MODULAR DATA FOR THE EXTENDED HAAGERUP SUBFACTOR

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Abstract. We compute the modular data (that is, the S and T matrices) for the centre of the extended Haagerup subfactor [BMPS12]. The full structure (i.e. the associativity data, also known as 6-j symbols or F matrices) still appears to be inaccessible. Nevertheless, starting with just the number of simple objects and their dimensions (obtained by a combinatorial argument in [MW14]) we find that it is surprisingly easy to leverage knowledge of the representation theory of $SL(2,\mathbb{Z})$ into a complete description of the modular data. We also investigate the possible character vectors associated with this modular data.

1. Introduction

The extended Haagerup subfactor provides perhaps the strangest currently known example of a *quantum symmetry*.

Fusion categories provide a suitable axiomatization for the notion of quantum symmetry: they are the finitely semisimple rigid tensor categories. The fundamental examples are the representation categories of finite groups (over $\mathbb{C}$), but there are many others. The semisimplified representation category of a quantum enveloping algebra $U_q\mathfrak{g}$ at a suitable root of unity gives another source of examples.

The remarkable discovery of an interesting classification of finite depth subfactors above index 4, initiated by Haagerup [Haa94], began to provide examples beyond these 'classical' ones. In particular, each finite depth subfactor $N \subset M$ gives a pair of Morita equivalent unitary fusion categories, as the categories of $N - N$ and $M - M$ bimodules. Haagerup and Asaeda constructed 'exotic' subfactors in [AH99], and the last missing case in Haagerup's classification between index 4 and $3 + \sqrt{3}$ was provided by the construction by Bigelow-Morrison-Peters-Snyder of the extended Haagerup subfactor [BMPS12]. Some of these fusion categories are distinctly different from those arising from finite groups or quantum groups: in particular the fusion categories coming from the Haagerup and extended Haagerup subfactors cannot be defined over any cyclotomic field [MS12].

Since the discovery of these examples, there has been some progress towards organising them. In particular, the theory of *quadratic* categories has been developed, particularly by Izumi [Izu01, Izu15] and Evans-Gannon [EG11, EG14]. These are categories with a group of invertible objects, and under the action of this group by left and right tensor product, just one other double coset. The category of $N - N$ bimodules of the Haagerup subfactor is a quadratic category. While the fusion categories coming from the Asaeda-Haagerup subfactor are not quadratic, work of Grossman-Izumi-Snyder [GIS15] shows that they are Morita equivalent to quadratic categories.

This leaves us with the following remarkable observation: the extended Haagerup fusion categories are the only known fusion categories not known to be related to finite groups, quantum groups, and quadratic categories. While this almost surely only reflects our feeble ability to discover and construct fusion categories, nevertheless these categories remain uniquely interesting objects.

Every fusion category has a braided centre, which is a modular tensor category. This paper tackles the problem of describing the braided centre of the extended Haagerup categories. While we do not give a full description (in particular the associators), we produce the modular data, that is, the S and T matrices.

Recently, Morrison-Walker discovered [MW14] that a purely combinatorial argument determines the number of simple objects, and their dimensions, in the centre of extended Haagerup. This paper uses that just that information, and by representation theoretic arguments determines the modular data.

More generally, fusion categories are notoriously difficult to classify, and we hope that the methods described here can be developed into part of a machine for analysing potential new examples. As a precedent, the classifications of rank 2 and of rank 3 fusion categories [Ost03, Ost13] have relied heavily on understanding the possible modular centres. In fact, the arguments Sections 6, 7, and 8 have been automated as part of a developing `Mathematica` package, which for example can also perform the analogous arguments for the Haagerup and Asaede-Haagerup categories.

It seems likely that every unitary modular tensor category can be realised as the representation category for some strongly rational vertex operator algebra, and as such a CFT would offer at least some 'explanation' for the existence of the extended Haagerup subfactor. We explore what can be said about such an object. In particular, we are able to describe the possible character vectors associated to such a CFT. For $c = 8$ or $c = 16$, we can completely enumerate them; for $c = 24$ we at least show that there are plausible candidates.

Both the Haagerup and extended Haagerup subfactors see the prime 13. Is this a coincidence? The 13 enters their modular data in apparently different ways: through the inequivalent irreps we call $\rho_5^{(13)}$ and $\rho_{14}^{(13)}$ for the Haagerup and the extended Haagerup respectively. However $\rho_{14}^{(13)}$ lies in the symmetric square of $\rho_5^{(13)}$, and we will see in Section 10.2 that the possible character vectors for both (at the smallest possible value of central charge, namely $c = 8$) are built from theta functions of the lattice $L = A_3 52[1, \frac{1}{4}]$, using notation of [CS99]. In particular, our work suggests that there may be a natural relation between the (still hypothetical) extended Haagerup VOA $\mathcal{V}_{EH}$ and the square $\mathcal{V}_{Haag} \otimes \mathcal{V}_{Haag}$ of the (still hypothetical) Haagerup VOA.

2. Background

Throughout we write $\xi_m = e^{2\pi i/m}$, $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$.

2.1. From subfactors to modular tensor categories.

A *fusion category* $\mathcal{C}$ is a $\mathbb{C}$-linear semi-simple rigid monoidal category with finitely many isomorphism classes of simple objects and finite-dimensional spaces of morphisms, such that the endomorphism algebra of the unit object 1 is $\mathbb{C}$. A $*$-operation on $\mathcal{C}$ is a conjugate-linear involution $\mathrm{Hom}(x, y) \to \mathrm{Hom}(y, x)$ satisfying $(fg)^* = g^* f^*$ and $(f \otimes h)^* = f^* \otimes h^*$ for all $f \in \mathrm{Hom}(x, y)$, $g \in \mathrm{Hom}(z, y)$ and $h \in \mathrm{Hom}(z, w)$. A

$*$-operation is called positive if $f^* f = 0$ implies $f = 0$. A category equipped with a positive $*$-operation is called *unitary* or C^*.

Given a finite index and depth subfactor $N \subset M$ of Type II_1 factors, we obtain two unitary fusion categories: the *principal even part* consisting of the N-N bimodules which occur as summands of tensor powers of ${}_N M_N$, and the *dual even part*, consisting of the M-M bimodules occurring as summands of tensor powers of ${}_M M \otimes_N M_M$.

Let $\mathcal{C}$ be any fusion category. Write $\Phi(\mathcal{C})$ for its set of isomorphism classes of simple objects. So $\operatorname{rank}\mathcal{C} = \|\Phi(\mathcal{C})\|$. The Grothendieck ring $K(\mathcal{C})$ of $\mathcal{C}$ is also called its fusion ring. Given $[x], [y] \in \Phi(\mathcal{C})$, the structure constants $N_{[x],[y]}^{[z]} \in \mathbb{Z}_{\geq 0}$ of the fusion ring defined by $[x][y] = \sum_{[z]} N_{[x],[y]}^{[z]}[z]$ are called the *fusion coefficients*. A *dimension* on $\mathcal{C}$ is a ring homomorphism from the fusion ring $K(\mathcal{C})$ to $\mathbb{C}$; the *Perron-Frobenius dimension* PFdim of $\mathcal{C}$ is the unique dimension taking positive real values on all non-zero objects.

A *modular tensor category* is a spherical braided fusion category $\mathcal{C}$ satisfying a certain nondegeneracy condition. Define a matrix $\tilde{S}$, with rows and columns indexed by $\Phi(\mathcal{C})$, by $\tilde{S}_{[x],[y]} = \mathrm{tr}_{x\otimes y}(c_{y,x} \circ c_{x,y})$, where $c_{x,y}$ is the braiding. Then $\tilde{S}$ is well-defined; the non-degeneracy condition is that $\tilde{S}$ be invertible. In a modular tensor category, $\tilde{S}$ is symmetric, and the values $\tilde{S}_{[1],[x]}$ define a dimension $\dim(x)$ on $\mathcal{C}$. If in addition $\mathcal{C}$ is unitary, dim$= \mathrm{PFdim}$.

Given a fusion category $\mathcal{C}$, the *(braided) centre* or *(quantum) double construction* associates to it a modular tensor category $Z(\mathcal{C})$. The forgetful functor $Z(\mathcal{C}) \to \mathcal{C}$ defines a ring homomorphism on the fusion rings and (hence) preserves dimensions. The forgetful functor has an adjoint called the *induction functor*. Given a finite index and depth subfactor $N \subset M$, we obtain a (unitary) modular tensor category by applying the centre construction to its principal even part. The modular tensor category associated to the dual even part will be equivalent, but the two induction functors can carry independent information, as we'll see.

Given a fusion category $\mathcal{C}$, or for that matter a subfactor $N \subset M$, it is very difficult to determine the centre $Z(\mathcal{C})$. A surprising discovery of Morrison–Walker is that it is often possible to determine a unique possibility for the induction functor at the level of the fusion rings.

Define a diagonal matrix $\tilde{T}$, with rows and columns indexed by $\Phi(\mathcal{C})$, by $\tilde{T}_{[x],[y]} = \delta_{[x],[y]}(\mathrm{tr}_x \otimes \mathrm{id}_x)(c_{x,x})$. Then $\tilde{T}$ is well-defined and unitary. The assignment $s \mapsto \tilde{S}$, $t \mapsto \tilde{T}$ defines a projective representation of the modular group $\mathrm{SL}(2,\mathbb{Z}) = \langle s, t\rangle$, where we put $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $t = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The permutation matrix defined by $C_{[x],[y]} = \delta_{[y],[x^\vee]}$ (where $x^\vee$ is the right or left dual of x) satisfies $C^2 = I$ and commutes with both $\tilde{S}$ and $\tilde{T}$ — it is often called *charge-conjugation*. *Verlinde's formula* computes the fusion coefficients of a modular tensor category $\mathcal{C}$ in terms of $\tilde{S}$:

$$N_{[x],[y]}^{[z]} = \mathcal{D}^{-2} \sum_{[w]\in\Phi(\mathcal{C})} \dim([w])^{-1} \tilde{S}_{[x],[w]} \tilde{S}_{[y],[w]} \tilde{S}_{[z^\vee],[w]} \tag{1}$$

where $\mathcal{D}^2 = \sum_{[x]\in\Phi(\mathcal{C})} \dim([x])^2$.

2.2. **Modular data and congruence representations.** A modular tensor category is a fairly complicated beast. Remarkably, a highly constrained combinatorial invariant of a modular tensor category seems close in practise to being a complete invariant.

Definition 2.1. *Let Φ be a finite set of labels, one of which (call it 1) is distinguished. By modular data we mean matrices $S = (S_{xy})_{x,y\in\Phi}$, $T = (T_{xy})_{x,y\in\Phi}$ of complex numbers such that*

(a) S is unitary and symmetric; T is unitary and diagonal;
(b) $S_{1,x} \in \mathbb{R}^\times$ for all $x \in \Phi$; there is some $o \in \Phi$ such that $S_{o,x} > 0$ for all $x \in \Phi$;
(c) $S^2 = (ST)^3$;
(d) the numbers defined by

$$N_{xy}^z = \sum_{w\in\Phi} \frac{S_{xw}S_{yw}\overline{S_{zw}}}{S_{1w}} \tag{2}$$

are nonnegative integers, where the bar denotes complex conjugation.

