Functoriality in S^3

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http://tqft.net/faro

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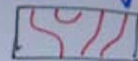
A parable

Let's begin with a parable, back in the world of the Temperley-Lieb category.

This is just

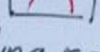
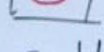
$\left\{\begin{matrix}\text{finite sets of}\\ \text{points on a line}\end{matrix}\right\} \circlearrowleft$ TL-diagrams



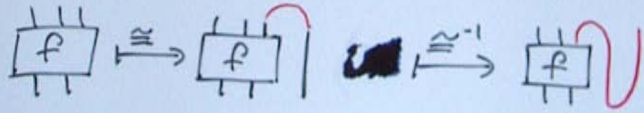
mod $\bigcirc = q + q^{-1}$ and isotopy.

You can go look up the axioms for a category with duals (or a 'rigid' category, or a 'pivotal' category or...) and notice that TL has duals:

- 'evaluation' and 'coevaluation' maps

 and 

- providing isomorphisms $\mathrm{Hom}(a,b) \cong \mathrm{Hom}(a+1, b-1)$



But I'd prefer you to think of it as a planar algebra.

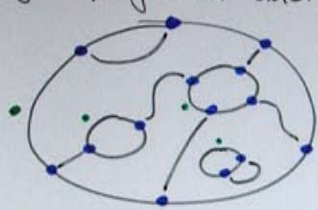
Temperley-Lieb as a planar algebra

- For each circle with marked points  there's the module of TL diagrams filling it:

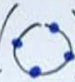
E.g.:

$\mathcal{P}($$) = \mathbb{C}[q, q^{-1}]\{$, $\}$

- For every 'spaghetti and meatballs' diagram



there's a map

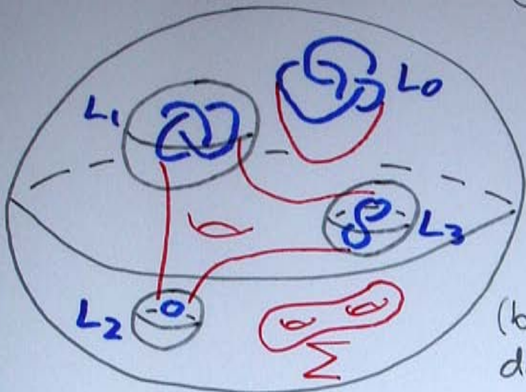
$\mathcal{P}($$) \otimes \mathcal{P}($$) \otimes \mathcal{P}($$) \longrightarrow \mathcal{P}($$)$

Sadly this isn't written down anywhere, but a planar algebra is exactly equivalent to a (strict) pivotal category.

Prejudice: planar algebras are <u>nicer</u> than pivotal categories.

A 4-d analogue for Khovanov homology

- Khovanov homology associates a bigraded vector space to each link in S^3.
  (no up-to-isotopy, no up-to-isomorphism)
- To each '4-ball with lasagna':



(be careful - one dimension suppressed)

a linear map $\bigotimes_i Kh(L_i) \longrightarrow Kh(L_0)$

depending on the surface Σ only up to isotopy in $B^4_0 / \cup B^4_i$.

The Khovanov 4-category.

Let's define

$$\mathrm{Objects}(\textbf{2-Tangles}) = \left\{ \begin{array}{l} \text{tangles (no isotopy) in } B^3 \\ \text{w/ bdy pts on the equatorial } S^1 \end{array} \right\}$$

$$\mathrm{Morphisms}(\textbf{2-Tangles}) = \left\{ \begin{array}{c} \text{cobordisms in } B^3 \times I \\ \text{mod isotopy} \end{array} \right\}$$

and another category K with the same objects, and

$$\mathrm{Hom}_K(T_1, T_2) = \mathbb{C}\left\{ \begin{array}{l} \text{chain maps } Kh(T_1) \to Kh(T_2) \\ \text{up to homotopy} \end{array} \right\}$$

