfrak{su}_3$ foam category. These isomorphisms are categorifications of the relations amongst $\mathfrak{su}_3$ webs.
- Today, it will just look like the local relations imply these isomorphisms; more honestly, I'd show you how the relations are forced upon us if we want the isomorphisms to exist.
- There are quite a few relations!

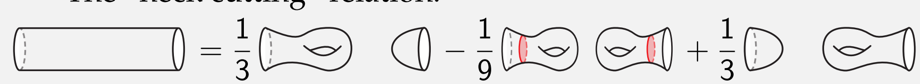
- "Closed foam" relations:



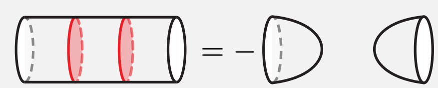
- "Sheet" relations:



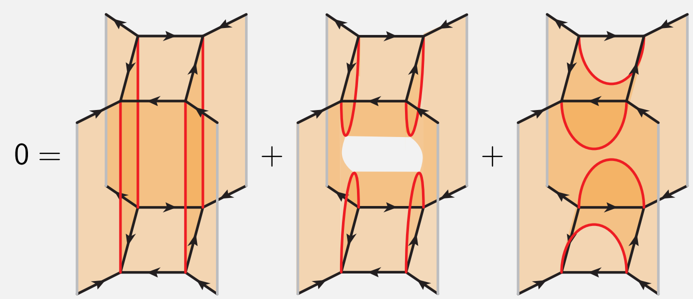
- The "neck cutting" relation:



- The "airlock" relation:



- The "three rocket" relation:



What is a "canopolis"?

A canopolis is something which is both a planar algebra and a category at the same time.

- We associate some category $\mathcal{C}_k$ to each disc D_k with k boundary points.
  - We think of the objects of $\mathcal{C}_k$ as living in the disc D_k,
  - and the morphisms as living in the cylinder $D_k \times [0, 1]$ – the 'can'. Morphisms are composed by stacking cylinders.
- We also insist that
  - the objects in all the different categories fit together as a planar algebra,
  - and the morphisms fit together as a (different!) planar algebra.
- Finally, there's a compatibility condition, saying that if you build a 'city of cans' using vertical and planar compositions, it didn't matter what order you did it in.

Example

The category of seamed cobordisms is actually a canopolis, with the objects forming the $\mathfrak{su}_3$ web planar algebra (just as sets; no linear combinations of diagrams, or relations).

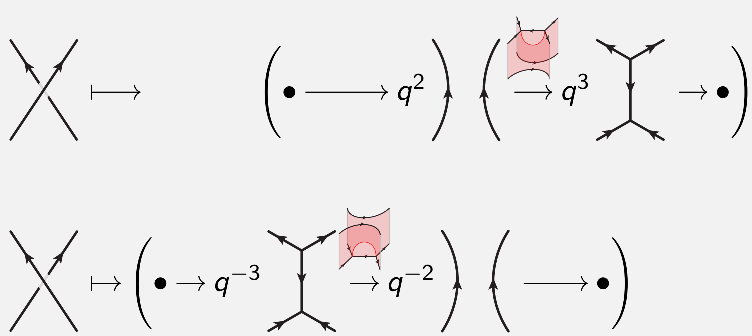
Example

Complexes in a canopolis form a planar algebra.
(See appendix for details.)

Finally we can say what the $\mathfrak{su}_3$ homology knot invariant is! It's a map of planar algebras, from tangles to up-to-homotopy complexes of foams,

$$\textbf{Tangles} \to \mathsf{Kom}_{/\text{htpy}}\,\textbf{Foam}\,(\mathfrak{su}_3)$$

defined by



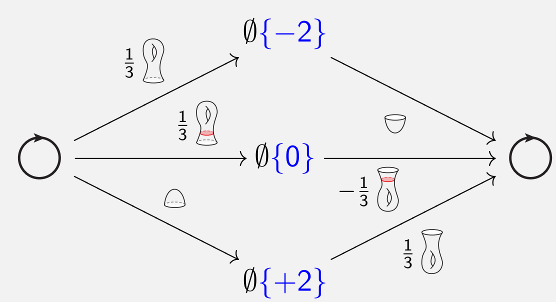
(See appendix for details.)