The matrices $\tilde{S}, \tilde{T}$ coming from a modular tensor category can always be rescaled so as to give modular data (with 1 being $[1]$) — in particular, $S = \mathcal{D}^{-1}\tilde{S}$ ($\mathcal{D}$ is defined only up to a sign, but the sign should be chosen so that $\mathcal{D}^{-1}\tilde{S}$ has a strictly positive row). When the modular tensor category is unitary, o in (b) is also $[1]$. When the modular tensor category is the centre of a fusion category, then $T = \tilde{T}$.

The surprising lesson of this paper is that, although it is very difficult in general to obtain the modular tensor category from a fusion category or subfactor, it can be surprisingly easy to obtain the corresponding modular data.

There are several easy consequences of the definition of modular data. One is that it defines a (unitary) $\mathrm{SL}(2,\mathbb{Z})$-representation ρ through $s \mapsto S, t \mapsto T$. We will often call this ρ modular data. Also, $C = S^2$ is a permutation matrix $C_{x,y} = \delta_{y,x^\vee}$ commuting with S and T, and satisfies $C^2 = I$ and

$$\overline{S_{x,y}} = S_{x^\vee,y} \qquad \forall x, y \in \Phi\,. \tag{3}$$

Hence $1^\vee = 1$ and $o^\vee = o$; moreover, $C = I$ iff S is real. The Perron–Frobenius dimensions are $\mathrm{PFdim}(x) = \frac{S_{xo}}{S_{0o}}$. When $\mathrm{PFdim}(x) = 1$, then $x\,x^\vee = 1$ in the fusion ring, and this has significant consequences for S and T (but as we won't use these, we won't write them down).

The numbers S_{xy} lie in some cyclotomic field $\mathbb{Q}[\xi_N]$. Then for each Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}[\xi_N]/\mathbb{Q})$, there is a permutation $x \mapsto x^\sigma$ of Φ and signs $\epsilon_\sigma : \Phi \to \{\pm 1\}$ such that

$$\sigma(S_{xy}) = \epsilon_\sigma(x) S_{x^\sigma,y} = \epsilon_\sigma(y) S_{x,y^\sigma}\,. \tag{4}$$

For example, complex conjugation corresponds to (3), i.e. to the permutation $x \mapsto x^\vee$ and signs $\epsilon(x) = +1$.

Verlinde's formula (2) tells us the ratios S_{xy}/S_{1y}, being eigenvalues of the integer matrix $N_x = (N_{xa}^b)_{a,b\in\Phi}$, must be algebraic integers. Hence for any Galois automorphism σ, both $S_{1^\sigma,1}/S_{11}$ and

$$\frac{\epsilon_\sigma(1^{\sigma^{-1}})}{\epsilon_\sigma(1)}\sigma\left(\frac{S_{1^{\sigma^{-1}}1}}{S_{11}}\right) = \left(\frac{S_{1^\sigma 1}}{S_{11}}\right)^{-1}$$

are algebraic integers. But recall that $\dim x = S_{x1}/S_{11}$ for any $x \in \Phi$. Thus we know that $\dim(1^\sigma)$ is an algebraic unit for all σ. This observation will help us identify later the Galois orbit of the unit 1.

Of course, $\mathrm{Gal}(\mathbb{Q}[\xi_N]/\mathbb{Q}) \cong \mathbb{Z}_N^\times$, where the correspondence $\sigma \leftrightarrow l$ is given by $\sigma(\xi_N) = \xi_N^l$. We'll write σ_l for the automorphism corresponding to $l \in \mathbb{Z}_N$. For example, complex conjugation is σ_{-1}. We can say much more for the modular data associated to a modular tensor category.

We let $\Gamma(N)$ denote the principal congruence subgroup

$$\{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv I \pmod N\}.$$

We call N the *conductor* of an $\mathrm{SL}(2,\mathbb{Z})$-representation ρ if N is the smallest positive integer such that $\Gamma(N)$ is in the kernel of ρ (and $N = \infty$ if no $\Gamma(M)$ is in the kernel). We call N the *conductor* of a field $K \supseteq \mathbb{Q}$ if N is the smallest positive integer such that $K \subseteq \mathbb{Q}[\xi_N]$ (and $N = \infty$ if no cyclotomic field contains K).

Proposition 2.2 (c.f. [NS10, Theorem 6.8] [CG99, Ban03])**.** *Let S, T, ρ be the modular data of a modular tensor category. Let N be the order of T. Then $N < \infty$, N equals the conductor of ρ, and N is a multiple of the conductor of the field $\mathbb{Q}[S]$ generated by all entries S_{xy}. Moreover,*

$$T_{x^\sigma,x^\sigma} = T_{xx}^{l^2} \qquad \text{for } \sigma = \sigma_l \tag{5}$$

for any $\sigma \in \mathrm{Gal}(\mathbb{Q}[\xi_N]/\mathbb{Q})$. If we define a signed permutation matrix G_σ by $(G_\sigma)_{x,y} = \epsilon_\sigma(x)\delta_{y,x^\sigma}$, then

$$G_\sigma = CST^{1/l}ST^lST^{1/l} \qquad \text{for } \sigma = \sigma_l\,, \tag{6}$$

where '$1/l$' denotes the inverse mod N of l.

Now, $\Gamma(N)$ is normal in $\mathrm{SL}(2,\mathbb{Z})$, with quotient $\mathrm{SL}(2,\mathbb{Z})/\Gamma(N) \cong \mathrm{SL}(2,\mathbb{Z}_N)$. Thus this fact tells us that ρ factors through to a representation of the finite group $\mathrm{SL}(2,\mathbb{Z}_N)$, which we will also denote by ρ. It also tells us that $G_\sigma = \rho(\gamma)$, where γ is any element in $\mathrm{SL}(2,\mathbb{Z})$ congruent mod N to $\left(\begin{smallmatrix}\ell & 0\\ 0 & 1/\ell\end{smallmatrix}\right)$, where $\sigma = \sigma_l$.

We write $\chi_i^{(N)}$ for the $\mathrm{SL}(2,\mathbb{Z}_N)$-character denoted X.i by GAP, and denote by $\rho_i^{(N)}$ the corresponding representation. This labelling is generally not unique, and depends on how the conjugacy classes are identified with the columns of GAP's character table, but for the $\mathrm{SL}(2,\mathbb{Z}_N)$ we need, we will make this explicit. For example, for $\mathrm{SL}(2,\mathbb{Z}_2)$ we assign the generators S, T to class 2a, while for $\mathrm{SL}(2,\mathbb{Z}_3)$ we assign S, T to class 4a and 3b, respectively. An $\mathrm{SL}(2,\mathbb{Z}_N)$-irrep ρ obeys $\rho(-I) = \pm I$. If it is $+I$ we call ρ *even*, in which case it factors through to an irrep of $\mathrm{PSL}(2,\mathbb{Z}_N)$; if $\rho(-I) = -I$, we call ρ *odd*.

Given a d-dimensional $\mathrm{SL}(2,\mathbb{Z})$-representation ρ, write $\mathcal{T}(\rho)$ for the multiset $\{t_1, \ldots, t_d\}$ where $\{e^{2\pi i t_j}\}$ is the list of eigenvalues of $\rho(t)$. For us, $\rho(t)$ will always have finite order, so the $t_j \in \mathbb{Q}/\mathbb{Z}$. One easy consequence of an $\mathrm{SL}(2,\mathbb{Z})$-representation ρ having finite conductor N is that the multiset of T^{l^2}-eigenvalues is independent of $l \in \mathbb{Z}_N^\times$:

$$\{t_1, \ldots, t_d\} = \{l^2 t_1, \ldots, l^2 t_d\} \qquad \text{for all } l \in \mathbb{Z}_N^\times\,. \tag{7}$$

To see this, note that in $\mathrm{SL}(2,\mathbb{Z}_N)$, t^{l^2} equals t conjugated by $\left(\begin{smallmatrix}\ell & 0\\ 0 & 1/\ell\end{smallmatrix}\right)$, and so T^{l^2} must have the same multiset of eigenvalues as T.

Let $\prod_p p^{\nu_p}$ be the prime decomposition of N. By the Chinese Remainder Theorem, the group $\mathrm{SL}(2,\mathbb{Z}_N)$ is isomorphic to the direct product of the $\mathrm{SL}(2,\mathbb{Z}_{p^{\nu_p}})$. This implies that the irreps of $\mathrm{SL}(2,\mathbb{Z}_N)$ are the tensor products $\otimes_p \rho_p$, where each ρ_p is an irrep of $\mathrm{SL}(2,\mathbb{Z}_{p^{\nu_p}})$.

For example, the even 1-dimensional $\mathrm{SL}(2,\mathbb{Z})$-representations are $\rho_1^{(1)}, \rho_2^{(3)}, \rho_3^{(3)}$, while the odd ones are $\rho_2^{(2)}, \rho_2^{(2)} \otimes \rho_2^{(3)}, \rho_2^{(2)} \otimes \rho_3^{(3)}$. These have

$$\mathcal{T}(\rho) = \{0\}, \left\{\frac{2}{3}\right\}, \left\{\frac{1}{3}\right\}, \left\{\frac{1}{2}\right\}, \left\{\frac{1}{6}\right\}, \left\{\frac{5}{6}\right\},$$

respectively.

Finally we have Cauchy's theorem for modular tensor categories, recently proved in [BNRW13]:

Proposition 2.3. *The primes dividing the conductor of a modular tensor category are the same primes that divide the norm of its global dimension.*

(This result is not actually essential to what follows. Our first derivation of the modular data did not use this, but it considerably simplifies the analysis.)

3. Galois actions

Lemma 3.1. *Suppose a simple object x has T_{xx} a root of unity with order $N_x = \prod_p p^{\mu_p}$. Then the number of distinct eigenvalues of T in the full Galois orbit of x is*

$$\begin{aligned} k(N_x) &= \max\{1, 2^{\mu_2 - 3}\} \prod_{2<p|N_x} p^{\mu_p - 1}(p-1)/2 \\ &= N_x 2^{-\min\{\mu_2, 3\}} \prod_{2<p|N_x} \frac{1}{2}\left(1 - \frac{1}{p}\right). \end{aligned}$$

The size of the full Galois orbit is thus a multiple of $k(N_x)$.

Proof. Suppose the order of T is N, some multiple of N_x. For any $\ell \in \mathbb{Z}_{N_x}^\times$, there is an $\ell' \in \mathbb{Z}_N^\times$ with $\ell' \equiv \ell \pmod{N_x}$. Now

$$T_{\sigma_{\ell'} x \sigma_{\ell'} x} = T_{xx}^{\ell'^2} = T_{xx}^{\ell^2}.$$

Thus there are as many distinct eigenvalues of T in the orbit of x as there are images of the squaring map in $\mathbb{Z}_{N_x}^\times$.