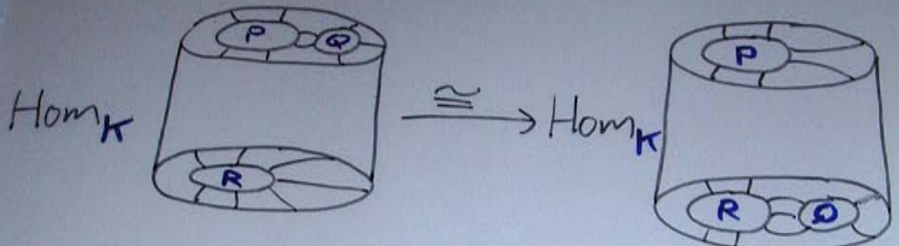
There's a functor $K \to$ **2-Tangles**;

that's just "functoriality of Khovanov homology"

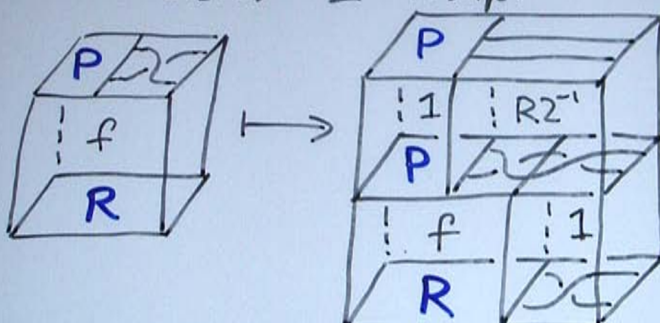
Composing tangles horizontally gives each of **2-Tangles** and K the structure of a 3-category, or, better, a **canopolis**.

More duals!

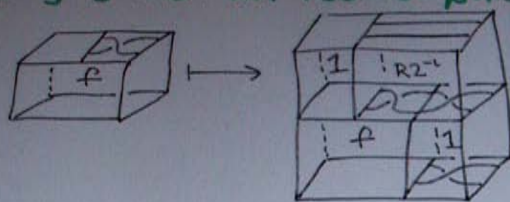
Now I want isomorphisms between different Hom spaces, allowing us to move subtangles between source and target.

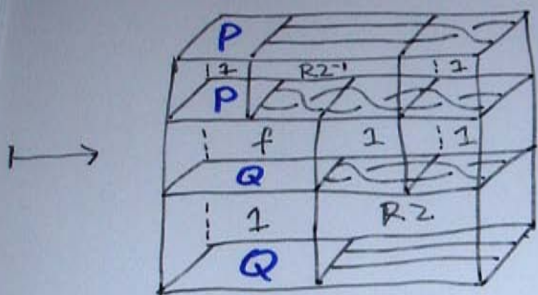


When Q is a single crossings, we use a Reidemeister 2 map:

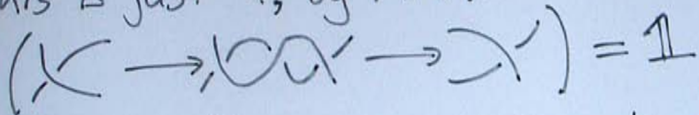


Why is that an isomorphism?

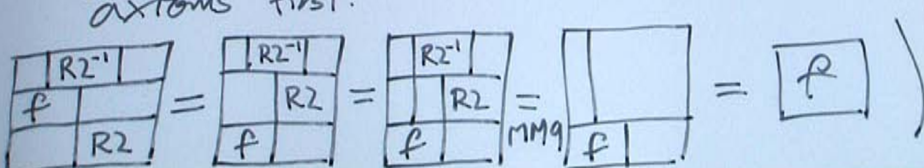




This is just f, by MM9:



(being more careful, we use some 2-category axioms first:

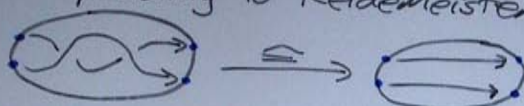


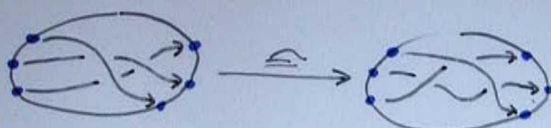
Even better, **K** is a **braided canopolis**

