KEVIN WALKER ON TQFTS

1. Introduction

Today we'll talk about stuff related to skein modules. Eventually we'll need something more general, but these will do for now. Next, if we want to think about TQFTs as a physicist would we'll see that this leads us back to the notion of skein modules. Furthermore, in some sense these two notions will be equivalent. With these ideas in mind we'll reach a certain axiomatic framework. TQFTs are all about locality, so the key ideas will be gluing along codimension 1. A little more exotic will be notions of gluing along codimension 2, which will lead to so-called Drinfeld doubles and Drinfeld centers. Up to this point, everything will be in some sense combinatorial topology; however the quantities we define will be manifestly invariant.

However, at the "top level" there are things to check (e.g., invariants under Kirby moves). This leads to the Path Integral Theorem, which is related to certain combinatorial constructions such as state sums. The history of this subject is therefore backwards to the approach we'll be taking. Historically, state sums are the starting point.

We'll try to make this an example-driven sequence of talks: Turaev-Viro theories are in some sense generic $2+1$ dimensional theories; Dijgraaf-Witten theories; generic s.s. (semi-simple?) $1+1$ theories; Crane-Yetter-Kauffman theories (which lead to Witten-Reshethkin-Turaev invariants). If there is time, we'll get to more exotic examples like contact structures and Khovanov homology.

In essence, we'll be constructing a machine that takes an n-category (of a certain type) and spits out a $(n+1)$-dimensional TQFT. Again, the history of this is backwards: people originally discovered various examples independently, and only recently has the general framework been understood.

2. Initial Examples

We'll probably spend the entire day describing the $\mathbf{Z}/2$-homology of a surface, but from a different point of view. Perhaps had math developed in an alternate universe—had topologists turned right rather than left in the 1930s—we'd all understand homology in terms of TQFTs rather than the usual.

Let Y^2 be some compact 2-manifold with $\partial Y = \emptyset$, and let

$$\mathcal{F}(Y) := \{1 - \text{submanifolds of } Y\}.$$

We'll be thinking of this as just a set (not a space), and we'll define an equivalence relation $\sim$ on this set, and let

$$A(Y) := \mathbf{C}[\mathcal{F}(Y)]/\sim.$$

[[[PICTURES.]]]

So if we look locally in Y and see an arc going through a little 2-ball, first we allow for isotopies rel boundary. The next equivalence allows us to replace a circle inside the 2-ball by δ multiplied by the empty picture. The last picture allows us to swap pairs of arcs in the same 2-ball.

If we set $\delta = 1$, as a set it's a nice exercise to see that

$$\mathcal{F}(Y)/\sim \cong H_1(Y; \mathbf{Z}/2).$$

Probably the easiest comparison is via singular homology. So we have that

$$A(Y) \cong \mathbf{C}[H_1(Y; \mathbf{Z}/2)].$$

Pretending that we don't know homology theory, our goal will be to understand $A(Y)$ as we glue 2-manifolds together.

[[[PICTURE]]]

So if we start with Y_{cut} which has $\partial Y_{\text{cut}} \cong S^1 \bigsqcup S^1$ and glue the two circles we get a 2-manifold we call Y_{glue}. So how is $A(Y_{\text{cut}})$ related to $A(Y_{\text{glue}})$? Define

$$\mathcal{F}(M) \cong \{\text{codim 1 submanifolds}\}$$

and then we have a natural restriction,

$$\mathcal{F}(Y^2) \xrightarrow{\partial} \mathcal{F}(\partial Y),$$

but it will turn out that these restriction aren't quite what we want; really we want restriction with respect to some natural boundary condition. So let

$$\mathcal{F}(Y; c) := \partial^{-1}(c)$$

for $c \in \mathcal{F}(\partial Y)$ some fixed boundary condition. Then

$$A(Y; c) := \mathbf{C}[\mathcal{F}(Y; c)]/\sim.$$

So let's restate our goal: we will have our cut surface with a specified boundary condition c and c' on the two copies of S^1. Then we want to relate $A(Y_{\text{cut}}; c, c')$ to $A(Y_{\text{glue}})$.

Now for all $c \in \mathcal{F}(S)$, we want to understand

$$A(Y_{\text{cut}}; c, c) \xrightarrow{\text{gl}_c} A(Y_{\text{glue}})$$

and then let c vary, giving a surjective map

$$\bigoplus_{c \in \mathcal{F}(S)} A(Y_{\text{cut}}; c, c) \xrightarrow{\text{gl}_c} A(Y_{\text{glue}}).$$

We'd like to determine the kernel of the map. We note that the sum is not over objects up to isotopy; in generally we'll see that isotopy only happens in the "top dimension." In higher codimension, we'll never want to do things up to isotopy. So in our example the sum is actually over some uncountable set of points on circles.

[[[PICTURES WITH COLLARS.]]]

Gluing our surface up along two circles will be the same as gluing in a little cylinder. But then there are two ways to glue in the cylinder. To denote this, let $e \in A(S \times I; c, c')$ and $x \in A(Y_{\text{cut}}; c, c')$. Then

$$x \cdot e \in A(Y_{\text{cut}}; c', c'), \quad e \cdot x \in A(Y_{\text{cut}}; c, c)$$

and we want to impose $gl_{c'}(x \cdot e) \sim gl_c(e \cdot x)$, so $x \cdot e - e \cdot x \in \mathsf{Ker}(gl)$.

Theorem 2.1 (Codimension 1 Gluing, Version 1)**.**

$$\mathsf{Ker}(gl) = \langle x\dot{e} - e \cdot x \rangle$$

We'll prove this later.