Now, $\mathbb{Z}_{mn}^\times \cong \mathbb{Z}_m^\times \times \mathbb{Z}_n^\times$ if $\gcd(m,n) = 1$. Moreover, $\mathbb{Z}_{p^n}^\times \cong \mathbb{Z}_{p^{n-1}(p-1)}$ for prime $p \neq 2$ and any n, and $\mathbb{Z}_{2^n}^\times \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$ for $n \geq 2$. Those well-known facts give us the structure of any $\mathbb{Z}_N^\times$, and hence the cardinality of the image of the squaring map, as given above. □

Corollary 3.2. *Let S, T be the modular data of some modular tensor category. Suppose a prime p divides the conductor of $\mathbb{Q}[d_x]$ for some $x \in \Phi$, and $\|\Phi\| < p(p-1)/2$. Then the order N of T is pM, where M is coprime to p.*

Proof. Since $d_x = S_{x1}/S_{11}$, $\mathbb{Q}[d_x] \subseteq \mathbb{Q}[S]$. Certainly p divides N, by Lemma 1.2. If p^2 divides N, then there would be some $y \in \Phi$ with root of unity T_{yy} having order N_y a multiple of p^2. Then Lemma 3.1 would imply $\|\Phi\| \geq k(N_y) \geq k(p^2) = p(p-1)/2$, a contradiction. □

4. Basic lemmas

Let S, T be the modular data coming from a modular tensor category, and let ρ be the corresponding $\mathrm{SL}(2,\mathbb{Z})$ representation. Write N for the order of T and Φ for the set of simple objects. Recall the multiset $\mathcal{T}(\rho) = \{t_x\}_{x\in\Phi}$ defined last section, so $T_{xx} = \exp(2\pi i t_x)$. As always, $1 \in \Phi$ denotes the unit and $x^\vee$ the dual.

In the following, we will assume for convenience that $T_{11} = 1$, and that $S_{x1} > 0$. Both are true for instance for the double of any subfactor (as categorical dimensions coincide with Frobenius-Perron dimensions, which are positive). All of our results can be easily generalised when those assumptions are dropped.

Because ρ is a representation of the finite group $\mathrm{SL}(2,\mathbb{Z}_N)$, it decomposes into a direct sum $\rho \cong \oplus_{i\in\mathcal{I}}\,\rho_i$ of irreps. Our strategy will be to control the possibilities for this decomposition. Write $S_i = \rho_i(s)$ and $T_i = \rho_i(t)$. Like ρ, each ρ_i is a matrix representation; bases Φ_i are chosen so that each T_i is diagonal. Then there will exist an invertible matrix Q, with entries $Q_{iz,x}$ for $i \in \mathcal{I}, z \in \Phi_i, x \in \Phi$, such that $S = Q^{-1}(\oplus_i S_i)Q$ and $T = Q^{-1}(\oplus_i T_i)Q$. Write $N = \prod_p p^{\nu_p}$ as before; then $\rho_i \cong \otimes_p \rho_{i,p}$ where $\rho_{i,p}$ is some irrep of $\mathrm{SL}(2,\mathbb{Z}_{p_p^\nu})$.

Call $i \in \mathcal{I}$ *even* resp. *odd* if the subrepresentation ρ_i is even resp. odd. Call a simple object x *unique* if t_x occurs with multiplicity one in $\mathcal{T}(\rho)$.

Let's collect some simple observations. (See [BNRW15, §3] for some related statements.)

Lemma 4.1.

(a) If $Q_{iz,x} \neq 0$ or $(Q^{-1})_{x,iz} \neq 0$, then $T_{i;\,zz} = T_{xx}$.

(b) Suppose $S_{xy} \neq 0$. Then both $t_x, t_y \in \mathcal{T}(\rho_i)$ for some index $i \in \mathcal{I}$.

(c) For each $x \in \Phi$, write $t_x = \sum_p \frac{m_p}{p^{\nu_p}}$ for $m_p \in \mathbb{Z}$; then there is a (not necessarily unique) index $i_x \in \mathcal{I}$ such that both $0, m_p/p^{\nu_p} \in \mathcal{T}(\rho_{i_x,p})$ for all p.

(d) Suppose $x \in \Phi$ is unique. Then i_x defined in (c) *is unique. Let $\hat{x}$ denote the unique index in Φ_{i_x} with $T_{i_x;\,\hat{x}\hat{x}} = T_{xx}$. Then for all $i \in \mathcal{I}, z \in \Phi_i, y \in \Phi$, $Q_{iz,x} = \mathcal{Q}_x \delta_{ii_x}\delta_{z\hat{x}}$ and $Q_{i_x\hat{x},y} = \mathcal{Q}_x\delta_{xy}$, for some nonzero $\mathcal{Q}_x$.*

(e) Suppose $x, y \in \Phi$ are both unique and that $i_x = i_y$. Then $S_{i_x;\,\hat{x}\hat{y}}\mathcal{Q}_y^2 = S_{i_x;\,\hat{y}\hat{x}}\mathcal{Q}_x^2$ and $S_{xy} = S_{i_x;\,\hat{x}\hat{y}}\mathcal{Q}_y/\mathcal{Q}_x$.

(f) Suppose $x \in \Phi$ is unique and $0 \in \mathcal{T}(\rho_{i_x})$ has multiplicity one. Write z_x for the unique index in Φ_{i_x} with $T_{i_x;z_xz_x} = 1$. Then for any $y \in \Phi$ with $T_{yy} = 1$, $S_{xy} = S_{i_x;\,\hat{x}z_x}Q_{z_x,y}/\mathcal{Q}_x = (Q^{-1})_{y,i_xz_x}\mathcal{Q}_x S_{i_x;\,z_x\hat{x}}$.

(g) For each $r \in \mathcal{T}(\rho)$, let

$$n_+(r) = \sum_{\text{even } i\in\mathcal{I}} \mathrm{mult}_{\mathcal{T}(\rho_i)}(r)$$

and

$$n_-(r) = \sum_{\text{odd } i\in\mathcal{I}} \mathrm{mult}_{\mathcal{T}(\rho_i)}(r).$$

Then $n_+(r) + n_-(r) = \mathrm{mult}_{\mathcal{T}(\rho)}(r)$ and $n_+(r) - n_-(r)$ is the number of $x = x^\vee \in \Phi$ with $t_x = r$. In particular, $n_+(r) \geq n_-(r)$.

Proof. Because both T and $\oplus_i T_i$ are diagonal, the (iz,x)-entries of $QT = (\oplus_i T_i)Q$ and $TQ^{-1} = Q^{-1}(\oplus_i T_i)$ give (a). To see (b), suppose $S_{xy} \neq 0$. Since $S_{xy} = \sum_{i,a,b}(Q^{-1})_{x,ia}S_{i;\,ab}Q_{ib,y}$, this means there is some indices $i \in \mathcal{I}$ and $a, b \in \Phi_i$ such that both $(Q^{-1})_{x,ia}, Q_{ib,y} \neq 0$. From (a), this gives (b). Part (c) now follows from (b) and $S_{x1} \neq 0$: $\mathcal{T}(\rho_{i;p}) \subset p^{-\nu_p}\mathbb{Z}/\mathbb{Z}$ and any $r \in \mathcal{T}(\rho)$ will have a unique (mod 1) expression as a sum $\sum_p m_p/p^{\nu_p}$. Part (d) is immediate from (a). Parts (e) and (f) now follow from $S_{yx} = S_{xy} = (Q^{-1}(\oplus_i S_i)\, Q)_{xy}$.

To see part (g), restrict charge-conjugation $S^2 = Q^{-1}(\oplus_i S_i^2)\, Q$ to the $x \in \Phi$ with $t_x = r$. The trace of that permutation submatrix will equal the number of self-dual x with $t_x = r$; since $S_i^2 = \pm I$ depending on whether ρ_i is even or odd, that trace will also equal $n_+(r) - n_-(r)$. □

5. Dimensions

In [MW14], the combinatorial data $A : K_0(Z(\mathcal{C})) \to K_0(\mathcal{C})$ of the restriction functor $Z(\mathcal{C}) \to \mathcal{C}$ was obtained, when $\mathcal{C}$ is both the principal even and dual even fusion categories of the extended Haagerup. These are respectively

$$A_{EH1} = \begin{pmatrix} 1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&1&2&1&0&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&2&1&1&1&3&2&2&2&2&2&2&2&2&1&1&1&1&1&1&1&1 \\ 0&4&1&2&4&2&3&3&3&3&4&4&4&4&3&3&3&3&1&1&1&1 \\ 0&5&1&4&2&3&3&3&3&3&5&5&5&5&4&4&4&4&1&1&1&1 \\ 0&3&1&1&2&2&1&1&1&1&3&3&3&3&2&2&2&2&1&1&1&1 \end{pmatrix}$$

$$A_{EH2} = \begin{pmatrix} 1&1&1&2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&2&1&2&1&2&2&2&2&2&2&2&2&2&1&1&1&1&1&1&1&1 \\ 0&4&1&1&4&3&3&3&3&3&4&4&4&4&3&3&3&3&1&1&1&1 \\ 0&4&1&3&2&2&2&2&2&2&4&4&4&4&3&3&3&3&1&1&1&1 \\ 0&4&1&3&2&2&2&2&2&2&4&4&4&4&3&3&3&3&1&1&1&1 \\ 0&1&0&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0 \\ 0&1&0&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0 \end{pmatrix}.$$

The matrices corresponding to the induction functors are the transposes. The 22 columns correspond to the 22 simple objects Φ in the centre $Z(\mathcal{C})$. The columns have been ordered so that the first column corresponds to the tensor unit. For reasons which will be clear shortly, we will name these 22 simple objects, in order, $\omega_0, \omega_1, \omega_2, \alpha_1, \alpha_2, \alpha_3, \beta_1, \ldots, \beta_4, \gamma_1, \ldots, \gamma_4, \delta_1, \ldots, \delta_4, \epsilon_1, \ldots, \epsilon_4$. Here ω_0 is the tensor identity.

Each restriction matrix tells us two things. First, the image of the tensor identity in $\mathcal{C}$ (namely the first row in $A_{\mathcal{C}}$) will be an eigenvector of both S and T, with eigenvalue 1 and T_{11} respectively [EG11, Theorem 1]. As in any centre, we can take $T_{11} = 1$ here; this tells us for instance that

$$T_{\omega_i\omega_i} = 1 = T_{\alpha_j\alpha_j} \tag{8}$$

for all $0 \leq i \leq 2$ and $1 \leq j \leq 3$. Second and far more important, we obtain the dimensions $\dim x$ for the simple x in $Z(\mathcal{C})$: these dimensions are the components of the Perron–Frobenius eigenvector of the matrix $A_{\mathcal{C}}^t A_{\mathcal{C}}$, normalised so that $\dim 1 = 1$. Its eigenvalue will be the global dimension $\mathcal{D} = \sqrt{\sum_{x\in\Phi} \dim(x)^2}$. Of course, $S_{1x} = S_{x1} = \frac{\dim(x)}{\mathcal{D}}$.

