10
Stochastic mechanics, hidden variables, and gravity

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10.1. Introduction

In this chapter I should like to report on some recent work which was motivated by the expectation that the ultimate solution to the problem of quantum gravity will require some modification in the fundamental ideas we hold about quantum mechanics. While it appears that the basic principles of quantum mechanics can be applied meaningfully to certain special situations in which gravitational interactions are relevant, such as in the description of Hawking radiation or the scattering of a finite number of quanta on a flat background, all of these successful applications depend on recourse to a preferred time coordinate which is either available in an asymptotic region or is picked out by a symmetry of the background space-time. In more general circumstances, regions of strong fields without asymptotically flat regions or special symmetries, where there are no preferred time co-ordinates, the standard quantization procedures become ambiguous (Ashtekar and Geroch 1974; Ashtekar and Magnon-Ashtekar 1975; Kay 1978; Fulling 1973; Unruh 1976; DeWitt 1975; Kuchar 1982). This is a serious problem which goes to the foundations of quantum field theory and which rests, ultimately, on the conflict between the very different roles time plays in quantum mechanics and general relativity (Kuchar 1982). Thus, the point of view which I will pursue here is that, while we may be able to learn something about quantum gravity by trying to construct perturbatively sensible quantum field theories to describe the interactions of gravitons with matter on flat backgrounds, ultimately, in order to construct a viable and completely general theory bringing together quantum and gravitational phenomena, we may have to be prepared to meddle with the foundations of quantum mechanics.

There are several different results which to me suggest that the standard principles of quantum mechanics are breaking down in the
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the distribution of energy in the field, as it is distributed among many degrees of freedom, and certainly not due to ignorance about the state of the field. It is instead due to the fluctuations which would be observed due to the uncertainty principle, if we measured some field operator near the horizon. (In this sense a black hole is like a microscope which magnifies and makes real the virtual quantum fluctuations in the field near the horizon.)

Now, whenever we have a situation in which the distinction between two kinds of physical phenomena becomes frame-dependent, such as electric and magnetic fields or the effects of gravity and inertia, Einstein taught us that the distinction must not reside in the phenomena, but only in the combination of the phenomena and a particular observer. In these cases we should search for a single physical concept to encompass both phenomena, along with a co-ordinate invariant description of it, such that the distinction can only be introduced with respect to an explicitly indicated frame. What I would like to suggest is that we should try to do the same for the distinction between the effects of quantum and statistical fluctuations: accept the fact that the absolute distinction which holds conventionally between quantum and thermal effects is valid only with reference to a set of globally preferred frames in the absence of gravity, and look for some more general notion of a fluctuation which will allow us a co-ordinate invariant description of quantum processes, while allowing us always the possibility of drawing the distinction with respect to some choice of frame, should we find it useful to introduce it.

Now, how are we to do this? To begin with we need to find a concept which can encompass both the concept of a virtual quantum fluctuation and that of a real statistical fluctuation. It seems to me that the most straightforward way to do this is to adopt an interpretation of quantum mechanics in which virtual quantum fluctuations are ordinary statistical fluctuations. Such an interpretation, called the statistical interpretation of the wave function, has been around for a long time; it was in fact the view advocated by Einstein (1949). (For an excellent presentation of the statistical interpretation, see Ballentine (1970)). Its fundamental tenet is that the wave function is not a description of an individual system, rather it is a description of an ensemble of similarly prepared systems, where similarly prepared means prepared by identical macroscopic devices. In this view the distinction between a virtual and a quantum fluctuation is not an ontological distinction, but only a distinction as to the cause of the fluctuation. One must assume the existence of some universal source of fluctuations, and this source must impart some unusual properties to the correlations of these quantum fluctuations—chief among them being the existence of non-local correlations. The advantage of this view is that the question as to why these correlations, which are at the heart of most that
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is strange about quantum mechanics, exist becomes something which might be explained by new physical hypotheses, rather than something which must merely be accepted as a consequence of the linear structure of Hilbert space.

In order to see that this is the most straightforward direction to pursue consider the following argument. In order for it to be possible for the distinction between two phenomena to be seen as observer dependent, they must be phenomena of the same type. Given the conventional Copenhagen interpretation of quantum mechanics, in which the quantum state is postulated to be a complete description of an individual system there is a difficulty, because thermal fluctuations refer to fluctuations within an ensemble of distinct physical systems, while quantum fluctuations refer to uncertainty in the outcome of a measurement which might be performed on a single individual system. These are very different kinds of entities, and the distinction between them must remain absolute unless either the interpretation of the quantum state or the interpretation of the thermodynamic ensemble is altered. However, the possible alterations we might consider are severely restricted by the requirement that the interpretation of both the thermal ensemble and the quantum state must remain consistent with the conventional usage in the absence of gravity, where the distinction may be chosen unambiguously. Clearly we cannot alter the interpretation of the thermodynamic ensemble so that it refers only to an individual system. The only sensible alternative I am aware of is to modify the interpretation of the quantum state so that it refers to an ensemble of physical systems.

Once we adopt this interpretation there is no conceptual barrier to the distinction between thermal and quantum fluctuations being observer-dependent, or to quantum fluctuations being the cause of thermal fluctuations at some later time. However, this is only the first step. In order to proceed we need a mathematical formulation of the quantum theory in which the quantum state is explicitly formulated as a description of an ensemble of systems and where the quantum fluctuations are explicitly described in terms of a more general class of fluctuations. At least one such formulation does exist (the stochastic quantum mechanics of Fényes (1952) and Nelson (1966)), and it is for this reason the basis of the work I will describe in the following. In particular, Nelson's derivation of the Schrödinger equation from the more general theory of Brownian processes gives several conditions which distinguish quantum behaviour from other types of stochastic processes, and it is to an explanation of these conditions that we may look for a deeper understanding of what distinguishes quantum from thermal fluctuations.

There is another reason which might be mentioned for basing an extension of quantum mechanics to circumstances in which gravitation is
present on the stochastic mechanics. The difficulties which have been encountered in the extension of quantum field theory to curved space-time may be seen to arise partly as a result of the rather indirect connection between the fundamental mathematical quantities on which quantum mechanics is based, which are an algebra of operators on Hilbert space, and what is measured, which are probability distributions. This connection involves several steps, one or more of which becomes problematic when gravitational fields are present. For example, one result of this is the lack of an invariant definition of a conserved inner product for quantum field theory on an arbitrary curved background (Ashtekar and Geroch 1974; and others). On the other hand, in stochastic mechanics the fundamental mathematical quantities on which the theory is based are the observed probability distributions. The usual order of things is reversed in that conservation of probability is built in from the beginning, and it is the linear Hilbert space structure which is derived in a series of steps. Thus, what we expect will happen if the steps connecting the Hilbert space to the observed probability distribution become problematic in the presence of gravitation is that in stochastic mechanics conservation of probability will be maintained, but it may not be possible to interpret what is measured in terms of a linear Hilbert space structure. Given a choice, this is clearly better than maintaining the linear Hilbert space structure, but at the cost of losing an unambiguous connection between it and physical measurements.

