Informal Minicourse on TQFTs: Lecture 2

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The plan for today:

- More on Temperley-Lieb type examples
- Finite group examples (Dijkgraaf-Witten theories)
- Topological + QFT $\rightsquigarrow$ (generalized) skein modules
- Axioms for "fields" and local relations

Last time we had various examples of functors of manifolds, and last time we mostly focused when $n = 2$, and we looked last time at the functor $\mathcal{F}(M^n) = \{\text{curves in } M\}$, and then we defined $A(M, c) = \mathbb{C}[\mathcal{F}(M, c)]/\sim$, the vector space of linear combinations of curves modulo certain local relations. There are many different collections of local relations one you use, but the idea is to prove a gluing theorem once and for all, and then look at examples.

In the particular case of last time ($n = 2$), the relations were: isotopy; remove trivial loops by multiplying by some $\delta \in \mathbb{C}$; ....

There are a family of examples that start out with isotopy and removal of trivial loops. These together are enough to show that $A(D^2, c)$ (the skein module of the disk with any boundary conditions) is finite-dimensional, but with just those two we still have infinite-dimensional spaces for other surfaces.

Then last time we focused on $\mathbb{Z}/2$-homology, where ****picture****. This makes sense when $\delta = 1$. We might call this "K=4", because the pictures have four boundaries.

Then we mentioned a more novel theory, from the last fifty years, which might be called "K=6". It is determined by ****picture****, where $\delta^2 = 2$.

How can we construct this? We can solve the system of equations given by asking for a sum of diagrams with K=even many edges, and the condition that capping any adjacent pair gives zero. This strategy leads to an infinite family of theories, which we will see are closely related to the representation theory of quantum SL(2).

Exercise: For $\delta^2 = 2$ K=6 theory, show that $\dim(A(T^2))$ is: (a) ≤ 15, (b) ≤ 10, (c) ≤ 9, (d) $= 9$. These exercises are listed in the order of difficulty. $\diamondsuit$

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Exercise: Solve the K=8 equations. If we have 8 points on the disk, then there are 14 possibly diagrams (the Catalan number), and you should find the linear combination that always caps to zero. This is a tedious exercise, but you can do it by hand, but we'd like a general machinery which we might discuss later. ♢

Exercise: The shaded version of the K=6 theory. **Matt:** So you do all possible shadings, so that it's a sum of 10 diagrams? **Kevin:** No. The shadings have to agree on the boundary, so it's only a sum of five terms.

Then again we can look at the skein module for the torus. Then that has at two distinguished elements: the black empty diagram; and the white empty diagram. The question is: are these two diagrams the same? ♢

Let T be a topological space (T for "target"). One example we're particularly interested in is when $T = \mathrm{B}G$, the classifying space of a finite group. Now, we can define another functor $\mathcal{F}$, which works for any n (fixed), given by $\mathcal{F}(Y^k) = \{\text{maps } Y \to T\}$. This works for any $k \leq n$. We need a local relation. The definition of $\mathcal{F}$ applies at any dimension, but for the local relation we just use top-dimensional things. So we say that if $f, g \in \mathcal{F}(M^n)$, then $f \sim g$ if f is homotopic (rel boundary) to g.

Lemma: *This is a local relation — it's enough to impose the relation in disks.*

Then we have the same gluing theorem as before. We will recall it now. To define the gluing theorem, we first introduce a category for each $(n-1)$-dim manifold. Namely, define $A(Y^{n-1})$ to be the 1-category whose objects are $\mathcal{F}(Y)$ (not up to homotopy!), and $\hom(a, b) = A(Y \times I; a, b)$, and composition is gluing. Then we consider the situation where we have M, and we glue it up to M_{glue} ****picture****. Then $A(Y) \otimes A(Y)^{\text{op}}$ acts on $A(M, \cdot)$, and:

Theorem: *$A(M_{\text{glue}})$ is the coend of this action.*

Corollary: *In the special case when $M = M_1 \sqcup M_2$ and $M_{\text{glue}} = M_1 \cup_Y M_2$, when $A(M_1 \cup_Y M_2) = A(M_1) \otimes_{A(Y)} A(M_2)$.*

In the special case when $T = \mathrm{B}G$, we can think of this as being about isomorphism classes of G-bundles.

Anyway, we still haven't proved this theorem, but it's just the same as in the curves case.

TQFT $\rightsquigarrow$ skein modules

I should warn you. I try to give this argument in talks, but I usually just get blank stares. The beginning I think is more or less known, but the end is less well-known, and that's where the heart is.

So, what is QFT? I won't try to say what it is, but one fundamental part of it is the Path Integral.

In this, one considers $(n+1)$-dimensional manifolds W, and considers:

$$Z(W^{n+1};c) = \int_{f\in\mathcal{F}(W;c)} T(f) \in \mathbb{C}$$

Here $c \in \mathcal{F}(\partial W)$, and $\mathcal{F}(W;c) =$ "fields on W which restrict to c on ∂W". See, when one tries to do physics, one finds that without boundary conditions, the integrals that one defines for closed manifolds don't work. Also, I will not be rigorous in this section, so don't sweat the analytic details.