Numerically, these dimensions are approximately 1, 177.701, 49.396, 114.049 (7 times), 176.701 (4 times), 128.304 (4 times), and 48.396 (4 times), respectively, and the global dimension $\mathcal{D}$ is approximately 570.246. When they are computed exactly, they are all found to lie in the degree-3 extension $\mathbb{Q}_{dim}$ of $\mathbb{Q}$ in $\mathbb{Q}[\xi_{13}]$. More precisely, $\mathbb{Q}_{dim}$ has a basis $1, \zeta = 2\cos(2\pi/13) + 2\cos(10\pi/13)$ and $\zeta' = 2\cos(4\pi/13) + 2\cos(6\pi/13)$ over $\mathbb{Q}$; then

$$S_{11} = \frac{7-5\zeta'}{65}, \qquad S_{1,\omega_1} = \frac{12+5\zeta+5\zeta'}{65}, \qquad S_{1,\omega_2} = \frac{7-5\zeta}{65}, \tag{9}$$
$$S_{1,\alpha_i} = S_{1,\beta_j} = \frac{1}{5},$$
$$S_{1,\gamma_j} = \frac{1+\zeta+2\zeta'}{13}, \qquad S_{1,\delta_j} = \frac{1+2\zeta+\zeta'}{13}, \qquad S_{1,\epsilon_j} = \frac{-\zeta+\zeta'}{13}.$$

6. Galois action and the conductor

Theorem 6.1. *Any modular data compatible with the restriction matrices given in the last section has*

(1) conductor $N = 5 \times 13$,

(2) first 3 rows and columns of S determined by

$$S_{\omega_i,x} = S_{x,\omega_i} = \sigma_{16}^i S_{1,x},$$

where $\sigma_l(\xi_{65}) = \xi_{65}^l$, and

(3) (i) the objects $\{\omega_i\}$ forming a single Galois orbit,

(ii) the objects $\{\alpha_1, \ldots, \alpha_3, \beta_1, \ldots, \beta_4\}$ forming a union of Galois orbits, and

(iii) the objects $\{\gamma_1, \ldots, \gamma_4, \delta_1, \ldots, \delta_4, \epsilon_1, \ldots, \epsilon_4\}$ either forming a single Galois orbit of size 12, or forming two Galois orbits of size 6, each of which containing two each of the γ_i, δ_i, and ϵ_i.

Proof. Define $\bar{\sigma}_l \in \mathrm{Gal}(\mathbb{Q}[\xi_{13}]/\mathbb{Q})$ by $\bar{\sigma}_l(\xi_{13}) = \xi_{13}^l$. We see that $\mathrm{Gal}(\mathbb{Q}_{dim}/\mathbb{Q}) = \{\bar{\sigma}_1, \bar{\sigma}_3, \bar{\sigma}_9\}$.

Then $\mathcal{D} = 295 + 125\zeta + 175\zeta'$, which has norm $\mathcal{D}\bar{\sigma}_3(\mathcal{D})\bar{\sigma}_3^2(\mathcal{D}) = 21125 = 5^3 13^2$. By Cauchy's theorem for modular tensor categories (Fact 2.3), the order N of T will be $5^a 13^b$ for some $a, b \geq 1$. By Corollary 3.2, $b = 1$.

Whatever the value of the conductor, we have a surjective map $\pi : \mathbb{Z}_N^\times \to \{\bar{\sigma}_1, \bar{\sigma}_3, \bar{\sigma}_9\}$, corresponding to the restriction of $\sigma \in \mathrm{Gal}(\mathbb{Q}[\xi_N]/\mathbb{Q})$ to $\mathbb{Q}_{dim}$, which we'll write $\pi\sigma_l = \bar{\sigma}_3^{\pi l}$.

From (4) and $S_{1x} > 0$, we obtain the sign $\varepsilon_l(x) = \mathrm{sign}(\sigma_l S_{1,x})$ for any $l \in \mathbb{Z}_N^\times$ and any $x \in \Phi$. Since $\bar{\sigma}_3(S_{1,1}) = S_{1,\omega_1}$ and $\bar{\sigma}_3^2(S_{1,1}) = S_{1,\omega_2}$, $\bar{\sigma}_3(S_{1,\alpha_1}) = S_{1,\alpha_1}$, $\bar{\sigma}_3(S_{1,\gamma_1}) = -S_{1,\delta_1}$, $\bar{\sigma}_3(S_{1,\delta_1}) = S_{1,\epsilon_1}$, we obtain

$$\varepsilon_l(\omega_i) = \varepsilon_l(\alpha_i) = \varepsilon_l(\beta_j) = \varepsilon_{\bar{\sigma}_3}(\delta_j) = +1\,,\ \varepsilon_{\bar{\sigma}_3}(\gamma_j) = \varepsilon_{\bar{\sigma}_3}(\epsilon_j) = -1\,,\ \forall i, l, j\,.$$

Moreover, $\mathbb{Z}_N^\times$ sends $\{\alpha_1, \ldots, \alpha_3, \beta_1, \ldots, \beta_4\}$ to itself, and

$$\omega_i^{\sigma_l} = \omega_{i+\pi(l)} \ \ \forall i, l\,.$$

When $\pi(l) = 1$, σ_l sends $\{\gamma_1, \ldots, \gamma_4\} \to \{\delta_1, \ldots, \delta_4\} \to \{\epsilon_1, \ldots, \epsilon_4\} \to \{\gamma_1, \ldots, \gamma_4\}$.

Using this Galois action, we obtain $S_{\omega_i,x} = \bar{\sigma}_3^i S_{1,x}$ for $i = 1, 2$. Thus we know the first 3 rows and columns of S (as well as the first 6 diagonal elements of T, of course).

Because we know $N = 5^a 13$, there will exist a unique order-3 element l in $\mathbb{Z}_N^\times$ with $\pi(l) = 1$, namely $l \equiv 3 \pmod{13}$ and $l \equiv 1 \pmod{5^a}$. We will also use $\bar{\sigma}_3$ to denote this element of $\mathrm{Gal}(\mathbb{Q}[\xi_N]/\mathbb{Q})$. As the elements of the sets $\{\gamma_1, \ldots, \gamma_4\}$, $\{\delta_1, \ldots, \delta_4\}$, and $\{\epsilon_1, \ldots, \epsilon_4\}$ are at this point indistinguishable, we may choose $\bar{\sigma}_3(\gamma_i) = \delta_i$, $\bar{\sigma}_3(\delta_i) = \epsilon_i$, and $\bar{\sigma}_3(\epsilon_i) = \gamma_i$.

Thus we see that the objects $\{\gamma_1, \ldots, \gamma_4, \delta_1, \ldots, \delta_4, \epsilon_1, \ldots, \epsilon_4\}$ form between one and four Galois orbits, with these orbits having size a multiple of 3. But (5) implies that the length of this Galois orbit must be even if T_{γ_i,γ_i} has order a multiple of 5, and the length must be a multiple of 6 if the order is a multiple of 13. This gives us (iii).

Suppose now that 5^2 divides N. Then there is a simple object x with T_{xx} a root of unity with order N_x divisible by 25. By Lemma 3.1, the Galois orbit containing x has size a multiple of 10. From the above, this is impossible. Thus we have proved that $N = 5 \times 13$.

Finally we see that $\bar{\sigma}_3$ is $\sigma_{16} \in \mathrm{Gal}(\mathbb{Q}[\xi_{65}]/\mathbb{Q})$. □

7. The group of 12

The character table of $\mathrm{SL}(2, \mathbb{Z}_{13})$ (computed from GAP) is given in Figure 1. The number $\overline{A} = (1 - \sqrt{13})/2$, so labelled because it is a Galois associate of $A = (1 + \sqrt{13})/2$. Class 2a is the central element, s and t correspond to class 4a and 13a respectively, while 12a generates the Galois group $\mathbb{Z}_{13}^\times$.

Proposition 7.1. *Let ρ be the $SL(2, \mathbb{Z})$-representation ρ coming from the modular data of the centre of the extended Haagerup. Then $\rho \cong \rho_{14}^{(13)} \oplus \rho_{(5)}$, where $\rho_{(5)}$ is some representation whose kernel contains $\Gamma(5)$.*

Proof. We learned in Theorem 6.1 that the full Galois group leaves invariant the sets $\{\omega_i\}$, $\{\alpha_i\} \cup \{\beta_i\}$, and $\{\gamma_i\} \cup \{\delta_i\} \cup \{\epsilon_i\}$ of simples. We also know that the order of T is $N = 5 \times 13$.

Consider Φ_{13}, the set of those simples x whose T_{xx} has order a multiple of 13. Because of Equation (5), the set Φ_{13} is a union of Galois orbits. By Lemma 3.1, each such orbit has size divisible by $\frac{13-1}{2} = 6$. The set Φ_{13} cannot contain an α_i or ω_i (because their T is 1), nor β_i (because those either have $T = 1$ or form Galois orbits of cardinality ≤ 4). So we have $\Phi_{13} \subseteq \{\gamma_i\} \cup \{\delta_i\} \cup \{\epsilon_i\}$.

From the character table we find that the only nontrivial irreps ρ' of $\mathrm{SL}(2, \mathbb{Z}_{13})$ for which $0 \in \mathcal{T}(\rho')$ are $\rho_4^{(13)}$, $\rho_5^{(13)}$, and the irreps $\rho_{12}^{(13)}$ to $\rho_{17}^{(13)}$.

First, suppose for contradiction that ρ contains a subrepresentation of the form $\rho_{13} \otimes \rho_5$, where ρ_{13} resp. ρ_5 are irreps with conductor exactly 13 resp. 5. If ρ_5 has dimension at least 3, then $(\rho_{13} \otimes \rho_5)(t)$ will have at least $6 \times 3 = 18$ diagonal entries with order a multiple of 13, contradicting $\|\Phi_{13}\| \leq 12$. Hence ρ_5 has dimension 2, so by the same argument all Φ_{13} is accounted for by $\rho_{13} \otimes \rho_5$, and any other subrepresentation of ρ must have conductor coprime to 13 (and hence dividing 5). But $\dim \rho_5 = 2$ implies $0 \notin \mathcal{T}(\rho_5)$ thanks to Equation (5), and this contradicts Lemma 4.1(c).

	$1a$	$26a$	$26b$	$2a$	$13a$	$13b$	$14a$	$7a$	$7b$	$7c$	$14b$	$14c$	$12a$	$3a$	$4a$	$6a$	$12b$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	6	A	$\overline{A}$	-6	$-A$	$-\overline{A}$	1	-1	-1	-1	1	1	.	.	.	.	.
χ_3	6	$\overline{A}$	A	-6	$-\overline{A}$	$-A$	1	-1	-1	-1	1	1	.	.	.	.	.
χ_4	7	A	$\overline{A}$	7	A	$\overline{A}$	.	.	.	.	.	.	-1	1	-1	1	-1
χ_5	7	$\overline{A}$	A	7	$\overline{A}$	A	.	.	.	.	.	.	-1	1	-1	1	-1
χ_6	12	-1	-1	12	-1	-1	B	D	C	B	D	C	.	.	.	.	.
χ_7	12	-1	-1	12	-1	-1	C	B	D	C	B	D	.	.	.	.	.
χ_8	12	-1	-1	12	-1	-1	D	C	B	D	C	B	.	.	.	.	.
χ_9	12	1	1	-12	-1	-1	$-B$	D	C	B	$-D$	$-C$	.	.	.	.	.
χ_{10}	12	1	1	-12	-1	-1	$-C$	B	D	C	$-B$	$-D$	.	.	.	.	.
χ_{11}	12	1	1	-12	-1	-1	$-D$	C	B	D	$-C$	$-B$	.	.	.	.	.
χ_{12}	13	.	.	13	.	.	-1	-1	-1	-1	-1	-1	1	1	1	1	1
χ_{13}	14	1	1	14	1	1	.	.	.	.	.	.	1	-1	-2	-1	1
χ_{14}	14	1	1	14	1	1	.	.	.	.	.	.	-1	-1	2	-1	-1
χ_{15}	14	-1	-1	-14	1	1	.	.	.	.	.	.	.	2	.	-2	.
χ_{16}	14	-1	-1	-14	1	1	.	.	.	.	.	.	E	-1	.	1	$-E$
χ_{17}	14	-1	-1	-14	1	1	.	.	.	.	.	.	$-E$	-1	.	1	E

$$A = -\zeta_{13}^2 - \zeta_{13}^5 - \zeta_{13}^6 - \zeta_{13}^7 - \zeta_{13}^8 - \zeta_{13}^{11}$$
$$B = -\zeta_7 - \zeta_7^6$$
$$C = -\zeta_7^3 - \zeta_7^4$$
$$D = -\zeta_7^2 - \zeta_7^5$$
$$E = -\zeta_{12}^7 + \zeta_{12}^{11}$$

FIGURE 1. The character table of $SL(2, \mathbb{Z}_{13})$.