We may express this situation more simply as follows. In ordinary quantum mechanics we have both the conservation of probability and the superposition principle. However, it may not be possible to maintain both when we generalize to an arbitrary curved space-time, because the connection between them makes use of special properties of Minkowski space-time such as the existence of global inertial frames. If this is the case then it is better to give up the superposition principle to maintain conservation of probability than the reverse.

Indeed, results which I will discuss later show that just this kind of thing does happen when stochastic mechanics is generalized to describe the motion of a free particle in a background gravitational field. Conservation of probability is maintained, and the probability distribution obeys well defined laws of evolution. But, in general, these laws can only be expressed as linear evolution equations in the weak field limit.

Before turning to a discussion of these results I would like to comment on the basic difficulties that I believe underlie the various problems that we have been discussing. Perhaps the most basic of these, which was touched on above, is the conflict between the need for a globally preferred time co-ordinate in quantum theory and the purely local character of time in general relativity. This problem has many aspects,
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some of which were touched on in presentations at the Oxford meeting by Ashtekar, Barbour, and Sparling. However, at least in part, this problem may be seen as closely related to another basic problem. This is the conflict between the purely local description of physics in general relativity and the purely global character of the pure quantum state. Indeed, it must be emphasized that even in ordinary quantum field theory, the distinction between a pure and a mixed state can only be made with respect to the entire global state at a given time. For example, many states can be constructed which appear to describe a thermal ensemble in some region, but which are actually pure states, because measurements made in that region are correlated through non-local EPR type correlations to measurements made in other regions (cf. Chapter 22 for a related discussion).

Indeed, these considerations suggests that it might be true that there is no way to distinguish a pure state of a quantum field from a mixed state by local measurements. A consequence of this is then that there is locally no operational way to distinguish the effects of a quantum fluctuation from those of a thermal fluctuation. If this is the case then it is not surprising if the distinction between them depends locally on a choice of frame, and becomes ambiguous in the absence of preferred global frames.

Several different topics are covered in the following sections. In the next section a review of the formalism which is used to describe Brownian motion is presented. This is not complete, but should allow the reader to understand the details of the following material. However, the reader will not need to be familiar with the details of this material to understand the essential points of what follows. In the third section a review of Nelson's formulation of quantum mechanics is given, and in the one following an experimental test of one of Nelson's assumptions is discussed. Section 10.5 contains the generalization of Nelson's formulation to a free particle propagating in an arbitrary background gravitational field, and the last section contains a brief presentation of a class of hidden variable theories.

10.2. Review of the theory of Brownian motion

Let us begin with a brief review of Nelson's formulation of stochastic mechanics. While there is not enough space here for a complete presentation of the theory of stochastic processes, I will define the basic quantities and write down the basic equations that they satisfy. For those interested in more detail there are some excellent reviews (Nelson 1966, 1967, 1979; for an introduction to the subject of stochastic processes, see
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In stochastic mechanics, quantum processes are seen to be special cases of Brownian processes, so we begin with an outline of the theory of Brownian motion. The basic assumption of the theory is that the motion of a particle, \( x_i(t) \), is subject to a random stochastic motion \( \Delta x_i(t) \). These random motions will be specified probabilistically. Thus, the basic quantities of the theory will be an ensemble of paths in an \( n \)-dimensional space, which will represent possible motions for the particle. The notation \( x_i(t) \) then represents an ensemble of motions, and for each member of the ensemble the change in position in a time \( dt (>0) \) is given by

\[
\Delta x_i = b_i(x(t), t) \, dt + \Delta x_i,
\]

where \( b_i(x(t), t) \) is a given function but \( \Delta x_i \) is a stochastic variable which varies over the ensemble. \( \Delta x_i \) is specified by its average over the ensemble; in particular the motion is called Brownian if

\[
\langle \Delta x_i \rangle = 0 \quad (10.2)
\]

\[
\langle \Delta x_i \Delta x_j \rangle = 2\nu \delta_{ij} \, dt \quad (10.3)
\]

where \( \nu \) is called the diffusion constant. In particular, eqn (10.3) implies that the random change in \( x_i(t) \) in a time \( dt \) is given by

\[
|\Delta x_i| = \sqrt{\langle \Delta x_i^2 \rangle} = \sqrt{\nu dt}.
\]

That the differentials of the spatial variables are proportional to the square root of the time differential is characteristic of the theory of Brownian motion. As a result the motions \( x_i(t) \) are represented by continuous but non-differentiable functions. Because of this it is important also to consider the change in the position of the particle in a small negative increment of time. This is given, for \( dt < 0 \), by

\[
D^* x_i(t) = -b_i^*(x(t), t) \, dt + \Delta^* x_i(t),
\]

where \( \Delta^* x_i(t) \) satisfy

\[
\langle \Delta^* x_i(t) \rangle = 0 \quad (10.5)
\]

and

\[
\langle \Delta^* x_i(t) \Delta^* x_j(t) \rangle = -2\nu \delta_{ij} \, dt \quad (10.6)
\]

for \( dt < 0 \). In general, for Brownian motion \( b_i \neq b_i^* \) and \( \Delta x_i \neq \Delta^* x_i \).

At any time \( t \) we can define the probability density \( \rho(x, t) \), which tells us the probability of finding the particle in a unit volume centred on the point \( x \). Thus, the expectation value, in the ensemble of paths of some function \( F(x, t) \), is given by

\[
\langle F(x(t), t) \rangle = \int d^n x \sqrt{q(x)} \, \rho(x, t) F(x, t), \quad (10.7)
\]
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where \( q \) is the determinant of the metric on \( n \)-space. In particular, \( \rho(x(t), t) \) satisfies

\[
\int d^n x \sqrt{q(x)} \rho(x, t) = 1. \tag{10.8}
\]

In addition, we may define the propagation kernel \( P(x, t_2 | y, t_1) \) which is the probability of finding the particle in a unit volume centred on the position \( x_i \) at time \( t_2 \) if it was in a unit volume around \( y_i \) at time \( t_1 \). By definition it must satisfy

\[
\sqrt{q(x)} \rho(x, t_2) = \int d^n y \sqrt{q(y)} P(x, t_2 | y, t_1) \rho(y, t_1) \tag{10.9}
\]

\[
\int d^n x \sqrt{q(x)} P(x, t_2 | y, t_1) = 1. \tag{10.10}
\]

In addition we shall assume that \( P(x, t_2 | y, t_1) \) satisfies the Smoluchowski equation

\[
P(x, t_3 | y, t_1) = \int d^n z \sqrt{q(z)} P(x, t_3 | z, t_2) P(z, t_2 | y, t_1) \tag{10.11}
\]

where \( t_3 > t_2 > t_1 \). In the rest of this section we shall assume for simplicity \( q = 1 \). However, we shall need these more general formulae in Section 10.5.