Let M^1 be a closed 1-manifold.

Definition 2.2. The cylinder category, denote $A(M)$, has object $\mathcal{F}(M)$ and morphisms $a \to B$ given by $A(M \times I; a, b$. Composition is given by gluing cylinders.

This gives us a 1-category, and in fact a linear 1-category.

Notice that for Y a 2-manifold, (∂Y) acts on $\{A(Y; c)\}$ where $c \in \mathcal{F}(\partial Y)$ where c runs through all boundary condition. So what does it mean for a category to act? First, we have a vector space for every object, and then for every morphism in the category we have a linear map. Said quickly, we have a functor from the linear category to the category of vector spaces.

[[[PICTURE.]]]]

The picture is that we can glue cylinders on to the boundary of Y (provided the boundary conditions match up), and stacking the cylinders is composition.

Note that so far what we have is a 1-category for every 1-manifold, and a vector space for every 2-manifold.

3. Abstracting the Example

Recall for $S, Y_{\text{cut}}, Y_{\text{glue}}$ we have

(1) $\forall a \in \mathcal{F}(S)$, $\text{gl}_a : A(Y_{\text{cut}}; a, a') \to A(Y_{\text{glue}})$

(2) $\forall e : a \to b$ $e^* : b \to a$, $e \in \mathsf{Mor}(A(S))$ we have

$$\begin{array}{ccc} A(Y_{\text{cut}}; a, b) & \overset{1 \times e^*}{\to} & A(Y_{\text{cut}}; a, a) \\ \downarrow e \times 1 & & \downarrow \text{gl}_a \\ A(Y_{\text{cut}}; b, b) & \overset{gl_b}{\to} & A(Y_{\text{glue}}) \end{array}$$

Theorem 3.1 (Codimension 1 Gluing Theorem, Version 2)**.** *$A(Y_{\text{gl}})$ is universal with respect to (1) and (2) above. $A(Y_{\text{gl}})$ is the coend of the $A(S) \times A(S)^{op}$-module $A(Y_{\text{cut}})$.*

The third version of the gluing theorem will use certain semisimplicity features of the categories, which will make more abstract examples computable.

Let's consider the category $A(S^1)$ in more detail. Notice that any object in $A(S^1)$ is isomorphic to either a circle with no points or a circle with 1 point: given a circle with two points, we can pair them on the cylinder, and this morphism is an isomorphism. We'll call these two objects 0 and 1. Now,

$$\mathsf{Hom}(0, 1) \cong 0 \cong \mathsf{Hom}(1, 0).$$

and

$$\mathsf{Hom}(0, 0) \cong \mathbf{C}(\emptyset, R)$$

[[[PICTURE]]] where $\emptyset$ denotes the map $0 \to 0$ with not arcs on the cylinder, and R is the map $0 \to 0$ with a single circle in the interior (R for "ring" on the cylinder). Lastly,

$$\mathsf{Hom}(1,1) \cong \mathbf{C}(1,Y)$$

[[[PICTURE]]] where the morphism 1 connects the points in a "straight line" and T is a twist of this line.

We can now find some idempotents by looking at these morphisms:

$$e_0^{\pm} := \frac{1}{2}(\emptyset \pm R), \quad e_1^{\pm} := \frac{1}{2}(1 \pm T).$$

Let $\mathcal{L} = \{e_0^+, e_0^-, e_1^+, e_0^-\}$. We claim that $A(S^1)$ is semisimple and that $\mathcal{L}$ is a complete set of minimal idempotents.

Before we considered $A(Y;c)$ for an arbitrary boundary condition c, and now we'll consider $A(Y;\alpha)$ where α is an idempotent.

Theorem 3.2 (Codimension 1 Gluing Theorem, Version 3)**.**

$$A(Y_{\text{glue}}) \cong \bigoplus_{\alpha \in \mathcal{L}} A(Y_{\text{cut}}, \alpha, \alpha)$$

The above is what we we're really after: the direct sum is over a finite (or at worst, discrete) set. Historically this was the first thing people wrote down, so why would you every bother with something more complicated? Well, there are other examples that we want to consider coming from semisimple linear categories, and then the above will prove useful.

4. Recap and Generalizations

The first two skein relations we have are more-or-less fixed; namely we require isotopy invariance and that circles can be replaced by the empty picture multiplied by some δ. After that, however, there is a lot of freedom.

For example, we we do one of the saddle moves we could multiply by -1. For this to be consistent, we need $\delta = -1$. As an (easy) exercise, one can show that

$$\mathsf{dim}(A_{\delta=-1}(Y)) \cong |H_1(Y;\mathbf{Z}/2)| = \mathsf{dim}(A_{\delta=1}(Y)).$$

Thus, these guys are isomorphic as vector spaces. However, we claim that there is no natural isomorphism (e.g. with respect to homeomorphisms or gluing. What is the $\delta = -1$ example isomorphic to? The hint is to think about spin structures.

Next, note that there is an exact sequence

$$0 \to U(D^2;c) \hookrightarrow \mathbf{C}[\mathcal{F}(D^2;c)] \to A(D^2;c) \to 0,$$

so classifying the possible gluing maps is the same as classifying $U(D^2;c)$. But these are just the local (skein) relations! So one might try to classify these. Let $b \in \mathcal{F}(\partial D^2)$ be minimal in $U(D^2;b) \neq 0$.

[[[PICTURES.]]]

For example, say b is six points; one can write down six equations for the coefficients of the various generating pictures, and it turns out there is a unique solution, up to scale.

The point is that we can think of these local relation pictures as providing generalizations of $\mathbf{Z}/2$-homology that only make sense on surfaces.