Also, $T : \mathcal{F}(W;c) \to \mathbb{T} \subset \mathbb{C}$, where $\mathbb{T} = U(1)$ is the circle group. Usually one writes $T(f) = e^{iS(f)}$, and calls S the "action" of the theory, but for what we're doing that's irrelevant.

What we'll do is look at locality, and see how far it gets. In real examples, it's notoriously difficult to make this integrals well-defined. What one typically does in TQFT is to bypass these difficulties, and we'll try to make as short a bypass as we can, preserving the story. But at the end of the day we will not use path integrals.

Let's suppose that the "field" functor $\mathcal{F}$ is *local* in the following sense. Given a situation where we're gluing $(n+1)$-dim manifolds, ****picture****, W with boundary conditions $\to W_{\text{glue}}$, then we have a restriction maps:

$$\mathcal{F}(W_{\text{glue}}; c_{\text{glue}}) \to \mathcal{F}(W;c,\cdot) \rightrightarrows \mathcal{F}(Y,d)$$

and the first request is that this presents $\mathcal{F}(W_{\text{glue}})$ as an equalizer. The second request is that $\mathcal{F}(W_1 \sqcup W_2) = \mathcal{F}(W_1) \times \mathcal{F}(W_2)$.

Let's also suppose that T is *local*, so that $T(x_1 \sqcup x_2) = T(x_1)\cdot T(x_2)$.

Then it follows that to compute things, we can chop up:

$$Z(W_{\text{glue}}) = \int_{x\in\mathcal{F}(Y^n;d)} \left[\int_{y\in\mathcal{F}(W;c,x,x)} T(y)\right]$$

Again: I'm not proving this is true. I'm saying that in reasonable examples we can expect it to be true.

Then we consider the vector space $V(M^n)$ to be the vector space of functions $\mathcal{F}(M^n) \to \mathbb{C}$. What kind of functions? Ł^2, smooth, ? I won't try to say.

Then we can consider $Z(W^{n+1}) \in V(\partial W)$, by $Z(W)(c) = Z(W;c)$.

Theo: Am I write to think that this vector space V is what we were calling A before? **Kevin:** No. The vector space V will be much bigger than A, and is in some sense dual to it.

Then we can think about locality for $V(M)$. I.e.: how does it behave under gluing? Consider gluing M^n along a boundary P^{n-1}, or cutting M_{glue} along a submanifold P. Then we expect

$$V(M_{\text{glue}}) \cong \text{``}\bigoplus_{y\in\mathcal{F}(P)}\text{''}\, V(M;y)$$

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for some notion of $\bigoplus$. Here $V(M;y) = \mathrm{Maps}(\mathcal{F}(M;y) \to \mathbb{C})$.

****picture**** So, now consider W^{n+1} with boundary broken into two pieces $\partial_{\mathrm{in}} W$ and $\partial_{\mathrm{out}} W$. Write $P = \partial(\partial_{\mathrm{in}} W) = \partial(\partial_{\mathrm{out}} W)$. Choose $d \in \mathcal{F}(P)$.

Then I claim that $\forall d$, we can think of the path integral as a map

$$Z(W)_d : V(\partial_{\mathrm{in}} W; d) \to V(\partial_{\mathrm{out}} W; d)$$

How? Given $f \in V(\partial_{\mathrm{in}} W; d)$ and $x \in \mathcal{F}(\partial_{\mathrm{out}} W; d)$, we define

$$Z(W)_d(f)(x) = \int_{y \in \mathcal{F}(W;x)} f(y|_{\partial_{\mathrm{in}} W}) \cdot T(y)$$

Then analogy is that we can think of the path integral as being like a matrix.

Theo: You've secretly snuck in an inner product on V. **Kevin:** Yes. I'm suppressing some of those details.

So this is a very standard idea. You look at the earliest papers of Atiyah where he coined the term "TQFT", and he does this. So I've said nothing new so far.

Also so far everything makes sense in the non-topological case. Then I'd be giving you a very vague introduction to QFT. But soon we'll use the assumption that it only depends on the topological type of the manifold.

So we're thinking of W as a bordism with corners between the different pieces of its boundary. So think about $Y^n \times I$. Now, you might think that $Y \times I$ looks like a rectangle, but I want you to think of it as a bigon. We want the *pinched product*, whereby we identify $(p,t) \sim (p,t')$ for $p \in \partial Y$. This is because we want $\partial Y \times I = Y \cup Y$ even when Y has boundary.

Then we've defined operators $Z(Y \times I)_d : V(Y;d) \to V(Y;d)$ for each $d \in \mathcal{F}(\partial Y)$. But a fundamental fact about $Y \times I$ is that there's an isomorphism $(Y \times I) \cup_Y (Y \times I) \cong (Y \times I)$ (not as Riemannian manifolds, but yes as smooth or PL or whatever).