From these equations, one may derive the Fokker–Planck equations

\[
\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_i} (\rho b_i) + \nu \nabla^2 \rho \tag{10.12}
\]

\[
\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_i} (\rho b_i^*) - \nu \nabla^2 \rho \tag{10.13}
\]

It is very useful to define the current velocity, \( v_i = \frac{1}{2}(b_i + b_i^*) \) and the osmotic velocity \( u_i = \frac{1}{2}(b_i - b_i^*) \). These satisfy the equations

\[
\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_i} (\rho v_i) \tag{10.14}
\]

\[
u v_i = \nu \nabla_i \nabla \rho. \tag{10.15}
\]

Finally, we need to define the forward and backwards stochastic time derivatives of a general function \( F(x(t), t) \). These are given by

\[
DF(x(t), t) = \lim_{dt \to 0} \frac{\langle F(x(t + dt), t + dt) - F(x(t), t) \rangle}{dt} \tag{10.16}
\]

\[
D^*F(x(t), t) = \lim_{dt \to 0} \frac{\langle F(x(t), t) - F(x(t - dt), t - dt) \rangle}{dt} \tag{10.17}
\]
where the averages are over all trajectories that go through the point $x(t)$ at the time $t$.

10.3. Nelson's derivation of quantum mechanics as a Brownian motion process

In order to understand the significance of Nelson's derivation, it is useful to begin by decomposing the Schrödinger equation into a conservation equation and a dynamical equation. If we write

$$\psi = \sqrt{\rho} \ e^{iS/\hbar} \quad (10.18)$$

then Schrödinger's equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + U\psi \quad (10.19)$$

decomposes into the conservation eqn (10.14), with the current velocity defined as

$$v_i = \frac{1}{m} \nabla \rho, \quad (10.20)$$

and the dynamical equation

$$\frac{\partial S}{\partial t} = -\frac{1}{2m} (\partial_i \rho)^2 + V + \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}. \quad (10.21)$$

This equation has the form of a Hamiltonian–Jacobi equation for the motion of a particle in a potential $V$, plus an additional term

$$V_{\text{quantum}} = \left(\frac{\hbar^2}{2m}\right) \nabla^2 \sqrt{\rho} / \sqrt{\rho}. \quad (10.22)$$

This term is rather strange from the point of view of probability theory, as it says that a quantum particle moves as if it were subject, in addition to external potentials, to a potential which is a function of its own probability distribution. However, as this term is now the only place where $\hbar$ appears, everything that distinguishes quantum mechanics from a probabilistic description of classical particle motion must be a consequence of the presence and form of this term. Thus, any attempt to explain quantum mechanics as arising from a probabilistic description of some more fundamental level of dynamics must come down to an explanation of this term.

In Bohm's (1952) hidden variable theory, the assumption is made that the particle and the wave function are separate and real entities. The wave function is assumed to obey Schrödinger's equation, and the particle is assumed to obey classical mechanics, with the additional
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assumption that it couples to the wave function through a potential which is exactly of the form of $V_{\text{quantum}}$ with $\rho$ assumed to be equal to $\psi^*\psi$. This works, in the sense that one can then show that under some plausible assumptions, $\rho$ will actually be the probability distribution of the particle. However, it is rather unsatisfying, as one just postulates that the wave function and the particle are coupled by $V_{\text{quantum}}$, and this is, to say the least, a very unusual form for a coupling between a particle and a field to have.

In stochastic mechanics, the term $V_{\text{quantum}}$ is derived, and to some extent, explained, from the general theory of Brownian motion outlined above, by specifying that the Brownian motion processes satisfy three additional conditions. These conditions are as follows:

1. The current velocity is irrotational. Thus, there exists a function $S(x, t)$ such that

$$m \mathbf{v}_i = \nabla S. \tag{10.23}$$

2. In spite of the fact that the particle is subject to random alterations in its motion there exists a conserved energy, defined in terms of its probability distribution, as

$$E = \int d^3x \rho(x, t)[\frac{1}{2}mv^2 + \frac{1}{2}mu^2 + V(x)]. \tag{10.24}$$

3. The diffusion constant is inversely proportional to the inertial mass of the particle, with the constant of proportionality being a universal constant $\hbar$:

$$\nu = \frac{\hbar}{m} \tag{10.25}$$

The derivation of Schrödinger's equation then proceeds as follows. One can show that condition (2) is equivalent to a stochastic Newton's law,

$$\frac{m}{2}(D^*D + DD^*)x_i = -\frac{\partial V}{\partial x_i}. \tag{10.26}$$

Using the identities in Section 10.2, one may derive the relation

$$\frac{1}{2}(D^*D + DD^*)x_i = \frac{\partial u_i}{\partial t} + (u_j \nabla_i)u_i - (u_i \nabla_j)u_i - v \nabla^2 u_i. \tag{10.27}$$

Using eqns (10.23) and (10.25), one finds an equation which is exactly the gradient of eqn (10.21). An argument about boundary conditions allows one to remove the gradient and this completes the derivation.

As this may seem a bit of black magic, several comments are in order. The most important is that this is not a constructive derivation. It says
that, given an ensemble of stochastic processes which satisfy conditions (1)–(3), the evolution of the probability distribution will be governed by the Schrödinger equation. To put it, perhaps loosely, into words, an ensemble of Brownian processes which are so delicately correlated that an exactly conserved energy of the form (10.24) may be defined in terms of their probability distribution (and which also obey conditions (1) and (2)) will behave as if each member of the ensemble is coupled to the probability distribution of the whole ensemble by eqn (10.21).

Thus, the quantum potential is explained, but at the cost of a very special set of assumptions which the ensemble of Brownian processes must obey. We might put it like this: consider the whole set of ensembles which satisfy the general conditions which define Brownian motion. Now, consider the very special subset of ensembles which also satisfy conditions (1)–(3). These ensembles may be labelled by the value of the conserved quantity $E$, and perhaps also by the values of other conserved quantities (although except for special cases these do not uniquely specify the ensemble). The paths which comprise each of these ensembles are very delicately correlated, so that (1) the evolution of the probability distribution is governed by a linear equation, in terms of the complex function $\psi$; and (2) the evolution of each path in the ensemble is governed by the probability distribution of the whole ensemble.

Clearly, such ensembles, if they exist, are very special. One thing that Nelson's derivation does not do is to tell us how to construct an ensemble which satisfies these properties, nor does it tell us if any such ensembles actually exist. Indeed, it is clear that any theory which aims to explain quantum mechanics as the statistical mechanics of some underlying dynamics must do exactly this. It must explain why ensembles with these special properties exist, and it must tell us how to construct them from some underlying dynamics. Clearly Bohm's theory does this, although by putting the result in by hand by hypothesizing that the particles are coupled to the quantum potential. In the last section I will describe another way to do this, at least to a certain degree of approximation.

10.4. Testability of Nelson's assumptions

It is interesting to ask whether we can loosen any of Nelson's assumptions to construct a class of theories which deviate from quantum mechanics in a controlled way (Smolin 1982). By subjecting these deviations to experimental constraints we can make a statement about how well Nelson's assumptions are satisfied in nature. Also, we shall find an intriguing analogy to the equivalence principle.