Hence $\rho \cong \rho_{13} \oplus \rho_{(5)}$, where every subrepresentation of ρ_{13} has conductor exactly 13, and every subrepresentation of $\rho_{(5)}$ has conductor coprime to 13 (hence dividing 5). Moreover, we know by Lemma 4.1(c) that $0 \in \mathcal{T}(\rho_{13})$. We will constrain ρ_{13} by considering the Galois matrix $G_{11} = \rho\left(\begin{smallmatrix} 11 & 0 \\ 0 & 11^{-1} \end{smallmatrix}\right)$, which we know from Proposition 2.2 is a signed permutation matrix. This permutation $x \mapsto x^{\sigma_{11}}$ permutes Φ_{13} without fixed points, since $\bar{\sigma}_3 = \sigma_{11}^8$ acts without fixed points. Likewise, σ_{11} permutes $\omega_0, \omega_1, \omega_2$ without fixed points, since $\bar{\sigma}_3$ does. Therefore σ_{11} leaves invariant the sets $\{\alpha_i\} \cup \{\beta_j\}$, as well as that part of $\{\gamma_i\} \cup \{\delta_i\} \cup \{\epsilon_i\}$ not in Φ_{13}. Of course, $\rho_{(5)}\left(\begin{smallmatrix} 11 & 0 \\ 0 & 11^{-1} \end{smallmatrix}\right) = I$ since $\rho_{(5)}$ has conductor dividing 5. Together, this means $\dim \rho_{(5)} + \chi_{13}(12a) = \operatorname{Tr} G_{11}$ is the trace of a signed permutation matrix with $22 - \|\Phi_{13}\| - 3$ rows, i.e.

$$\dim \rho_{13} - \|\Phi_{13}\| - \chi_{13}(12a) \in \{3, 5, 6, 7, \ldots\}. \tag{10}$$

Here, '$12a$' refers to the conjugacy class of $\left(\begin{smallmatrix} 11 & 0 \\ 0 & 11^{-1} \end{smallmatrix}\right)$; the value 4 is excluded because the trace of a signed permutation matrix of size $n \times n$ cannot equal $n - 1$ (nor be larger than n).

Suppose next for contradiction that $\|\Phi_{13}\| < 12$. Then $\|\Phi_{13}\| = 6$, and ρ_{13} is $\rho_4^{(13)}$ or $\rho_5^{(13)}$. In this case, $\dim\rho_{13} - \|\Phi_{13}\| - \chi_{13}(12a) = 2$, a forbidden value. Similarly, if $\|\Phi_{13}\| = 12$ but ρ_{13} is not irreducible, then $\rho_{13} \cong \rho' \oplus \rho''$, where $\rho' \in \{\rho_4^{(13)}, \rho_5^{(13)}\}$ and $\rho'' \in \{\rho_2^{(13)}, \rho_3^{(13)}, \rho_4^{(13)}, \rho_5^{(13)}\}$. But then $\dim\rho_{13} - \|\Phi_{13}\| - \chi_{13}(12a)$ equals 2 (if $\rho'' \in \{\rho_2^{(13)}, \rho_3^{(13)}\}$) or 4 (if $\rho'' \in \{\rho_4^{(13)}, \rho_5^{(13)}\}$), both of which are forbidden.

Thus ρ_{13} is irreducible and of dimension ≥ 13 (since $0 \in \mathcal{T}(\rho_{13})$), so ρ_{13} is one of $\rho_{12}^{(13)}, \ldots, \rho_{17}^{(13)}$. We can dismiss $\rho_i \cong \rho_{15}^{(13)}, \rho_{16}^{(13)}, \rho_{17}^{(13)}$ out of hand, because these are odd, contradicting Lemma 4.1(g). Moreover, $\rho_{13} \cong \rho_{12}^{(13)}$ resp. $\rho_{13}^{(13)}$ have $\dim\rho_{13} - \|\Phi_{13}\| - \chi_{13}(12a)$ equal to 0 resp. 1, so also must be dismissed. The only remaining possibility is $\rho_{13} \cong \rho_{14}^{(13)}$. □

We give an explicit matrix realisation of $\rho_{14}^{(13)}$ in Appendix B.

8. The group of 4

So far, we have accounted for the 12 simples $\{\gamma_i\} \cup \{\delta_i\} \cup \{\epsilon_i\}$, as well as 4 simples x with $T_{xx} = 1$: namely 2 appearing in the $\rho_{14}^{(13)}$ (recall Proposition 7.1), and 2 trivial $\mathrm{SL}(2,\mathbb{Z})$-irreps associated with the two modular invariants

$$(1\,1\,1\,1\,1\,1\,0\,\ldots\,0)^{\intercal}, \quad (1\,1\,1\,2\,1\,0\,\ldots\,0)^{\intercal}, \tag{11}$$

coming from the induction functors. That leaves unaccounted 6 simples (amongst $\{\omega_i\} \cup \{\alpha_i\} \cup \{\beta_i\} =: \mathcal{R}$). We also know $\rho \cong \rho_{14}^{(13)} \oplus \rho_{(5)}$, where $\rho_{(5)}$ has conductor exactly 5. Our goal in this section is to identify $\rho_{(5)}$.

In Figure 2 we give the character table of $\mathrm{SL}(2,\mathbb{Z}_5)$ (computed in GAP). Class $5a$ contains t, class $4a$ contains both s and $\begin{pmatrix} 2 & 0 \\ 0 & 2^{-1}\end{pmatrix}$, while class $2a$ contains $-I$. The number $\overline{A} = -2\cos(4\pi/5)$ is the unique nontrivial Galois associate of $A = -2\cos(2\pi/5)$.

	$1a$	$10a$	$10b$	$2a$	$5a$	$5b$	$3a$	$6a$	$4a$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	2	A	$\overline{A}$	-2	$-A$	$-\overline{A}$	-1	1	.
χ_3	2	$\overline{A}$	A	-2	$-\overline{A}$	$-A$	-1	1	.
χ_4	3	$\overline{A}$	A	3	$\overline{A}$	A	.	.	-1
χ_5	3	A	$\overline{A}$	3	A	$\overline{A}$	.	.	-1
χ_6	4	-1	-1	4	-1	-1	1	1	.
χ_7	4	1	1	-4	-1	-1	1	-1	.
χ_8	5	.	.	5	.	.	-1	-1	1
χ_9	6	-1	-1	-6	1	1	.	.	.

$$A = -\zeta_5 - \zeta_5^4$$

Figure 2. The character table of $SL(2,\mathbb{Z}_5)$.

Proposition 8.1. *Let $\rho_{(5)}$ be as in Proposition 7.1. Then $\rho_{(5)} \cong \rho_8^{(5)} \oplus 1 \oplus 1 \oplus 1$.*

Proof. We can write $\rho_{(5)} = \rho_5 \oplus \rho_1$, where every subrepresentation of ρ_5 has conductor exactly 5, and ρ_1 consists of exactly $(8 - \dim \rho_5)$ copies of the trivial representation 1. The only T_{xx} we need to constrain are the four β_i, because the other entries in $\mathcal{R}$ all have $T_{xx} = 1$. Recall from Theorem 6.1 that $N = 5 \times 13$. Let Φ_5 consist of those $x \in \{\beta_1, \ldots, \beta_4\}$ with $t_x \neq 0$. Then $t_x \in \frac{1}{5}\mathbb{Z}$ for all $x \in \Phi_5$, and any $x \notin \Phi_5 \cup \Phi_{13}$ has $t_x = 0$.

Suppose for contradiction that $\|\Phi_5\| < 4$. Then $\rho_5 \cong \rho_4^{(5)}$ or $\rho_5^{(5)}$, since by Lemma 4.1(c) $0 \in \mathcal{T}(\rho_5)$. Suppose $\rho_5 \cong \rho_4^{(5)}$ (the argument handling its Galois associate $\rho_5^{(5)}$ is identical). The irrep $\rho_4^{(5)}$ is generated by matrices

$$S_4^{(5)} = \frac{1}{5}\begin{pmatrix} c - c' & \sqrt{2}c - \sqrt{2}c' & \sqrt{2}c - \sqrt{2}c' \\ \sqrt{2}c - \sqrt{2}c' & 2c + 3c' & -3c - 2c' \\ \sqrt{2}c - \sqrt{2}c' & -3c - 2c' & 2c + 3c' \end{pmatrix}, \quad T_4^{(5)} = \operatorname{diag}(1, \xi_5, \xi_5^4)\,,$$

where $c = 2\cos(2\pi/5)$, $c' = 2\cos(4\pi/5)$. These have Galois matrix (recall (6))

$$G_{2;4}^{(5)} = ST^3ST^2ST^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Write x, x' for the unique simples with $T_{xx} = \xi_5, T_{x'x'} = \xi_5^4$. Then by Lemma 4.1(e),(f) and $S_{1x} = S_{1x'} > 0$, we would obtain $\mathcal{Q}_x = \mathcal{Q}_{x'}$ and $\varepsilon_2(x) = (G_2)_{xx'} = -\mathcal{Q}_{x'}/\mathcal{Q}_x = -1$, contradicting that we know $\epsilon_\sigma(\beta_i) = +1$ for all Galois automorphisms σ.

Therefore, $\|\Phi_5\| = 4$, so there are exactly 6 simples $x \in \Phi$ with $T_{xx} = 1$, namely $\{\omega_i\} \cup \{\alpha_i\}$. From (5), the Galois automorphism σ_{-12} fixes each $x \in \Phi_{13}$, and permutes Φ_5 without fixed points. From Theorem 6.1(3)(i) and the values $S_{1,\omega_i} \in \mathbb{Q}[\xi_{13}]$, we know σ_{-12} fixes each ω_i. The modular invariant $(1\,1\,1\,2\,1\,0\,0^{16})^\intercal$ must by definition be an eigenvector of all $\rho(\gamma)$, and hence $\rho\left(\begin{smallmatrix} -12 & 0 \\ 0 & -12^{-1} \end{smallmatrix}\right)$, with eigenvalue 1, which implies σ_{-12} fixes each α_i. We already knew all $\varepsilon_{-12}(x) = +1$. Therefore, exactly as in the derivation of (10), we obtain $\operatorname{Tr} G_{-12} = 22 - 4 = \chi_{13}(1a) + \chi_5(4a) + \chi_1(1a)$, i.e.

$$\dim \rho_5 - \chi_5(4a) = 4\,. \tag{12}$$

Consider now that ρ_5 is not irreducible; then $\rho_5 \cong \rho' \oplus \rho''$ where $\rho' \in \{\rho_4^{(5)}, \rho_5^{(5)}\}$ and $\rho'' \in \{\rho_2^{(5)}, \rho_3^{(5)}, \rho_4^{(5)}, \rho_5^{(5)}\}$ and $\dim \rho_5 - \chi_5(4a) = 6$ or 8, contradicting (12).