(Tautologically) we have objects called

 and 

and a collection of isomorphisms corresponding to Reidemeister moves

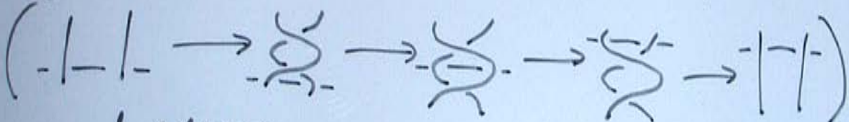




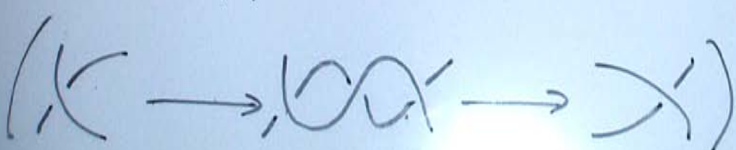
etc.

While these isomorphisms aren't identities, they satisfy nice coherence axioms coming from Carter & Saito's movie moves.

Eg MM6

 $= \mathbb{1}$

and MM9

 $= \mathbb{1}$

Coherence

We now have many isomorphisms of Hom-spaces
- moving a subtangle between source & target
- pre- (or post-) composing with an isotopy of the source (or target) tangle.

How do these all fit together?

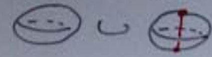
Think of each such isomorphism as induced by its 'wake', a surface in $S^3 \times I$.
- draw the initial source & target tangles on the hemispheres of the inner S^3, and the final tangles on the outer S^3.
- in the bulk, draw the cylinder which isotopes the subtangle past the equator.

Theorem (Morrison, Walker)

If two sequences of such isomorphisms have 'wakes' which are isotopic in $S^3 \times I$, then the compositions are equal, as isomorphisms of Hom-spaces.

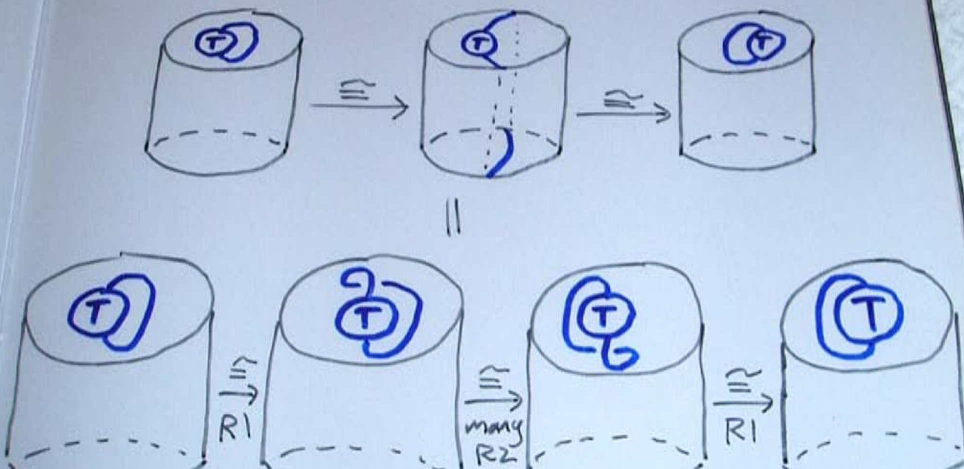
Proof

Most of the time, the wake avoids the 'axis of the southern hemisphere':

$S^3 =$ 

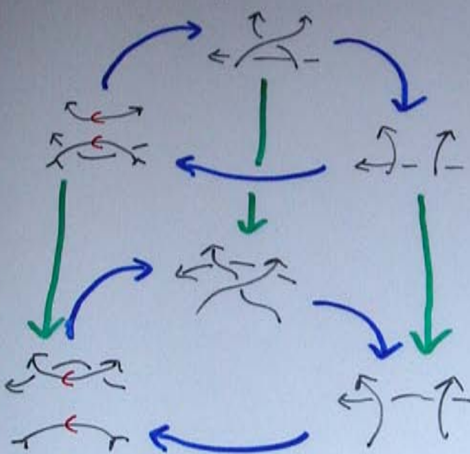
When it does, the result is just functoriality in B^3 (and some 3-category nonsense).

When it passes through the axis, we need to check:



Why does this work?

We have an extension of Kh to 'disoriented tangles', in which the exact triangle is functorial:



The square faces commute, not just up to sign.

Now the result follows easily, by induction of the number of crossings, and the five lemma.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