Informal Minicourse on TQFTs: Lecture 2

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Today's outline:

- quick review of last lecture
- Path integral $\rightsquigarrow$ skein module
- variety of axiomatics for TQFT
- our axioms

So, last time we talked about a field functor $\mathcal{F}(X^k) =$ "fields", $k \leq n+1$, which is something we fixed, and we imagined having a function $T : \mathcal{F}(W^{n+1}) \to \mathbb{C}$, $T = e^{iS}$ in the usual notation, and we had the *path integral* for $c \in \mathcal{F}(\partial W)$

$$Z(W^{n+1}; c) = \text{“}\int_{\mathcal{F}(W;c)}\text{”}\, T(x)$$

Then we set

$$V(M^n; c) = \{\text{functions } \mathcal{F}(M; c) \to \mathbb{C}\}$$

and then we can think of c as an argument, and then $Z(W^{n+1}) \in V(\partial W)$. Also, we might assume there's some kind of inner product on V, say deriving from the path integral on $M \times I$. Then we can divide ∂W into $\partial_{\text{in}} \cup \partial\text{out}$, with $\partial\partial_{\text{in}} = \partial\partial_{\text{out}} = P$, and then for $d \in \mathcal{F}(P)$ we can think of $Z(W)_d : V(\partial_{\text{in}}; d) \to V(\partial_{\text{out}}; d)$.

In particular, we get a projection $\Pi_{M,d} = Z(M \times I)_d : V(M; d) \to V(M; d)$. I should remind you that when I write $M \times I$ when $\partial M \neq \emptyset$, then I mean the pinched boundary conditions. I don't want to have $\partial M \times I$ parts of the boundary.

Then we define the *physical Hilbert space* to be $Z(M, d) = \text{Im}(\Pi_{M,d})$. Typically V will be very large but Z will be finite-dimensional. Then we saw that $Z(W^{n+1}) \in Z(\partial W) \subseteq V(\partial W)$.

Then we also considered B a ball in M^n, and choose $d \in \mathcal{F}(\partial B)$. Then we say that for any such ball, there was a projection $\Pi_B = \text{“}\bigoplus_{d \in \mathcal{F}(\partial B)}\text{”}\Pi_{B,d} : V(M) \to V(M)$. Since gluing a ball onto $M \times I$ doesn't change it, it follows that $\text{Im}(\Pi_M) \subseteq \text{Im}(\Pi_B)$. This is true for all B, and the Π_Bs commute, and so $Z(M) \subseteq \bigcap_{B \subseteq M} \text{Im}(\Pi_B)$.

Actually, something stronger was true. We have this procedure, which we will make more precise later, for turning manifolds into algebras. Using this, we see that if we have $\{B_i\}$ a cover of M, then $M \times I = \bigcup_i B_i \times I$, by adding lumps of clay to build up the whole thing. This implies that $\boxed{Z(M) = \bigcap_{B \subseteq M} \mathrm{Im}(\Pi_B)}$. This was the punchline from the last lecture.

Kevin Lin: I don't understand your proof with the lumps of clay that $M \times I = \bigcup_i B_i \times I$? **Kevin:** It's a partition of unity argument. You choose a partition of unity that's compatible with your cover, and the height of the lump is given by the value of the partition of unity.

So, now we'll consider the dual of this picture. Choose the predual of V to be $V(M)^* = \mathbb{C}[\mathcal{F}(M)]$. (We're still at a physical level of rigor, so we won't be too worried about duals and so on — the path integral is not well defined except in a few cases.)

Then we should have $Z(M)^*$ a quotient of $V(M)^*$. What we do is: for a ball B and $c \in \mathcal{F}(\partial B)$, we set

$$U(B;c) = \{x \in \mathbb{C}[\mathcal{F}(B;c)] \text{ s.t. } f(x) = 0 \, \forall f \in Z(B;c)\}$$

These are the fields that are invisible to the path integral. Then:

$$Z(B;c)^* = \mathbb{C}[\mathcal{F}(B;c)]/U(B;c)$$

So far this is all tautological, but using that $Z(M)$ is the image of a bunch of commuting projections, we define for a general manifold $U(M^n;c) \subseteq \mathbb{C}[\mathcal{F}(M;c)]$, generated by all

$$u \cdot r \text{ s.t. } B \subseteq M, u \in U(B;c), r \in \mathcal{F}(M \smallsetminus B;c)$$

The $\cdot$ denotes the gluing of the fields.

Then we have:

$$\boxed{Z(M;c)^* \cong A(M;c) = \mathbb{C}[F(M;c)]/U(M;c)}$$

So what's the plan for the rest of the course? The path integral is notoriously hard to define. What we will axiomatize is the above definition — fields and local relations — and try to build the rest of the theory. Why do we want to do it this way? Well, we can't do the path integral, but it's also more general in interesting examples with applications to quantum computing.