Thus ρ_5 must be irreducible, with $0 \in \mathcal{T}(\rho_5)$, of dimension ≥ 5, and even. The only possibility is $\rho_5 \cong \rho_8^{(5)}$. □

A matrix realisation of $\rho_8^{(5)}$ is given in Appendix B.

9. End game

We have obtained in Propositions 7.1 and 8.1 that the modular data ρ of the centre of the extended Haagerup satisfies $\rho \cong \rho_{14}^{(13)} \oplus \rho_8^{(5)} \oplus 1 \oplus 1 \oplus 1$. Explicit matrix realisations of $\rho_{14}^{(13)}$ and of $\rho_8^{(5)}$ are in Appendix B. Define S' to be the corresponding block diagonal matrix and T' to be the corresponding diagonal matrix. The statement that $\rho \cong \rho_{14}^{(13)} \oplus \rho_8^{(5)} \oplus 1 \oplus 1 \oplus 1$ is that there is an invertible 22-by-22 matrix Q so that $QS = S'Q$ and $QT = T'Q$.

We have established that the simples $\{\beta_i\}$ have T-eigenvalues the four primitive 5-th roots of unity; as there is nothing to distinguish the β_i amongst themselves we may assume the eigenvalues appear in any convenient order. Similarly, we know that the simples $\{\gamma_i\} \cup \{\delta_i\} \cup \{\epsilon_i\}$ have T-eigenvalues which are all the primitive 13-th roots of unity. The T-eigenvalues for γ_i determine the T-eigenvalues for δ_i and ϵ_i since

$$T_{\delta_i\delta_i} = T_{\gamma_i^{\sigma_{16}}\gamma_i^{\sigma_{16}}} = (T_{\gamma_i\gamma_i}^{16^2})$$

and

$$T_{\epsilon_i\epsilon_i} = T_{\gamma_i^{\sigma_{16}^2}\gamma_i^{\sigma_{16}^2}} = (T_{\gamma_i\gamma_i}^{16^4}).$$

However it remains to decide which four of the 13-th primitive roots appear as the T-eigenvalues for the γ_i. We look at top left entry of the equation $STS = CT^*S^*T^*$. The right hand side is simply $\mathcal{D}^{-1}$, while the left hand side becomes $\sum_{x\in\Phi} \frac{\dim(x)^2}{\mathcal{D}^2} T_{xx}$. We find that this is only true if the

$$\frac{1}{2\pi i}\log(T_{\gamma_i\gamma_i}) = \left(\frac{9}{13}, \frac{6}{13}, \frac{4}{13}, \frac{7}{13}\right)$$

(up to the permutation, which is fixed as shown). This we may take

(13)
$$\frac{1}{2\pi i}\log(T_{xx}) = \left(0,0,0,0,0,0,\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},\frac{9}{13},\frac{6}{13},\frac{4}{13},\frac{7}{13},\frac{3}{13},\frac{2}{13},\frac{10}{13},\frac{11}{13},\frac{1}{13},\frac{5}{13},\frac{12}{13},\frac{8}{13}\right).$$

We know the 16 simples $\{\beta_i\} \cup \{\gamma_i\} \cup \{\delta_i\} \cup \{\epsilon_i\}$ are all unique, in the sense of Section 4, and so most entries of Q are determined from Lemma 4.1. The equation $QT = T'Q$ tells us that Q is the product of a permutation and a block diagonal matrix with all blocks 1-by-1 except for one, corresponding to 1-eigenvalues of T, which is 6-by-6. Much of that 6-by-6 block is irrelevant.

We also have learned much about S, some of which is collected in hypotheses (a)-(g) in the following Theorem (e.g. we know $S^2 = I$, since all simples are self-dual, so (f) is its (x,x)-entry for $x \in \{\gamma_i, \delta_i, \epsilon_i\}$).

Theorem 9.1. *Suppose*

(a) S' and T' are the explicit matrices for $\rho_{14}^{(13)} \oplus \rho_8^{(5)} \oplus 1 \oplus 1 \oplus 1$ appearing in Appendix B,
(b) T is the 22-by-22 diagonal matrix with entries given by Equation (13)*,*
(c) S is a 22-by-22 matrix whose first three rows and columns are given by Equation (9) *and Theorem 6.1(2),*
(d) we have the modular invariants appearing in Equation (11)*,*
(e) S is symmetric,
(f) $\sum_y S_{xy}S_{xy} = 1$ for $x \in \{\gamma_i, \delta_i, \epsilon_i\}$, and
(g) Q is invertible and $QS = S'Q$ and $QT = T'Q$.

Then S is given by

$$S = \left(\begin{array}{ccc|ccccccc|ccc}
 & & & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & & & \\
 & U & & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & & V & \\
 & & & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & & & \\
\hline
1/5 & 1/5 & 1/5 & 4/5 & -1/5 & -1/5 & -1/5 & -1/5 & -1/5 & -1/5 & & & \\
1/5 & 1/5 & 1/5 & -1/5 & 4/5 & -1/5 & -1/5 & -1/5 & -1/5 & -1/5 & & & \\
1/5 & 1/5 & 1/5 & -1/5 & -1/5 & 4/5 & -1/5 & -1/5 & -1/5 & -1/5 & & & \\
\cline{7-10}
1/5 & 1/5 & 1/5 & -1/5 & -1/5 & -1/5 & \multicolumn{1}{|c}{} & & & & & 0 & \\
1/5 & 1/5 & 1/5 & -1/5 & -1/5 & -1/5 & \multicolumn{1}{|c}{} & & W & & & & \\
1/5 & 1/5 & 1/5 & -1/5 & -1/5 & -1/5 & \multicolumn{1}{|c}{} & & & & & & \\
1/5 & 1/5 & 1/5 & -1/5 & -1/5 & -1/5 & \multicolumn{1}{|c}{} & & & & & & \\
\hline
 & & & & & & & & & & A & B & C \\
 & V^t & & & & & 0 & & & & B & -C & A \\
 & & & & & & & & & & C & A & -B
\end{array}\right)$$

with U, V, W, A, B, and C given below.

Proof. This calculation appears in `code/EndGame.nb`, bundled with the `arXiv` sources of this article. We write $S = (S_{xy})_{x,y \in \Phi}$, and $Q = (Q_{ix})_{1 \le i \le 22, x \in \Phi}$. The following simple steps completely identify S.

1. Solve the linear equations in the $\{S_{xy}\}$ coming from the modular invariants and symmetry.
2. Solve the linear equations in the $\{Q_{ix}\}$ coming from $QT = T'Q$ (this just shows that Q is the product of a permutation and a block diagonal matrix, as mentioned above).
3. Look at entries of $QS - S'Q$ which do not involve any of the remaining unknown S_{xy}; these are linear equations in the $\{Q_{ix}\}$, which we can solve.
4. Observe that $\det Q$ has a factor of Q_{1,ω_0}, so this must not be zero. Find all the equations coming from $QS - S'Q$ of the form $Q_{1,\omega_0} X = 0$, where X is a linear combination of the $\{S_{xy}\}$, and set $X = 0$ for each.
5. Now, the equations $\sum_y S_{xy} S_{xy} = 1$ for $x \in \{\gamma_i, \delta_i, \epsilon_i\}$ simplify to $6 S^2_{\alpha_1 x} = 0$ for these same x, so all these entries of the S-matrix must be zero.
6. Observe that $\det Q$ has a factor of Q_{15,ω_0}, so this must not be zero. Find all the equations coming from $QS - S'Q$ of the form $Q_{15,\omega_0} X = 0$, where X is a linear combination of the $\{S_{xy}\}$, and set $X = 0$ for each.
7. Finally, treat the equations $QS - S'Q$ as quadratics in $\{S_{xy}\}$ and $\{Q_{ix}\}$ jointly, and solve them; there are only 5 solutions, of which 4 make $\det Q = 0$. The remaining solution is the one described in the statement of the Theorem. □

In fact, the same argument works if we disregard the modular invariant

$$(1\,1\,1\,1\,1\,1\,0\,\ldots\,0),$$

although then at the final step the quadratics have 64 solutions, of which only one allows $\det Q \neq 0$. We make this observation because there is a candidate third fusion category $EH3$ in the Morita equivalence class of the even parts of extended Haagerup. One can determine the fusion rules of this category, if it exists. The argument of [MW14] determines the dimensions of the irreducibles in

$Z(EH3)$ (exactly the same as the dimensions here), and that $Z(EH3)$ would have the modular invariant $(1\,1\,1\,2\,1\,0\,\ldots\,0)$, but not necessarily $(1\,1\,1\,1\,1\,1\,0\,\ldots\,0)$. Thus the fact that the argument here does not rely on this second modular invariant shows that the centre of any fusion category with the fusion rules of $EH3$ would have the same S and T matrices as the centre of extended Haagerup. Of course, the S and T matrices are not known to be complete invariants of the centre. If they were, however, this discussion would allow one to establish the existence of a third category, Morita equivalent to $EH1$ and $EH2$, merely by constructing any fusion category with the appropriate fusion ring.