Nelson's second assumption is just the conservation of energy, and I will not try to weaken that. The first assumption is difficult to weaken in a
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controlled way as, as soon as \( v \) is no longer a gradient, the four equations which determine the evolution of the probability density and the current density cannot be reduced to a single complex equation. (In the next section we will see an example where this happens in the presence of the gravitational field.) The third condition, however, is easy to relax. Indeed, it is quite natural to do this, as \textit{a priori} there is no reason to believe that the two quantities—the inertial mass that comes into the dynamical condition (10.24) (or (10.26)) and the diffusion constant, which determines the correlation length over which the quantum fluctuations of the particle’s motion are strongly correlated—are at all related. If we assume instead of eqn (10.25) that \( v = \hbar/(m + b) \), where \( b \) is a new constant with the dimensions of mass, then we find, instead of the Schrödinger equation, a nonlinear equation,

\[
\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi - \frac{b\hbar^2}{2m(m + b)} \frac{\nabla^2 \sqrt{\psi} \psi}{\sqrt{\psi} \psi}. \tag{10.28}
\]

If \( b/m \ll 1 \), then the nonlinear term will produce a shift in the energy of a stationary quantum system, which can be estimated by expanding eqn (10.28) in powers of \( b/m \). We find that the first-order shift in energy is

\[
\Delta E = \frac{\hbar^2 b}{8m^2} \int d^3x \frac{(\nabla \rho)^2}{\rho}. \tag{10.29}
\]

This term will make a contribution to the Lamb shift, from which we can get an experimental bound on \( b/m \). Using eqn (10.28) the splitting of the \( 2s \) from the \( 2p \) orbital is found to be

\[
\Delta E_{2s} - \Delta E_{2p} = -\frac{11}{12} \frac{\hbar^2 b}{m^2} \frac{1}{a_0^2} + O\left(\frac{b^2}{m^2}\right) \tag{10.30}
\]

where \( a_0 \) is the Bohr radius. If we demand that this effect is less than the present experimental uncertainty \( \Delta_{\text{exp}} \) for the Lamb shift in hydrogen, we find that,

\[
\left| \frac{b}{m} \right| \leq \left( \frac{12}{11a^2} \right) \frac{\Delta_{\text{exp}}}{m e^2}. \tag{10.31}
\]

From the present experimental uncertainty of \( \pm 0.06 \text{ MHz} \) (Triebwasser \textit{et al.} 1953; Robiscoe and Shyn 1970) we find that

\[
\left| \frac{b}{m} \right| \leq 4 \times 10^{-13}. \tag{10.32}
\]

Thus, to an extremely high degree of accuracy, we know that one constant, the inertial mass of a particle, actually determines two of its
other properties—its coupling to gravitation, and the correlation length associated with its quantum fluctuations. It is intriguing to contemplate whether this could happen were all these phenomena not closely connected in some fundamental sense.

10.5. Stochastic mechanics in a background gravitational field

In this section I would like to indicate briefly what happens when we try to extend the stochastic formulation of quantum mechanics to the case of a free particle propagating in an arbitrary background gravitational field. What we would like to do can be described in the following: we seek a set of equations which determine the evolution of a probability density for a particle moving freely in an arbitrary background space–time which satisfies three conditions: (1) probability conservation is maintained; (2) when the space–time becomes flat and velocities become non-relativistic the equations become equivalent to Schrödinger’s equation for an appropriate choice of diffusion constant; (3) when the diffusion constant is set to zero the equations describe an ensemble of non-interacting particles moving geodesically.

We find that this can be done, but at a cost. This is that we must explicitly specify a preferred frame with respect to which the motion is Brownian. This is necessary because, for a Brownian process, \(|dx| = \sqrt{dt}\); for this to be true we have to have an invariant distinction between \(dx\) and \(dt\). Thus the dependence of the quantization procedure on a choice of time is present in the stochastic formulation as well.

This means that we do not have a complete theory unless we can introduce some equations which allow the preferred frames to be determined in terms of the other dynamical variables of the theory. We will see that there is not an obvious way to do this for the single particle theory in an arbitrary background gravitational field. For this reason, and because what we are generalizing is a single particle equation, it is best to regard the following as a warm-up for an extension of a stochastic formulation of field theory to the case of a background gravitational field.

We begin by extending the formalism presented in Section 10.2 to describe Brownian motion in a background space–time \((M, g)\). In order to do this we shall give a preferred global \(3 + 1\) slicing, which will be specified by a unit time-like hypersurface-orthogonal vector field \(W^\alpha\). We define \(q_{uv} = -g_{uv} + W_u W_v\) to be the three-metric on the slices orthogonal to \(W^\alpha\). Coordinates in the surfaces will be denoted by \(x^i\) and \(dt\) will refer to an interval of proper time in the rest frame of \(W^\alpha\). We define a probability density \(\rho(x)\) relative to the slicing such that \(\sqrt{q(x)} \rho(x, t)\) (where \(q = \det q_{ij}\)) is the probability of finding the particle in a unit spatial volume around the point \(x^i\) on the slice labeled by \(t\). We
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will require that $\rho$ satisfy eqn (10.8) on each of these slices. Similarly, we
can carry over the definition of the propagation kernel $P(x, t_1 | y, t_2)$ to
this context, and it will satisfy, by definition eqns (10.9) and (10.10). We
will also demand that the Smoluchowski eqn (10.11) is satisfied for any
three slices orthogonal to $W^\mu$. The ensemble of Brownian processes is
defined by giving the forwards and backwards drift equations:

$$dx^\mu = b^\mu(x, t) \frac{d\tau}{b \cdot W} + \Delta x^\mu, \quad d\tau > 0 \quad (10.33)$$

$$d^*x^\mu = -b^*\mu(x, t) \frac{d\tau}{b^* \cdot W} + \Delta^* x^\mu, \quad d\tau < 0 \quad (10.34)$$

where $b \cdot W = b_\mu W^\mu$ and $b^\mu$ and $b^*\mu$, the forward and backwards drift
velocities, have been defined so as to transform as four-vectors. The
fluctuating terms now satisfy

$$\langle \Delta x^\mu \rangle = \langle \Delta^* x^\mu \rangle = 0 \quad (10.35)$$

$$\langle \Delta x^\mu \Delta x^\nu \rangle = -\langle \Delta^* x^\mu \Delta^* x^\nu \rangle = 2\nu_{\mu\nu} d\tau. \quad (10.36)$$

We see that we have Brownian motion only for the components of the
trajectories which are orthogonal to $W^\mu$. In addition, since the difference
between $b^\mu$ and $b^*\mu$ is due only to the action of these Brownian
fluctuations we shall assume that $W^\mu(b_\mu - b^*_\mu) = 0$. Using these relations
we can derive a pair of relativistic Fokker-Planck equations,

$$\frac{d\sqrt{q} \rho}{d\tau} = -\sqrt{q} D_i (\rho b^i) + \nu\sqrt{q} q^\mu D_i \frac{\rho}{b \cdot W} \quad (10.37)$$

$$\frac{d\sqrt{q} \rho}{d\tau} = -\sqrt{q} D_i (\rho b^*^i) - \nu\sqrt{q} q^\mu D_i \frac{\rho}{b^* \cdot W}, \quad (10.38)$$

where $D_i$ is the covariant associated with the three-metric $q_{ij}$.