Then if we assume that Z is topologically invariant — and this is a very strange thing to assume (Witten emphasizes how strange it is for his physicist friends in his earliest papers on the subject) — then $Z(Y \times I)_d \circ Z(Y \times I)_d = Z(Y \times I)_d$. Oh, I forgot to say, we have *composition* property, that $Z(W_1 \cup W_2)_d = Z(W_2)_d \circ Z(W_1)_d$, which is a restatement of previous properties.

Then $Z(Y \times I)_d$ is a projector. Then we define the *Hilbert space* for the theory to be $Z(Y;d) = \mathrm{Im}(Z(Y \times I)_d)$.

Then an obvious observation: for all W^{n+1}, we can glue collars onto it, so that $W \cup_{\partial W} (\partial W \times I) \cong W$. This implies that $Z(W)$, which originally lives in $V(\partial W)$, in fact lies in $Z(\partial W)$.

This is roughly the difference, when we were doing skein modules, of passing from all linear combinations of pictures to a finite-dimensional quotient.

Theo: I could imagine that in examples, this construction can help handle analytic difficulties, e.g. because I start with $V =$ all functions, and then Z consists just of molified functions? **Kevin:** In Chern–Simons theory, an important piece of work is to get down to these almost-finite-dimensional spaces.

So, what I've said so far is essentially in Atiyah's papers. I take a big W^{n+1}, and cut it into slices. But what I want to say now is that I can use these locality arguments to cut even further.

So consider an n-manifold M^n, which I want to think of as the boundary of some $n+1$ manifold, and it has a subspace B, which might be a copy of the disk. Then I can glue on $B \times I$. ****picture****

Then I find out that $M = B \cup_S (M \smallsetminus B)$, and then $V(M^n) = \bigoplus_{d \in \mathcal{F}(S)} V(B;d) \otimes V(M \smallsetminus B; d)$. Oh, one thing I should have said when talking about locality for V is that $V(M_1 \sqcup M_2) = V(M_1)$"$\otimes$"$V(M_2)$, with the tensor in quotes because of analytic difficulties.

Then, writing $\Pi_{(Y,d)} = Z(Y \times I)_d$, which is the projector from before, we have:

$$V(M^n) = \bigoplus_{d \in \mathcal{F}(S)} V(B;d) \otimes V(M \smallsetminus B; d) \xrightarrow{\bigoplus_d \Pi_{(B,d)} \otimes \mathrm{id}} V(M)$$

Set $\Pi_B = \bigoplus_d \Pi_{(B,d)} \otimes \mathrm{id}$.

Now, consider $M \times I$, and consider the union along B of $B \times I$ (always with pinched boundary conditions). Then $(M \times I) \cup_B (B \times I) \cong M$. And so it follows that $\Pi_M \circ \Pi_B = \Pi_M$. This is very similar to what we wrote down before, but now it's only a partial or relative thing.

So those are both projections, and it implies that $Z(M) = \mathrm{Im}(\Pi_M) \subseteq \mathrm{Im}(\Pi_B)$, and this is true for all disks $B \subseteq M$. And this implies:

$$Z(M) \subseteq \bigcap_{B \subseteq M} \mathrm{Im}(\Pi_B) \qquad (\star)$$

Then the point is that the $\mathrm{Im}(\Pi_B)$s are local.

See, Atiyah et al said "A TQFT is a type of functor", but that definition underspecifies things: the vector spaces associated to boundaries should satisy their own type of locality.

We'd like now to improve on $(\star)$, by proving equality. So, take an open cover of M by balls, and order the opens in the cover. Then I can glue on little balls in some order. When we're done, we find that, if we do it right, then $\bigcup_i (B_i \times I) = M \times I$. What we've been doing all along is anytime we have a union, we get an equality of path integrals. So then we find out that $Z(M \times I) = \Pi_{B_1} \circ \dots$.

Theo: Should it be completely obvious to me why all these projectors commute with each other? **Kevin:** Maybe not completely obvious, but they do. That's because, schematically, we have $(B_1 \times I) \cup (B_2 \times I) \cong (B_2 \times I) \cup (B_1 \times I) \cong (B_1 \cup B_2) \times I$. ****picture**** **Theo:** And another question: I see how to do what you said in the compact case, by taking a finite cover. Do you have the machinery to do this in the noncompact case? **Kevin:** In these lectures, every manifold is compact. It's an interesting question how to do the noncompact case.

Next time, we will take this all-important equation

$$Z(M) = \bigcap_{B \subseteq M} \mathrm{Im}(\Pi_B) \tag{$\star\star$}$$

and deduce from it the dual picture: we will consider the dual space to $Z(M)$, and we will see that it is a quotient of the dual space to $V(M)$.

Also, we will try to be a bit more axiomatic and precise about some of the analytics.

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