Isomorphisms

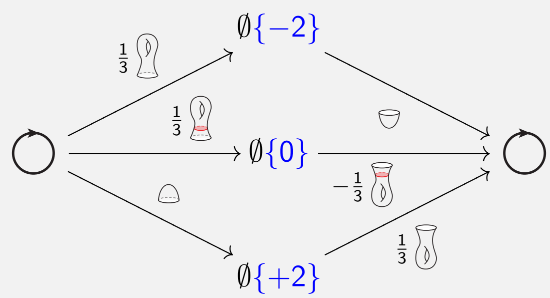
To begin, let's exhibit an isomorphism

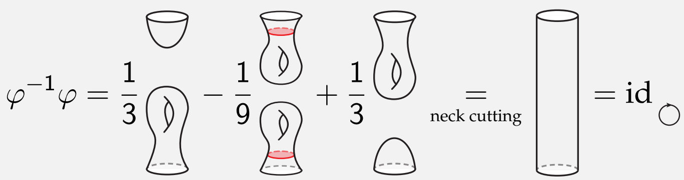
$$\varphi : \bigcirc \xrightarrow{\cong} \emptyset\{-2\} \oplus \emptyset\{0\} \oplus \emptyset\{+2\}$$

along with its inverse.

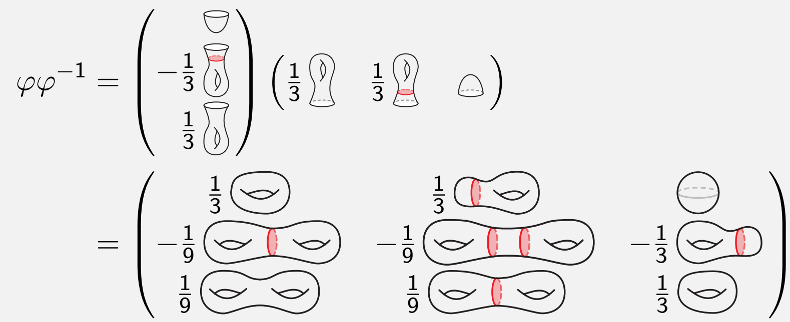


Now, let's check it really is an isomorphism!



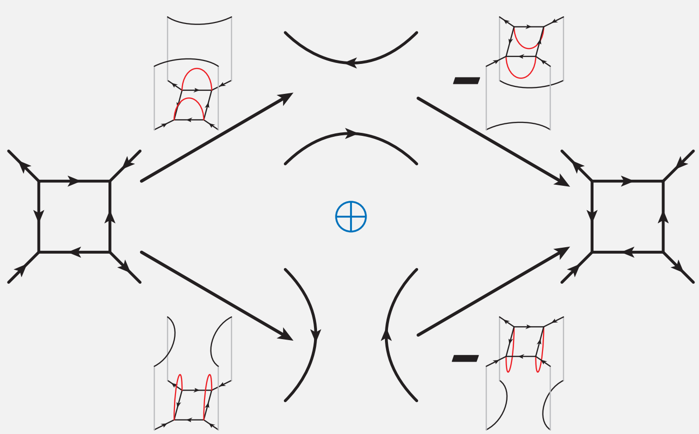


And now the other way:



$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{id}_{\emptyset\{-2\}\oplus\emptyset\{0\}\oplus\emptyset\{+2\}}$$

There's another isomorphism which 'explains' the rocket relation.