Taking the sum and difference of eqns (10.37) and (10.38), we find

$$\frac{d\sqrt{q} \rho}{d\tau} = -\sqrt{q} D_i (\rho \frac{v^i}{v \cdot W}) = -\sqrt{q} q^\mu \nabla_\mu (\rho \frac{q_{\mu\nu} v^\nu}{v \cdot W}) \quad (10.39)$$

$$u^\mu = \nu q^\mu \nabla_\mu LN\left(\frac{\rho}{v \cdot W}\right), \quad (10.40)$$

where we have defined $v^\mu = b^\mu + b^*\mu$ and $u^\mu = b^\mu - b^*\mu$.

These equations allow us to define a covariantly conserved probability
current. Recalling that $\rho$ was defined to be the probability density seen
by an observer moving with $W^\mu$, we write

$$\rho = \hat{\rho} \cdot W, \quad (10.41)$$

where $\hat{\rho}$ is independent of $W^\mu$. The reader may then verify that eqn
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(10.39) can be rewritten as

\[ \nabla_\mu (\hat{\rho} v^\mu) = 0. \]  

(10.42)

We now want to extend Nelson's three conditions to this case. To begin with, we shall define the diffusion constant as we did before by eqn (10.25). However, we should note that for a free particle this equation does not have the significance it had before, as \( m \) does not appear in the equation of motion.

We now want to go on to generalize the dynamical condition, expressed by eqns (10.24) and (10.26). In order to do this we need to define appropriate covariant generalizations of the stochastic time derivatives (10.16) and (10.17). It turns out that the correct definition, which gives the correct reduction to the Schrödinger equation in the non-relativistic limit, is to choose to take the forward stochastic derivative in terms of the proper time seen by \( b' \) and the backwards time derivative in terms of \( b'^* \). This definition gives us \( D x^\mu = b^\nu \) and \( D^* x^\mu = b'^* \mu \), and in general gives

\[ D F(x) = b^\nu \nabla_\mu F + \nu q^\alpha \nabla_\mu q^\nu \nabla_\alpha F \]  

(10.43)

\[ D^* F(x) = b'^* \nabla_\mu F - \nu q^\alpha \nabla_\mu q'^* \nabla_\alpha F. \]  

(10.44)

We can then posit a stochastic geodesic equation

\[ \frac{1}{2} (D^* D + D D^*) x^\mu = 0. \]  

(10.45)

Working this out, we find that

\[ v^\nu \nabla_\nu v^\mu - u^\nu \nabla_\nu u^\mu + \nu q^\alpha \nabla_\nu q^\beta \nabla_\alpha \mu = 0. \]  

(10.46)

If we plug in eqn (10.40) and set \( \nu = h/m \) we find an evolution equation for \( v^\mu \):

\[ v^\alpha \nabla_\alpha v^\mu = \frac{\hbar^2}{m^2} q^\alpha [ (\nabla_\beta LN \hat{\rho}) [ \nabla_\mu q^\nu \nabla_\beta LN \hat{\rho} ] - \nabla_\alpha q^\gamma \nabla_\nu q^\beta \nabla_\alpha LN \hat{\rho} ] . \]  

(10.47)

This equation and eqn (10.42) together determine the evolution of \( \hat{\rho} \) and \( v^\mu \), given a choice of initial data on a three-surface orthogonal to \( W^\alpha \). As a check on the derivation we may ask what these equations give in the limit of weak fields and low velocities. In this approximation we may choose co-ordinates so that the metric has the form \( g_{\alpha \beta} = \text{diag}(1 + 2 \Phi/c^2, -1, -1, -1) \). In addition we choose \( W^\mu \) to be the unit time-like vector orthogonal to flat three-space. In the non-relativistic limit one can then choose \( v_i = D S/m \). Making these choices one then finds that eqns (10.42) and (10.47) are equivalent to

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} D^2 \psi + m\Phi \psi, \]  

(10.48)
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where $\psi = \sqrt{\rho} \exp iS/h$. Thus, we have achieved our goal, which is the formulation of a theory (albeit one dependent on a preferred choice of slicing) which reduces to ordinary quantum mechanics in the presence of a Newtonian gravitational field in the weak field non-relativistic limit, and which reduces to a description of ordinary geodesic motion when Planck's constant is set to zero.

One thing we might like to know is whether, in the general case, eqn (10.47) reduces to an equation describing the evolution of a phase $S$ so that, as in the flat space case this equation, together with eqn (10.42), describes the evolution of a complex wave function $\psi = \sqrt{\rho} \exp iS/h$. That this is not the case can be seen as follows. If it were, then at least locally we could choose an $S$ such that $V_\mu = \nabla_\mu S$. In this case the left-hand side of eqn (10.47) becomes $\frac{1}{i} \nabla_\mu \left[ (\nabla_i S)(\nabla^i S) \right]$, and thus has vanishing curl. However, as the reader may verify, the curl of the right-hand side of eqn (10.47) does not, in general, vanish, unless $R^\lambda_{\mu\nu\rho} = \nabla_\nu W^\rho = 0$. Thus, in general, $\nu^\mu$ cannot be chosen orthogonal to any surfaces of constant phase, and the evolution described by eqns (10.42) and (10.47) cannot be reduced to the evolution of a complex wave function. However, as the evolution they generate conserves probability and is well defined they may be considered to be a sensible generalization of the Schrödinger equation to the case of a general background field.

One unsatisfactory feature which these equations possess is that the initial value problem is only well defined for data given on three surfaces orthogonal to $W^\mu$. This is not surprising, as eqn (10.47) is a generalization of the non-relativistic Schrödinger equation, and even in the absence of gravity the initial value problem for the Schrödinger equation is only well defined in terms of a preferred time variable. This again underscores the fact that the present theory is only a warm-up exercise before looking for an extension of stochastic field theory to background gravitational fields.

One thing we would like to try is to see if there is a simple way to determine the preferred frame $W^\mu$ in terms of the variables of the theory. For example, the simplest possibility would be if we could determine $W^\mu$ by a condition that the particle motion is always Brownian in that frame in which its probability distribution is at rest. This would be possible if we could consistently set $V^\nu = f W^\mu$, where $f$ is some function. This is, however, not in general possible, because, as $W^\mu$ has been assumed to be hypersurface orthogonal, in order to make this choice we need to be able to guarantee that if $V^\mu$ is chosen to be hypersurface orthogonal its

† This formulation then differs from one previously given by the author (Gravity Research Foundation essay, 1984 and EFI preprint) in which a small nonlinear term coupling $\Phi/c^2$ to $S$ was found. That formulation differed from the present one in that the $b \cdot W$ factors were missing from the stochastic differential eqns (10.33) and (10.34).
evolution according to eqn (10.47) must preserve the hypersurface orthogonality. However, as we have just shown, this is not in general the case.