In the above theorem describing S we have, with $c_k = \cos(2\pi k/65)$,

$$U = \begin{pmatrix} u_1 & u_2 & u_3 \\ u_2 & u_3 & u_1 \\ u_3 & u_1 & u_2 \end{pmatrix},$$

the u_i are the roots of $21125\lambda^3 - 8450\lambda^2 + 585\lambda - 1$,

$$\begin{aligned} u_1 &= \frac{1}{65}\left(7 - 10c_{20} - 10c_{30}\right) \\ &\simeq 0.00175363 \\ u_2 &= \frac{1}{65}\left(7 + 10c_8 + 10c_{18} - 10c_{21} + 10c_{25} - 10c_{31}\right) \\ &\simeq 0.311623 \\ u_3 &= \frac{1}{65}\left(12 - 10c_8 - 10c_{18} + 10c_{20} + 10c_{21} - 10c_{25} + 10c_{30} + 10c_{31}\right) \\ &\simeq 0.0866238, \end{aligned}$$

$$V = \begin{pmatrix} v_1 & v_1 & v_1 & v_1 & -v_2 & -v_2 & -v_2 & -v_2 & -v_3 & -v_3 & -v_3 & -v_3 \\ v_2 & v_2 & v_2 & v_2 & -v_3 & -v_3 & -v_3 & -v_3 & -v_1 & -v_1 & -v_1 & -v_1 \\ v_3 & v_3 & v_3 & v_3 & -v_1 & -v_1 & -v_1 & -v_1 & -v_2 & -v_2 & -v_2 & -v_2 \end{pmatrix},$$

the v_i are the roots of $169\lambda^3 - 13\lambda - 1$,

$$\begin{aligned} v_1 &= \frac{2}{13}\left(c_8 + c_{18} + c_{20} - c_{21} + c_{25} + c_{30} - c_{31}\right) \\ &\simeq 0.30969 \\ v_2 &= \frac{1}{13}\left(1 - 4c_8 - 4c_{18} + 2c_{20} + 4c_{21} - 4c_{25} + 2c_{30} + 4c_{31}\right) \\ &\simeq -0.224999 \\ v_3 &= \frac{1}{13}\left(-1 + 2c_8 + 2c_{18} - 4c_{20} - 2c_{21} + 2c_{25} - 4c_{30} - 2c_{31}\right) \\ &\simeq -0.0848702, \end{aligned}$$

$$W = \frac{1}{10}\begin{pmatrix} 3-\sqrt{5} & -2-2\sqrt{5} & -2+2\sqrt{5} & 3+\sqrt{5} \\ -2-2\sqrt{5} & 3+\sqrt{5} & 3-\sqrt{5} & -2+2\sqrt{5} \\ -2+2\sqrt{5} & 3-\sqrt{5} & 3+\sqrt{5} & -2-2\sqrt{5} \\ 3+\sqrt{5} & -2+2\sqrt{5} & -2-2\sqrt{5} & 3-\sqrt{5} \end{pmatrix}$$

and A, B, and C are band matrices, so $A_{ij} = a_{i+j-1 \pmod 4}$, etc., and $\{a_1, a_3, b_1, b_3, d_1, d_3\}$ are the roots of $28561\lambda^6 - 28561\lambda^5 + 8788\lambda^4 - 507\lambda^3 - 169\lambda^2 + 26\lambda - 1$, while $\{a_2, a_4, b_2, b_4, d_2, d_4\}$ are the roots of $28561\lambda^6 - 6591\lambda^4 - 507\lambda^3 + 338\lambda^2 + 39\lambda - 1$:

$$
\begin{aligned}
a_1 &= \frac{1}{13}\left(2c_8 - 2c_{10} + 2c_{18} - 2c_{20} - 2c_{21} + 4c_{25} - 4c_{30} - 2c_{31} + 1\right) \\
&\simeq 0.07470114748 \\
a_2 &= \frac{1}{13}\left(-2c_8 + 2c_{10} - 2c_{18} + 6c_{20} + 2c_{21} - 2c_{25} + 2c_{31} + 1\right) \\
&\simeq 0.2714005479 \\
a_3 &= \frac{1}{13}\left(2c_8 + 2c_{10} + 2c_{18} - 2c_{20} - 2c_{21} - 2c_{31} + 2\right) \\
&\simeq 0.3520512456 \\
a_4 &= \frac{1}{13}\left(-2c_{10} - 2c_{20} + 4c_{30}\right) \\
&\simeq -0.1865303711 \\
b_1 &= \frac{1}{13}\left(2c_{10} - 2c_{20} - 2c_{25} - 2\right) \\
&\simeq -0.1595713243 \\
b_2 &= \frac{1}{13}\left(4c_8 + 4c_{18} + 2c_{20} - 4c_{21} - 2c_{25} - 4c_{31}\right) \\
&\simeq 0.3315913069 \\
b_3 &= \frac{1}{13}\left(-2c_{10} - 2c_{20} + 2c_{25} - 4c_{30} - 3\right) \\
&\simeq -0.4369214224 \\
b_4 &= \frac{1}{13}\left(-2c_8 - 2c_{18} + 2c_{21} + 4c_{25} + 2c_{30} + 2c_{31}\right) \\
&\simeq -0.02172236355 \\
d_1 &= \frac{1}{13}\left(2c_8 + 2c_{10} + 2c_{18} - 2c_{21} + 2c_{30} - 2c_{31} - 2\right) \\
&\simeq 0.1502976190 \\
d_2 &= \frac{1}{13}\left(-6c_8 + 6c_{10} - 6c_{18} + 4c_{20} + 6c_{21} - 4c_{25} + 4c_{30} + 6c_{31} + 2\right) \\
&\simeq 0.2171593392 \\
d_3 &= \frac{1}{13}\left(2c_8 - 2c_{10} + 2c_{18} - 2c_{21} + 4c_{25} - 2c_{30} - 2c_{31} - 3\right) \\
&\simeq -0.12705247914 \\
d_4 &= \frac{1}{13}\left(2c_8 - 6c_{10} + 2c_{18} - 2c_{20} - 2c_{21} - 2c_{30} - 2c_{31} - 1\right) \\
&\simeq -0.4421581056.
\end{aligned}
$$

Finally, the matrix Q is not uniquely determined; a nice choice is

$$Q = \begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

As a consistency check, we offer:

Lemma 9.2. *The Verlinde formula gives non-negative integer fusion multiplicities, which are consistent with the restriction functor $Z(EH) \to EH$.*

10. Character vectors

A natural question is whether there is a vertex operator algebra (see e.g. [LL04]) corresponding to the centre of the even part of the extended Haagerup. This is at present too difficult to answer. However, in this section we obtain all possible *character vectors* with central charge $c \leq 24$ compatible with the modular data computed in this paper. This should be information crucial for constructing the hypothetical vertex operator algebra, or showing it cannot exist. Because the procedure for doing this is difficult to extract from the literature, we will include here a more pedagogical treatment.

10.1. The general theory.

By definition, a vertex operator algebra and its modules carry actions of the Virasoro algebra, so the vertex operator algebra characters are expressible as combinations of Virasoro ones. The Virasoro characters relevant to our discussion are given next. When $c > 1$ and $h > 0$, there is a Virasoro irrep $V(c, h)$ with character

$$\mathrm{ch}_{V(c,h)} = q^{h-c/24} \prod_{n=1}^{\infty} (1 - q^n)^{-1} = \sum_{m=0}^{\infty} p(m) q^{m+h-c/24} .$$

When $c > 1$ and $h = 0$, the Virasoro irrep $V(c, h)$ has character

$$\mathrm{ch}_{V(c,0)}(\tau) = q^{-c/24} \prod_{n=1}^{\infty} (1 - q)(1 - q^n)^{-1} = \sum_{m=0}^{\infty} (p(m) - p(m-1)) q^{m-c/24} ,$$

where $p(m)$ is the mth partition number, and where $q = e^{2\pi i \tau}$.

Definition 10.1. *Suppose ρ is a d-dimensional representation of $SL(2, \mathbb{Z})$ with $T = \rho\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ a diagonal matrix. By a character vector $\mathbb{X}(\tau)$ for ρ, we mean:*

(i) $\mathbb{X} : \mathbb{H} \to \mathbb{C}^d$ is holomorphic throughout the upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}\, \tau > 0\}$;

(ii) there is a diagonal rational matrix λ *such that*

$$e^{-2\pi i\tau\lambda}\mathbb{X}(\tau) = \sum_{n=0}^{\infty} \mathbb{X}_n e^{2\pi i n\tau} \tag{14}$$

converges absolutely in $\mathbb{H}$*, and* $\sum_{n=0}^{\infty} \mathbb{X}_n q^n$ *is holomorphic at* $q = 0$*;*

(iii) for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ *and all* $\tau \in \mathbb{H}$*,*

$$\mathbb{X}\left(\frac{a\tau + b}{c\tau + d}\right) = \rho\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\mathbb{X}(\tau)\,; \tag{15}$$

(iv) each coefficient $\mathbb{X}_n$ *takes values in* $\mathbb{Z}^d_{\geq 0}$*,* $\lambda_{11} < \lambda_{jj}$ *for all* $j \neq 1$*, and* $(\mathbb{X}_0)_1 = 1$*. Moreover, each component* $\mathbb{X}_{(j)}(\tau)$ *is nonzero and can be written* $\mathbb{X}_{(j)}(\tau) = \sum_{n=0}^{\infty} \mathbb{X}'_{n;j}\ \mathrm{ch}_{V(-24\lambda_{11},\lambda_{jj}-\lambda_{11}+n)}(\tau)$ *where each* $\mathbb{X}'_{n;j} \in \mathbb{Z}_{\geq 0}$*.*

It is common to write q^λ for $e^{2\pi i\tau\lambda}$. Note that $e^{2\pi i\lambda} = T$. A function $\mathbb{X}(\tau)$ satisfying (i)-(iii) is called a *weakly holomorphic vector-valued modular function* for ρ ('weakly holomorphic' means holomorphic in $\mathbb{H}$ and meromorphic at all cusps $\mathbb{Q} \cup \{i\infty\}$). A consequence of the fact that the coefficients $\mathbb{X}_n$ are rational, is that T has finite order (hence that λ is rational). The condition $\mathbb{X}'_n \in \mathbb{Z}^d_{\geq 0}$ implies $\mathbb{X}_n \in \mathbb{Z}^d_{\geq 0}$, but in practice isn't usually much stronger. We impose the condition $\lambda_{11} < \lambda_{jj}$ here because we seek a unitary vertex operator algebra; if ρ is modular data with $o \neq 1$ (recall Definition 2.1) then this condition would become $\lambda_{oo} < \lambda_{jj}$.

Most representations ρ will possess no character vectors; for example it is elementary to verify that it requires the first column of S to be strictly positive, and an old conjecture of Atkin–Swinnerton-Dyer [ASD71] implies that the existence of a character vector is only possible when ker ρ contains some $\Gamma(N)$.

The modules of a (unitary) strongly-rational vertex operator algebra $\mathcal{V}$ form a (unitary) modular tensor category [Hua05], where 'strongly-rational' means regular, simple, equivalent as a $\mathcal{V}$-module to its contragredient $\mathcal{V}^\vee$, $\mathcal{V}_0 = \mathbb{C}1$ and $\mathcal{V}_n = 0$ for $n < 0$. The modules are infinite-dimensional, but the operator L_0 in $\mathcal{V}$ acts semi-simply on the modules, and the eigenspaces are all finite-dimensional. For each irreducible module M of $\mathcal{V}$, define the character $\chi_M(\tau) = q^{-c/24}\mathrm{tr}_M q^{L_0} = q^{h_M - c/24}\sum_{n=0}^{\infty} \dim M_{h_M+n}\, q^n$, where $M = \coprod_{n=0}^{\infty} M_{h_M+n}$ and $M_{h'}$ is the L_0-eigenspace with eigenvalue h'. The numbers c, h_M are called the *central charge* of $\mathcal{V}$ and the *conformal weight* of M. Then Zhu [Zhu96] proved that these χ_M together form a weakly holomorphic vector-valued modular function for some representation ρ of $\mathrm{SL}(2,\mathbb{Z})$; this representation is given by the modular data of the modular tensor category [DLN12] (up to a third root of unity to be discussed shortly). One irreducible $\mathcal{V}$-module will be $\mathcal{V}$ itself, which we make the first module. The characters of the irreducible modules of a unitary strongly-rational vertex operator algebra, will form a character vector (hence the name).