Theo: Why didn't we just take the definition of $Z(M;c)^*$ the same as for Bs? **Kevin:** That's like defining $Z(M)$ the way we originally did, but failing to notice the locality result. That would be a true definition, but it fails to express how to cut and glue. Especially when looking for axiomatics, we should choose the stronger-looking definition. Let's define $U'(M) = \{x \text{ s.t. } f(x) = 0 \, \forall f \in Z(M)\}$. Then it follows from the previous discussion that $U'(M) = U(M)$.

So, now we'll talk about definitions and the various ways that one might define TQFT. This will be a bit philosophical or editorializing, but it will help situate our definition within what you might see in the literature.

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1. **Physicist definition:** There are fields $\mathcal{F}$, an action S, $T = e^{iS}$, and a path integral, and we go from there.

   But it's hard to make the path integral rigorous.

2. **Atiyah (following Segal):** One of the consequences of what the physicists do is they get a functor on the cobordism category, so let's take that as the definition. Namely, for a closed n-manifold we get a vector space $Z(M)$, and for a cobordism W between M_1 and M_2 we get a map $Z(W) : Z(M_1) \to Z(M_2)$. And this is functorial on cobordisms.

   This is a very influential definition, but it has a lot of problems. Namely, it really underdescribes the situation. First of all, it doesn't give any hint about how to actually construct TQFTs — you have to use some other means to cook up a vector space and an operator. But the bigger problem is: suppose we want to describe the vector space $Z(M)$ in terms of cutting M into pieces. Some of the earliest examples, Reshetikhin–Turaev, and emphasized in Witten's paper on the Jones polynomial, have the structure that you can compute $Z(M)$ by cutting M into pieces.

3. **Freed (?):** Fully extended Atiyah–Segal axioms.

   What Freed said roughly is that if X is a (closed) k-manifold, you want to associate to it an $(n-k)$-category. Then if we have a bordism, we should have a kind of functor / bimodule for $(n-k)$-categories, whatever those are, and they should compose. At the bottom level we should assign an n-category to 0-dimensional manifolds.

   This is the dominant definition today. But it's a bit complicated. To define precisely what an $(n-k)$-category is is quite hard, whereas the actual examples are much simpler than the pages and pages of n-category definitions.

   Put another way, we have very concrete descriptions of the Hilbert spaces, and the "extended TQFT" definition ignores that. See, the Path Integral is a scary monster standing in our way, between us and TQFTs. The Atiyah–Segal way is to take a very long route around the monster, and our route is to walk much closer to the monster.

   In mathematics, you have some examples, and you try to axiomatize their properties. You can write down their properties, but how do you know you've written down enough? Atiyah vastly underdescribed what a TQFT actually is, and Freed does a much better job, but I think we can go even further.

4. **Fields and local relations:** This will be our main definition. We'll specify what we mean by fields, local relations, and that will be enough. Indeed, it's pretty easy to see that from this style of TQFT you can get one in the Freed style. Conversely, we think we can go back the other way, if you have a Freed-style theory which is good enough. By analogy: if you want to define a von Neumann algebra, you can give it concretely in terms of $\mathcal{B}(H)$, or you can give an abstract definition but it's not so easy to work with.

In order not to get too bogged down, I'll give a somewhat detailed definition of what we mean by a "field functor", but I won't give all the technical details.

Fields.
Think about manifolds with dim $\leq n$. What structure do they have? They have boundaries, we can glue them, we can add collar neighborhoods. A field functor should be some kind of mapping from all this structure.

So, let $\mathscr{M}_k$ denote the category of k-manifolds and homeomorphisms (or diffeomorphisms, or oriented things, or whatever). Then we want:

1. a collection of functors $\mathcal{F}_k : \mathcal{M}_k \to \textsc{Set}$. We will do this for $0 \leq k \leq n$. Why not $n+1$? Because we will never do a path integral, so the very top level we will treat a little differently.
2. maps $\partial : \mathcal{F}_k(X^k) \to \mathcal{F}_{k-1}(\partial X)$, a natural transformation of functors, i.e. commuting with homeomorphisms. (How do we really do this, say in the oriented case. We think of all our lower-dimensional manifolds as coming with a germ of thickening.)

   Then we can define $\mathcal{F}(X; c) = \partial^{-1}(c)$ for $c \in \mathcal{F}(\partial X)$.