We might try to drop the condition that \( W^\mu \) be itself hypersurface orthogonal. However, in this case it is not clear whether there are in general any choices of surfaces for which eqns (10.42) and (10.47) have a well defined initial value formulation. Thus it seems that in this simple case there is no way to eliminate the need to give \( W^\mu \) separately to define the theory in an arbitrary background gravitational field. One may hope that in a stochastic formulation of field theory it will be possible to determine the preferred frames dynamically in terms of the other variables of the theory.

However, in spite of these weaknesses, we may ask what we have learned from this formulation that may still be true in an extension to field theory. The most important thing is that while the equations define sensible time-evolution for \( v^\mu \) and \( \rho \) in the presence of an arbitrary background field, consistent with probability conservation, this evolution can only be described by a linear equation on a linear space of states in the weak field non-relativistic limit. In the more general case, in which \( v^\mu \) cannot be assumed to be hypersurface orthogonal, the equations are nonlinear and do not reduce to a single complex equation. Thus, in spite of the fact that probability conservation is maintained, the superposition principle does not apply in these more general circumstances. It seems likely that this will also be a feature of the stochastic field theory.

### 10.6. A non-local hidden variable theory

In the previous section we have seen that the equivalence between the stochastic formulation of quantum mechanics and the Hilbert space formulation of quantum mechanics may break down in the presence of gravitational fields. As the stochastic formulation seems to remain well defined this is good reason to try to take it seriously as a possible vehicle for extending quantum physics to domains in which gravity is important. At the same time the stochastic formulation has a few peculiar features, which, if we are to take it seriously as a possible fundamental theory, we would like to understand better. Chief among them is the high degree of correlation which must hold between the individual trajectories which comprise any ensemble which satisfies Nelson's three conditions, as we discussed above. Two things we would like to know are whether there in fact exist any Brownian ensembles of trajectory which satisfy Nelson's conditions, and if so, how to construct some examples of them.

It is clear that to answer this last question we should essentially have to construct a hidden variable theory, because we would be explicitly
constructing an ensemble of classical trajectories (albeit not differentiable) which have the property that the evolution of their probability density and probability current density are governed by Schrödinger's equation. This is one important motivation for seeking to construct an example of a hidden variable theory.

Strictly speaking, it is not necessary to believe that there exists an underlying dynamics to make use of either the statistical interpretation or the stochastic formulation of quantum mechanics. However, as soon as one believes that these may contain more truth than the usual interpretation or the usual formulation, then it is hard to avoid curiosity about whether it is possible to make some description of an individual quantum system which is more complete than that which can be adduced from the wave function. Similarly, we want to know what sort of dynamics could give rise to ensembles which satisfy Nelson's conditions, and hence be responsible for the unusual properties exhibited by quantum fluctuations. Thus, if one is willing to entertain the possibility that there might be some truth in the kinds of arguments I discussed in Section 10.1, then it seems that one has at least to entertain the notion that quantum physics and gravity are tied up in a fundamental way, not at the level of quantum mechanics, but at the level where quantum mechanics is itself explained in terms of a more elementary theory.

Before proceeding, I would like to mention briefly some additional reasons which might motivate us to look for hidden variable theories. First, and most obviously, if quantum mechanics can be explained by a hidden variable theory then many of the puzzles concerning quantum mechanics can be resolved without doing violence to our common sense notions of realism, ontology and epistemology. For example, the measurement problem evaporates because quantum states simply describe ordinary statistical ensembles of systems.

Secondly, I believe that the history of physics shows that progress in physics is more often achieved through developments of physics rather than through developments in philosophy. If we have the possibility of dissolving the philosophically troubling implications of quantum mechanics by inventing new theories without these difficulties, then this seems to me to be preferable to struggling with radical solutions to the basic philosophical questions which are motivated by the fact that they make quantum mechanics more rational. Furthermore, once we challenge ourselves with the task of inventing a new theory which explains the peculiar features of the quantum world then we find ourselves engaged in trying to ask new questions about physics and invent new hypotheses about nature to answer them, and this is probably what, as physicists, we are more likely to succeed at doing. The philosophers have plenty of time
to mull over the peculiarities of quantum mechanics, if they turn out to be unavoidable.

Now, we will pay a price for sticking with our common sense ideas about reality and our relation to it, and this is that physics will have to become explicitly non-local. However, the experimental evidence (Chapter 1; Freedman and Clauser (1972); Clauser (1976); Fry and Thomson (1976); Aspect et al. (1981, 1982a,b)) against the Bell inequalities (Bell 1965) is now so strong that we can state confidently, and independently of any specific theory, that nature is full of non-locality. All around us are occurring space-like separated events whose joint probability distributions, were we to measure them, would not factor into products of local probabilities. Thus, we have a choice only between non-locality hidden in the linear structure of the Hilbert spaces which describe many particle systems and non-locality explicitly built into the dynamics of a hidden variable theory. To go from the former to the latter seems a cheap price to pay to avoid having to alter radically the basic tenets of classical realism.

Of course, I do not hope, or wish, to convince those who have found an interpretation of quantum mechanics which they find satisfying, such as the adherents of the many-worlds interpretation. At this stage of ignorance it is undoubtably for the best if our different feelings impel us in different directions; I am merely stating mine. In any case, all of the above means nothing if the attempt to construct hidden variable theories does not lead to theories which are compelling, beautiful, and surprising on their own merits. Nothing remotely of this sort presently exists. However, what does exist are some examples of explicitly non-local hidden variable theories which show that, at least, theories of this type do exist. Furthermore these theories answer the questions about stochastic mechanics we mentioned above by giving explicit constructions of ensembles which, at least to a certain degree of approximation, satisfy Nelson's three conditions.

I would like briefly to describe the particular motivations which led to the construction of these theories. First of all, Bell's theorem seems to indicate the possibility that non-local correlations may be found between any pair of systems which have previously interacted. This indicates that, in order to explain these correlations, a hidden variable theory will have to have at least one hidden variable for every pair of particles that potentially could interact. This leads to the question of whether a hidden variable theory could be made by inventing new dynamical variables which are attributed to pairs of particles, in addition to the usual dynamical variables which describe properties of individual particles. The simplest possibility which might be tried is then to represent each dynamical
variable of an $N$-particle system by an $N \times N$ matrix in which the diagonal elements are attributed to individual particles, while the off-diagonal elements are attributed to pairs of particles.

Let us suppose that we do assign a dynamical variable to every pair of particles, and that the dynamics will involve both these non-local variables and the local dynamical variables associated with individual particles. Then it follows that any local description of the physics must be statistical, even if the basic dynamics is deterministic. Even if we could invent experiment arrangements which controlled the variables associated with pairs of particles, each of which are nearby, the motion of the particles in our neighbourhood will be influenced by those variables assigned to pairs containing a nearby particle and a far away particle, and there is no way these variables could possibly be controlled in local experiments. These variables will have to be described probabilistically and, as there are many more particles which are far away than there are which are nearby, they will most likely swamp any contributions coming from the pairs of particles, both of which are local.