Decategorification

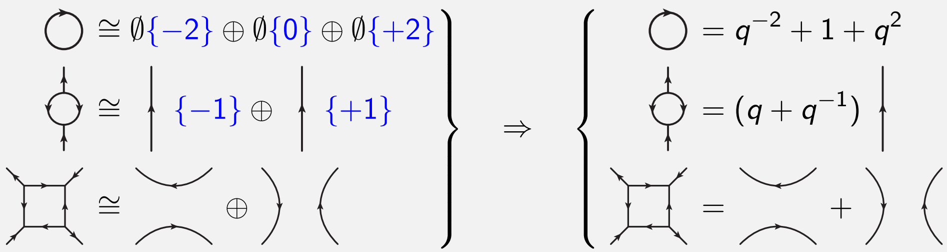
- We can "decategorify" a category, (not quite the usual way *(See appendix for details.)*) obtaining an abelian group:

$$\mathcal{C} \rightsquigarrow \left\langle \mathrm{Obj}(\mathcal{C}) \,\middle|\, \begin{matrix} A = B + C \text{ whenever} \\ A \cong B \oplus C \end{matrix} \right\rangle$$

- We can perform the same trick on a canopolis, obtaining a planar algebra of abelian groups.

Decategorifying **Foam** ($\mathfrak{su}_3$) to recover **Spider** ($\mathfrak{su}_3$)

- What is the planar algebra coming from the canopolis of $\mathfrak{su}_3$ foams?
- We've seen enough to be sure it's some quotient of the planar algebra of $\mathfrak{su}_3$ webs:



Every relation we expect appears. But are they any isomorphisms we haven't noticed?

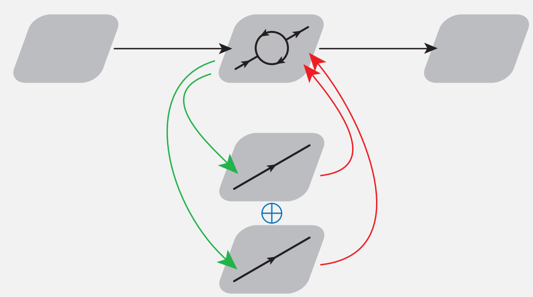
- We're still working on this! We can prove the corresponding result for $\mathfrak{su}_2$ by working out standard forms for morphisms.
- There's probably also an explanation in terms of $\mathfrak{su}_3$ representation theory: at generic values of q, there are no possible quotients.

An algorithm for simplifying complexes

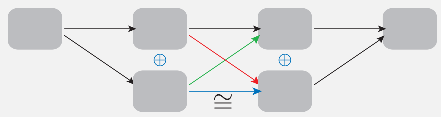
- The invariant of a tangle is an up-to-homotopy complex of seamed cobordisms.
- We'd like to have an algorithm for finding 'simpler' representatives of the homotopy class. With an effective algorithm we can
  - reduce proving Reidemeister invariance to some calculations; just apply the algorithm to both sides of the Reidemeister move, and
  - write a computer program!
- Further, this algorithm should, at least for knots and links, produce a 'standard form', independent of all choices.

The algorithm has two steps, 'decomposition' and 'cancellation'.

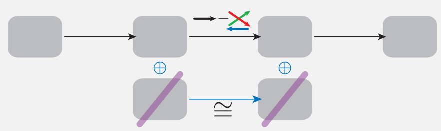
Decomposition Replace diagrams containing loops, bigons, or squares with their direct sum decompositions:



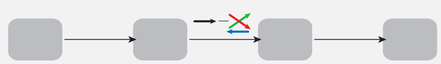
Cancellation Repeatedly cancel any isomorphisms appearing in the complex, using 'Gaussian elimination':



is isomorphic, as a complex, to



which has a contractible direct summand, that can be homotoped away.