Which reminds me, what are our examples? We can take $\mathcal{F}(X) = \{\text{maps } X \to \text{target space}\}$, e.g. B$G$. Or $\mathcal{F}(Y) = \{\text{codimension-1 submanifolds}\}$, or more generally, $\mathcal{F}(Y)$ could be the set of string diagrams on Y. (What do I mean by a "string diagram"? Assuming you have some other notion of n-category, which in $n = 2$ is not so hard, and then a string diagram is dual to a pasting diagram.)

3. Gluing. We will do this in three stages: gluing along $\emptyset$, along closed manifolds, and along manifolds with corners. (By the way, gluing with corners is important and useful, and it wasn't until later that this was included, and that's one way the early definitions are inadequate. **Kevin Lin:** In Witten's Jones polynomial paper, did he use gluing with corners?**Kevin:** No, but if you look at my '91 notes trying to understand Witten's paper, I use them a lot. Physicists haven't traditionally thought about higher categories.)
   (a) $\mathcal{F}(X_1 \sqcup X_2) = \mathcal{F}(X_1) \times \mathcal{F}(X_2)$, natural for homeomorphisms and also for taking boundaries. (Originally we were writing subscripts, but we will omit them — they should be clear from context.)
   (b) Suppose $\partial X = Y_1 \sqcup Y_2 \sqcup S$, and we have some homeomorphism $f : Y_1 \to Y_2$. Then we can draw the following diagram:

$$\mathcal{F}(X) \xrightarrow{\partial} \mathcal{F}(S) \times \mathcal{F}(Y_1) \times \mathcal{F}(Y_2) \underset{f\circ \mathrm{pr}_1}{\overset{\mathrm{pr}_2}{\rightrightarrows}} \mathcal{F}(Y_2)$$

   and we can ask that $\mathcal{F}(X_{\text{glue}})$ is its equalizer. But is this true in examples? Yes for maps into spaces, but for the labeled graphs / submanifolds, there's an annoying technicality that fields on X_{glue} might not be transverse to the cutting. So we can take the equalizer, and the axiom is that there is a natural map from the equalizer into $\mathcal{F}(X_{\text{glue}})$, and the intuition is that this should be almost all of them.

   Theo: So are we going to fix this by, say, asking it to be a homotopy colimit? **Kevin:** At the top level, we will impose local relations, like isotopy, and then everything will be locally related to something in the equalizer. One level down, we will have a category, and we won't say that every object is in the equalizer, but will be isomorphic to something in the equalizer. And so on.

(c) Gluing with corners. Suppose $\partial X = Y_1 \cup Y_2 \cup S$, and these pieces have common boundary. We want to somehow relate fields on the glued up manifold with fields on the cut one. We will leave it as an **exercise**: it's another equalizer.

4. Products and collars. We want a functor $\times I : \mathcal{F}_k \to \mathcal{F}_{k+1}$. What is it supposed to be? In the first example, we may have a map $X \to T$ the target space, and then we have a projection $X \times I \to X$, and we just extend the field to be constant in the I-direction. We can do something similar in the other case. To do this, we want to assume that things are somehow standard at the boundary.

To make this all precise we really should make precise what we mean by germs of structures and so on.

The final thing we need in our list of axioms are
Local relations.
For any $c \in \mathcal{F}(\partial B^n)$ — we only do this in top dimension — we'd like $U(B; c) \subseteq \mathbb{C}[\mathcal{F}(B; c)]$. What properties should these subspaces have?

1. They form an ideal under gluing. Namely, if we decompose a ball $B = B_1 \cup B_2$, then given $u \in U(B_1; c)$ and $r \in \mathcal{F}(B_2; c')$, then $u \cdot r \in U(B_1 \cup B_2; c'')$.
2. If $x, y \in \mathcal{F}(B; c)$ and x is isotopic to y (i.e. $f : B \to B$ is isotopic rel boundary to the identity, and $y = f(x)$) then $x - y \in U(B; c)$.

So for us, a *TQFT* consists of *fields* satisfying the above axioms, and *local relations* satisfying the above axioms. It's pretty easy to write down examples like this, but what's hard is that if you write down something randomly, then usually the skein modules will be 0-dimensional. So the art is in finding local relations where things don't totally collapse.

A physicist would define a TQFT as being some fields sort of like the above, and an path integral. What we're saying is that: the local relations are supposed to come from doing the path integral on an $(n+1)$-ball, and so the point is we say "OK physicists, just tell me how to do the path integral on an $(n+1)$-ball with all possible boundary conditions, and I'll give you the rest."

What we will do next: we will reconstruct all the pieces of an Atiyah-style TQFT. For all k-manifolds with $k \leq n$, this is pretty straightforward. We'd also like to have the invariant for $(n+1)$-manifolds, and that's a bit trickier, and requires a combinatorial argument.