It is then intriguing to wonder if the statistical fluctuations seen in the local variables of quantum systems are due precisely to their couplings to these non-local variables. Perhaps the usual quantum mechanics, in which one has only local dynamical variables, results from doing ordinary statistical mechanics on the non-local variables, where all of the variables, local and non-local are coupled through some deterministic dynamics. Such a theory would be globally deterministic, but locally statistical.

There is, however, a difficulty with this that must be confronted. If there are many more non-local variables in which a particular particle is paired with particles far away than there are in which it is paired with nearby particles, then there are also many more non-local variables than there are local variables. If the whole system is treated according to statistical mechanics, then what is to prevent the system from reaching an equilibrium behaviour in which, by the equipartition theorem, the non-local variables completely dominate the energetics of the system such that the local physics is completely swamped. The system must be very special so that the non-local physics contributes what is, on a cosmological scale, a small perturbation to the local physics, rather than the reverse.

One way in which this problem might be resolved is the following. Suppose the non-local theory had the property that in the thermodynamic limit in which $N$, the number of particles, is taken to infinity, the physics became local and classical. However, suppose also that $N$, while very large, is finite. Then there will be statistical fluctuations in the evolution of these classical variables. These fluctuations will occur on scales which
are of order of $1/\sqrt{N}$ the dimensions of the whole system. Now, it is interesting to note that the ratio of the Compton wavelength of a stable elementary particle (which determines the rough scale of quantum fluctuations) to the radius of the universe is approximately $10^{-47}$, which is, roughly, $1/\sqrt{N}$, where $N$ is the number of stable particles observed in the universe.

This relation, which is related to Dirac's large number hypothesis (Dirac 1937, 1938), may be only a coincidence, or it may be an important clue as to why there are quantum fluctuations. In any case it gives a clue as to how to arrange things in a non-local hidden variable theory so that, even though there are many particles, the non-local physics produces fluctuations in the local physics rather than swamping it altogether. In the example to be discussed below this is achieved by scaling the non-local variables by appropriate powers of $1/N$ so that in the thermodynamic limit their effects disappear. However, one might imagine that in some real theory this was a deep and important property of the limit. More ambitiously, we might imagine that the underlying theory is completely non-local in that it makes no reference to local properties in space–time, but that the description of local physics in a space–time of small dimension emerged in the thermodynamic limit in a way analogous to that in which thermodynamics emerges from the classical mechanics of systems with large numbers of degrees of freedom.

I shall now describe a class of hidden variable theories constructed to exhibit these ideas. I shall do this by listing the assumptions of the theory and then stating a theorem which gives a correspondence between systems satisfying these assumptions and systems satisfying Schrödinger's equation. There is not space to present the details of the proof, but this is given elsewhere (Smolin 1983).

The system to be described is a non-relativistic $N$ particle system in two spatial dimensions where, as should be apparent from the above, $N$ is large. I will discuss several extensions to other systems in the last section. Our system will be described by an $N \times N$ complex matrix, $M_{ab}(t)$. The positions of the $N$ particles are given in the complex plane by its eigenvalues $\lambda_a = x_a + i b_a$. We make the following hypotheses as to the evolution of this system.

(A) Scaling hypothesis. If we decompose $M_{ab}$ into a diagonal and off-diagonal part:

\[ M_{ab} = \delta_{ab} D_a + N_{ab}; \quad N_{aa} = 0, \quad (10.49) \]

and

\[ \frac{N_{ab}}{D_a} = \frac{N_{ab}}{D_a - D_b} = \frac{1}{N^\delta}, \quad (10.50) \]
where the double bar denotes here an average over the magnitudes of all of the elements. (A single bar will denote the complex conjugate in what follows.) $q$ will be chosen so that the dynamics becomes purely local in the limit $N \to \infty$. In addition, this choice will permit us to make a perturbation expansion for the positions of the eigenvalues, $A_a$, in terms of powers of $1/N$. To second order we have,

$$
\lambda_a = D_a + \sum_{\mu a} \frac{N_{ab}N_{ba}}{D_a - D_b}.
$$

The second term describes a random walk in the complex plane of $N-1$ steps. The sum then contributes a term of order $1/N^{a-1/2}$, from which we infer that in order for classical mechanics to be restored in the $N \to \infty$ limit we must have $q > \frac{1}{4}$.

(B) Rules about the off diagonal elements. Each of the $N_{ab}$ is in one of three states:

1. Off: $N_{ab} = 0$.
2. Dormant: $N_{ab} \neq 0$, but with some fixed or slowly varying value.
3. Dynamical-varying on some short dynamical time scale.

For example, we might construct a model in which the dormant $N_{ab}$ are at random points in a region of the complex plane while the dynamical $N_{ab}$ move around in the region. There are two additional rules these variables must obey:

- **Basic rule:** For a fixed $a$ and $b$, only one of $N_{ab}$ and $N_{ba}$ may be dynamical at any one time.
- **Sufficiency rule:** For a given $a$, let $n_a$ be the number of $N_{ab}$ which are not off. Then for all $a$, $n_a > 1$.

(C) Underlying deterministic dynamics: We assume that the evolution of the matrix $D_{ab}$ is governed by a classical action principle of the form

$$
L_M = \sum_a m_a \dot{D}_a D_a + \frac{1}{2} m_a \sum_{a \neq b} \dot{N}_{ab}N_{ab} - F(N_{ab}) - V(\lambda_a). 
$$

Here, $F(N_{ab})$ is some potential energy function which describes interactions between the elements of $N_{ab}$, as well as any forces needed to constraint them to regions of the appropriate size in the complex plane. There are also contributions to the potential energy which are functions only of the eigenvalues $\lambda_a$. These may be of the form

$$
V(\lambda_a) = \sum_{a \neq b} V_2(|\lambda_a - \lambda_b|) + \sum_a V_{\text{external}}(\lambda_a),
$$

where the two terms describe, respectively, two-body forces between the particles and the effects of external potentials on the particles. The $m_a$ are
just the masses of the particles, as can be seen by taking the $N \rightarrow \infty$ limit in which the $N_{ab}$ go away, and we assume for simplicity that all of the off-diagonal elements have the same 'mass' $m_N$.

(D) **Mass fudge rule.** We assume that, for each $a$ and $b$,

$$\frac{n_a m_b}{n_b m_a}$$

This is motivated strictly by the fact that it guarantees Nelson's condition (3), eqn (10.25). It is in order to accommodate particles with different masses that some of the $N_{ab}$ must be off.

(E) **Brownian character of the motion of the dynamical $N_{ab}$.** We assume that the dynamics and initial conditions are chosen such that an appropriate timescale $\Delta t$ exists such that the motion of each dynamical $N_{ab}$, coarse grained over $\Delta t$, satisfies the conditions of Brownian motion eqns (10.1)-(10.6). For example, this can be achieved if there are short range elastic forces between the $N_{ab}$ and if $\Delta t$ is longer than the mean free time between collisions. The diffusion constant $\nu$ of the dynamical $N_{ab}$ is then computable in terms of the initial conditions and the parameters of the theory. The parameters must also be arranged so that $\Delta t$ is shorter than $\hbar/E$, where $E$ is the energy of the quantum system defined by the correspondence theorem below.