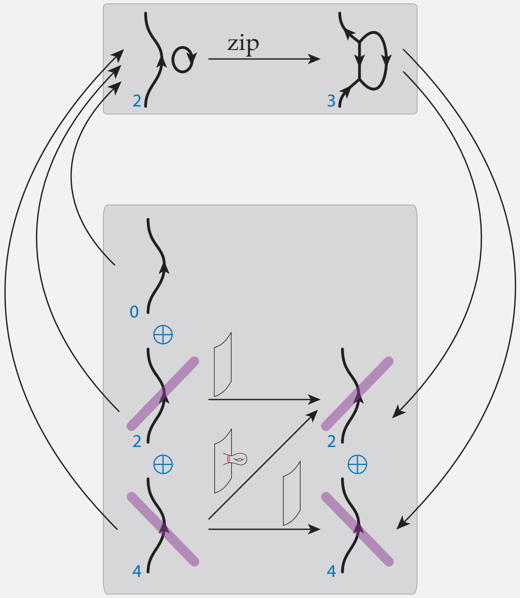
- This algorithm makes choices along the way; we can potentially get different answers by cancelling isomorphisms in different orders.
- For knots and links (that is, *closed* tangles), we eventually reach a complex in which only the *empty diagram* appears, because any closed non-empty $\mathfrak{su}_3$ web contains a loop, a bigon, or a square.
- Over $\mathbb{Z}_{(3)}$, the final form is a complex of matrices with entries in $\mathbb{Z}_{(3)}$.
- Over $\mathbb{Q}$, the final form is a complex with all morphisms being zero (because any non-zero matrix entry is invertible).

The Hopf link — oops not quite done yet!

Reidemeister invariance

- Previous proofs of Reidemeister invariance required constructing explicit homotopies.
- The simplification algorithm 'automates' the process.
  - Simplifying the complexes associated to each side of a Reidemeister moves always produces identical results.

Let's schematically simplify the complex associated to a positive twist, $\mathcal{F}\left(\ldots \right)$:



Appendix: link cobordisms

- Tangles form a planar algebra; in a natural way, tangle cobordisms (in $\mathbb{R}^4$) form a canopolis.
- Complexes in a canopolis form a planar algebra; complexes along with chain maps between them again form a canopolis.
- The $\mathfrak{su}_3$ homology invariant is actually functorial — it extends to a map between these canopolises.

Appendix: grothendieck groups

- Usually, one "decategorifies" by taking the Grothendieck group:

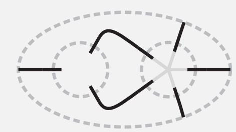
$$\mathcal{G}(\mathcal{C}) = \left\langle \mathrm{Obj}(\mathcal{C}) \;\middle|\; \begin{array}{c} A = B + C \text{ whenever} \\ (0 \to B \to A \to C \to 0) \\ \text{is an exact sequence} \end{array} \right\rangle$$

- This doesn't work, as **Foam** $(\mathfrak{su}_3)$ is not an abelian category; there are no notions of kernel, image or exactness.
- Instead we take the "split group":

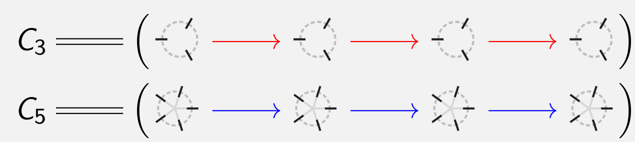
$$\mathcal{G}^{\text{split}}(\mathcal{C}) = \left\langle \mathrm{Obj}(\mathcal{C}) \;\middle|\; \begin{array}{c} A = B + C \text{ whenever} \\ A \cong B \oplus C \end{array} \right\rangle$$

Appendix: planar composition of complexes

- Given a quadratic (two internal discs) tangle,



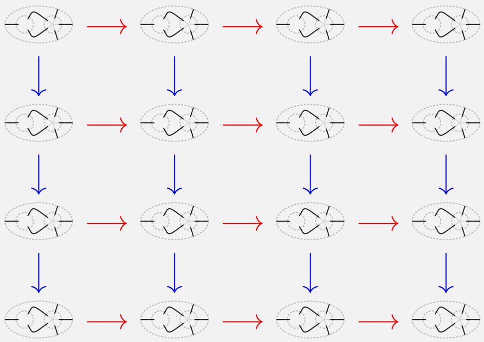
and a pair of complexes associated to the inner discs,



we need to define a new complex associated to the outer disc.

- We'll imitate the usual tensor product operation on complexes, making use of the planar tangle to combine objects and morphisms.

- First construct a double complex.



  - Each red arrow is the planar composition of an original red arrow with the identity on the right disc.
  - Each blue arrow is the planar composition of an original blue arrow with the identity on the left disc.

- Then collapse the double complex to a complex, by taking direct sums along the diagonals.