Another example of a system that satisfies this condition is, a variant of the model the Ehrenfests used in their explanation of Boltzmann's work (Ehrenfest and Ehrenfest 1959). We have two kinds of particles in a region of the complex plane. The dormant $N_{ab}$ are fixed randomly at sites in the region. The dynamical $N_{ab}$ move through the region with fixed velocity $v$, colliding elastically with the dormant $N_{ab}$ and with the walls. The diffusion constant is $\nu = iv$, where $i$ is the mean free path.

(F) **Assumptions concerning the statistical distributions of the $N_{ab}$.** We need to make some assumptions concerning how the ensemble of $N_{ab}$ are distributed. These will be made in terms of the probability distribution which describes the ensemble, $\rho(N_{ab})$, which is defined so that it satisfies eqn (10.8). In order to ensure that the theory is symmetric under rotations in the complex plane we will assume that

$$\int dN_{ab} \rho(N_{ab}) N_{ab} = 0$$

(10.53)

for all $N_{ab}$. We also need to assume that, for given $a$ and $b$, the
More scaling rules. The motion of the dynamical $N_{ab}$ may then be described by the formalism of stochastic processes described in Section 10.2. However, we can show that the coupling of the $D_a$ to the fluctuations of the $N_{ab}$, while non-vanishing, through the terms (10.53), may be neglected for the present purposes. Thus, the $D_a$ will be described by differentiable trajectories. We, however, do need to require that relations similar to eqn (10.50) hold between the velocities and accelerations of the $D_a$ and the $N_{ab}$:

$$|V_{ab}| \leq \frac{1}{N^q} |D| \left| \frac{\partial^2 D}{\partial t^2} \right| \left| \frac{\partial D}{\partial t} \right|^{-1}$$  \hspace{1cm} (10.58)$$

$$D^* B_{ab} + D B^*_{ab} < \frac{1}{N^q} \left| \frac{\partial^2 D}{\partial t^2} \right|.$$  \hspace{1cm} (10.59)$$

We are now ready to state a correspondence theorem:
Let $M(t)_{ab} = \delta_{ab} D(t) + N(t)_{ab}$ be an $N \times N$ complex matrix evolving in time such that conditions (A) to (D) and (G) are satisfied, and let us be given a statistical ensemble of such matrices, which satisfies conditions (E) and (F). Then,

(1) The eigenvalues of the matrix $M_{ab}$, $\lambda_{ab}$, undergo Brownian motion (a special case of a theorem of Dyson (1962)), such that the diffusion constant for each $\lambda_a$, $C_a$, satisfies,

$$C_a m_a = \hbar + O(1/\sqrt{N}).$$  \hspace{1cm} (10.60)$$

This defines a new constant $\hbar$ which is computable as a function of the diffusion constant for the off-diagonal elements, $\nu$:

(2) There exists a real function, $S(\lambda, \tilde{\lambda})$, such that, through first order in $1/N^{2q-1/2}$,

$$m_a v_{a,x} = \frac{\partial S}{\partial x_a},$$  \hspace{1cm} (10.61)$$

$$m_a v_{a,y} = \frac{\partial S}{\partial y_a}. 

(10.62)$$

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where \( \lambda_v = x_v + iy_v \) and \( u_v = u_{v,x} + iv_{v,y} \), with \( u_v \) the current velocity for \( \lambda_v \).

(3) If we form the complex function

\[
\psi(x_v, y_v) = \sqrt{\rho(x_v, y_v)} e^{i\frac{\varphi}{\hbar}}
\]

with \( \rho(x_v, y_v) \) the joint probability distribution function for the \( \lambda_v \), then \( \psi \) satisfies the equation

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x_v^2} + \frac{\partial^2}{\partial y_v^2} \right) \psi + \sum_{a \neq b} V(|\lambda_a - \lambda_b|) \psi
\]

\[
+ O(1/N^{2q-1/2}) + O(1/N^{1/2}).
\]

We may note that the errors come both from the neglect of terms in the velocities and accelerations of the eigenvalues which are smaller by powers of \( 1/N^{2q-1/2} \) than the dominant terms coming from the interparticle forces and from the fluctuations in the value of Planck’s constant from eqn (10.60).

While we do not have space here to give the proof of this theorem we may say one word about the method. The basic equations, which follow from our assumptions, relate expectation values of functions of the \( \lambda_v \) to expectation values of functions of the \( D_a \) and \( N_{ab} \). For example, the probability distributions are related by

\[
\rho_\lambda(\lambda) = \int (dD_a)(dN_{ab}) \rho_\lambda(D) \rho_D(D) \Pi_\lambda \delta \left( \lambda_v - D_a - \sum_{b \neq a} \frac{N_{ab}N_{ba}}{D_a - D_b} + \ldots \right),
\]

(10.65)

where \( \rho_\lambda \) and \( \rho_D \) are, respectively, the probability distributions for \( \lambda \) and \( D \). Similarly, given a quantity \( F(D, N) \) as a function of \( D_a \) and \( N_{ab} \) we may find the expectation value of that quantity as a function of \( \lambda \)

\[
F(\lambda) = \int (dN_{ab})(dD_a) F(D, N) \rho_\lambda(D) \rho_D(D) \Pi_\lambda \delta \left( \lambda_v - D_a - \sum_{b \neq a} \frac{N_{ab}N_{ba}}{D_a - D_b} + \ldots \right) / \rho_\lambda(\lambda).
\]

(10.66)

Using these relations one can show that the conditions (A) to (G) guarantee that Nelson’s three conditions are satisfied to the orders indicated in the theorem.

I would like to close by mentioning, without details, three extensions of this result which are presently being pursued.

(1) An extension to a lattice quantum field theory for a complex scalar field in any number of space dimensions in which each eigenvalue \( \lambda_v \) gives the value of the field on one site of the lattice. In this case one discovers that the continuum limit is even more tricky than in the usual lattice.
quantum field theory, because one must scale the parameters of hidden variable theory carefully with the lattice spacing so that \( h \) is finite in the limit. This is also interesting because one finds that, if the construction succeeds, Lorentz invariance will be recovered in the continuum limit in spite of the presence of non-local hidden variables defined with respect to some particular frame.

(2) An extension to a relativistic particle model in 3 + 1 dimensions based on replacing the complex elements of \( M_{ab} \) by elements of the Hermitian quaternions. One can then try to put interactions in by adding to the action Wheeler–Feynman type terms. Singularities are then encountered because division by Hermitian quaternions corresponding to light-like intervals is not defined. These may be the usual singularities of quantum field theory in a new guise.

(3) A kind of a 'Machian' version of the theory presented here in which the off-diagonal elements are all constant and the universal stochastic fluctuations necessary to derive quantum mechanics arise from the transmittal, through the non-local terms of eqn (10.66), of thermal fluctuations due to the fact that some of the particles may be in hot regions (for example in stars.) One then finds that Planck's constant is only non-zero when some parts of the system are hot. While almost certainly wrong, this is an amusing idea to contemplate!

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References