The modular data of a modular tensor category determines T up to a third root of unity. This ambiguity means that the central charge is only determined up to a multiple of 8. In particular, if some vertex operator algebra realises a modular tensor category, so will infinitely many others; once we've found a character vector, we've found infinitely many others. For example, tensor arbitrary many copies of the E_8 lattice vertex operator algebra to $\mathcal{V}$; this doesn't change the category, but each copy increases the central charge by 8 and multiplies the character vector by $J(\tau)^{1/3}$.

Thus the first step to trying to recover a strongly-rational VOA $\mathcal{V} = \coprod_{n=0}^{\infty} \mathcal{V}_n$ from a modular tensor category is to select a possible c, and then determine the possible character vectors $\chi_M(\tau)$. The second step would be to identify the space $\mathcal{V}_1$. It will be a reductive Lie algebra, and all homogeneous spaces M_h of all $\mathcal{V}$-modules M will be $\mathcal{V}_1$-modules. The key formula for this purpose is Proposition 4.3.5 of [Zhu96], which says that for all $u, v \in \mathcal{V}_1$ and all $\mathcal{V}$-modules M,

(16)
$$\sum_{n=0}^{\infty} \kappa_{M_{h_M+n}}(u,v)\, q^{n+h_M-c/24} = \mathrm{tr}|_M o(u[-1]v)\, q^{L_0-c/24} + \frac{\langle u,v\rangle}{24} E_2(\tau)\, \chi_M(\tau)\,.$$

Here and elsewhere, $E_n(\tau)$ denotes the weight n Eisenstein series for $\mathrm{SL}(2,\mathbb{Z})$, normalised to have leading term 1. Also, $\kappa_{M_h}(u,v) = \mathrm{tr}|_{M_h} o(u)u(v)$ is the Killing form of the $\mathcal{V}_1$-module M_h; in particular, $\kappa_{\mathcal{V}_1}$ is the Killing form of $\mathcal{V}_1$ itself. Closely related to the Killing form is the bilinear form $\langle u, v\rangle$, which is always nondegenerate and invariant. The first term on the right side is a vector-valued modular form of weight 2, for the same multiplier ρ. By itself, $\mathcal{V}_1$ generates a vertex operator subalgebra of $\mathcal{V}$, of affine algebra type. The coset or commutant of $\mathcal{V}$ by this subalgebra should itself be a strongly-rational vertex operator algebra with small central charge and trivial Lie algebra part and explicitly known character vector. Constructing $\mathcal{V}$ then largely comes down to identifying that coset vertex operator algebra.

For both steps 1 and 2, constructing vector-valued modular forms is crucial. In this paper we will restrict our attention to determining the possible character vectors, although the same method determines the possible weight-2 forms. The following treatment is developed in [BG07, Gan14].

Fix an $\mathrm{SL}(2,\mathbb{Z})$-representation ρ with T diagonal and of finite order. Let $\mathcal{M}^!(\rho)$ denote the space of all weakly holomorphic vector-valued modular functions for ρ. Let $J(\tau) = q^{-1} + 744 + 196884q + \cdots$ denote the Hauptmodul for $\mathrm{SL}(2,\mathbb{Z})$. In particular, $\mathcal{M}^!(1) = \mathbb{C}[J(\tau)]$. Note that $\mathcal{M}^!(\rho)$ is a module for the ring $\mathbb{C}[J(\tau)]$. Note that if $\rho' = Q\rho Q^{-1}$, then $\mathbb{X}(\tau) \in \mathcal{M}^!(\rho)$ iff $Q\mathbb{X}(\tau) \in \mathcal{M}^!(\rho')$.

A simple observation: if ρ is an *odd* $\mathrm{SL}(2,\mathbb{Z})$-irrep, then $\mathcal{M}^!(\rho) = 0$. This is because (15) applied to $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ gives $\mathbb{X}(\tau) = -\mathbb{X}(\tau)$. For this reason, in the following we'll restrict (without loss of generality) to even representations ρ by first projecting away any odd summands. Conveniently, the modular data we obtain from the extended Haagerup subfactor is already even.

The first fact is that $\mathcal{M}^!(\rho)$ is a free module of rank d over $\mathbb{C}[J(\tau)]$. Given d generators, it is convenient to collect them together as columns of a d-by-d matrix we'll call $\Xi(\tau)$; then there is a bijection between $\mathbb{X}(\tau) \in \mathcal{M}^!(\rho)$ and vectors $\mathbb{Y}(\tau) \in \mathbb{C}^d[J(\tau)]$ given by $\mathbb{X} = \Xi\mathbb{Y}$. We can choose the generators (hence Ξ) in such a way that there is a diagonal matrix Λ such that

$$\Xi(\tau) = q^{\Lambda}\left(I + \sum_{n=1}^{\infty} \Xi_n q^n\right). \tag{17}$$

Identifying any such Ξ is equivalent to identifying the full space $\mathcal{M}^!(\rho)$.

A word of warning: the convention of (17) differs from that of [BG07, EG11] which used $\Xi = q^{\Lambda}(Iq^{-1} + \chi + \ldots)$, but is the same as in [Gan14]. This notation change cleans up the formulas a little. It is not completely trivial that generators can be chosen so that (17) holds, but indeed it is true for all $\mathbb{C}[J]$-submodules M of $\mathcal{M}^!(\rho)$ of full rank. Once one has d vector-valued modular forms $\mathbb{X}^{(i)}(\tau)$ in M forming a matrix Ξ of shape (17) for some Λ, it is then elementary to find algorithmically d free generators for M with shape (17) (for a larger Λ) as desired. The (nonconstructive) existence of such $\mathbb{X}^{(i)}(\tau)$ is an immediate consequence of Theorem 3.1 of [Gan14].

The second fact is that $\Xi(\tau)$ is the solution to a first-order Fuchsian differential equation. The reason is that $\frac{E_{10}(\tau)}{\Delta(\tau)}\frac{d}{d\tau}$ is a differential operator on $\mathcal{M}^!(\rho)$, and so applied to each of the free generators (i.e. columns of Ξ) gives a vector-valued modular form which lies in the $\mathbb{C}[J(\tau)]$-span of the generators. That differential equation implies the recursion

$$[\Lambda, \Xi_n] + n\Xi_n = \sum_{l=0}^{n-1} \Xi_l \left(f_{n-l}\Lambda + g_{n-l}(\Xi_1 + [\Lambda, \Xi_1])\right) \tag{18}$$

for $n \geq 2$, where we write $(J(\tau) - 984)\Delta(\tau)/E_{10}(\tau) = \sum_{n=0}^{\infty} f_n q^n = 1 + 0q + 338328q^2 + \cdots$ and $\Delta(\tau)/E_{10}(\tau) = \sum_{n=0}^{\infty} g_n q^n = q + 240q^2 + 199044q^3 + \cdots$. Here, $\Delta = \eta^{24}$ where η is the Dedekind eta. We require $\Xi_0 = I$. Note that the ij-entry on the left-side of (18) is $(\Lambda_{ii} - \Lambda_{jj} + n)\, \Xi_{n\,ij}$, so (18) allows us to recursively identify all entries of Ξ_n, at least when all $|\Lambda_{jj} - \Lambda_{ii}| \neq n$. Indeed, it can be shown that $\Lambda_{jj} - \Lambda_{ii}$ can never lie in $\mathbb{Z}_{\geq 2}$.

This recursion means that $\mathcal{M}^!(\rho)$ is completely identified, i.e. $\Xi(\tau)$ is determined, once the matrices Λ and Ξ_1 are given. The matrices Ξ_1 and Λ are heavily constrained. In particular, Λ is diagonal, satisfying $e^{2\pi i \Lambda} = T$ as well as

$$\operatorname{Tr}\Lambda = -\frac{7d}{12} + \frac{1}{4}\operatorname{Tr} S + \frac{2}{3\sqrt{3}}\operatorname{Re}\left(e^{\frac{-\pi i}{6}}\operatorname{Tr} ST^{-1}\right). \tag{19}$$

When ρ is irreducible and $d < 6$, then any diagonal matrix satisfying both $e^{2\pi i \Lambda} = T$ and (19) will work, but in general these conditions won't always suffice.

Any $\Xi(\tau) \in \mathcal{M}^!(\rho)$ whose components are linearly independent over $\mathbb{C}$ gives us all of $\mathcal{M}^!(\rho)$ via [Gan14, Proposition 3.2]:

$$\mathcal{M}^!(\rho) = \mathbb{C}[J(\tau), \nabla_1, \nabla_2, \nabla_3],$$

where

$$\begin{aligned}
\nabla_1 &= \frac{E_4 E_6}{\Delta} q\frac{d}{dq} \\
\nabla_2 &= \frac{E_4^2}{\Delta}\left(q\frac{d}{dq} - \frac{E_2}{6}\right) q\frac{d}{dq} \\
\nabla_3 &= \frac{E_6}{\Delta}\left(q\frac{d}{dq} - \frac{E_2}{3}\right)\left(q\frac{d}{dq} - \frac{E_2}{6}\right) q\frac{d}{dq}
\end{aligned}$$

The building blocks of all of these differential operators is the operator $q\frac{d}{dq} - \frac{k}{12}E_2$, which sends weight k modular forms to weight $k+2$ ones.

We may take $\Lambda_{\rho_1\oplus\rho_2} = \Lambda_{\rho_1} \oplus \Lambda_{\rho_2}$ and $(\Xi_{\rho_1\oplus\rho_2})_1 = (\Xi_{\rho_1})_1 \oplus (\Xi_{\rho_2})_1$. Moreover, Λ and Ξ for the weakly-holomorphic vector-valued modular forms at weight 2 for the contragredient representation ρ, is $-I-\Lambda$ and $E_4(\tau)^2E_6(\tau)\Delta(\tau)^{-1}(\Xi(\tau)^\intercal)^{-1}$. (Definition 10.1 can be extended to forms of arbitrary even weight in the obvious way; Proposition 4.1 of [Gan14] tells how to convert $\Xi(\tau)$'s for different weights but the same ρ.)

We can find $\Xi_{\rho_1\otimes\rho_2}(\tau)$ from $\Xi_{\rho_1}(\tau)$ and $\Xi_{\rho_1}(\tau)$ using the fact that

$$\mathcal{M}^!(\rho_1\otimes\rho_2) = \mathbb{C}[J(\tau),\nabla_1,\nabla_2,\nabla_3](\mathcal{M}^!(\rho_1)\otimes\mathcal{M}^!(\rho_2)).$$

Suppose $\mathcal{M}^!(\rho_1)$ is free of rank d_1 over $\mathbb{C}[J(\tau)]$, and $\mathcal{M}^!(\rho_2)$ is free of rank d_2 over $\mathbb{C}[J(\tau)]$. Starting with the matrix $\tilde{\Xi} = \Xi_{\rho_1}(\tau)\otimes\Xi_{\rho_2}(\tau) \in M_{d_1d_2\times d_1d_2}(\mathbb{C}((q)))$, we form the $d_1d_2 \times 4d_1d_2$ matrix $\left(\tilde{\Xi}\ \nabla_1\tilde{\Xi}\ \nabla_2\tilde{\Xi}\ \nabla_3\tilde{\Xi}\right)$ and then find a $\mathbb{C}[J(\tau)]$ basis for the columns. Replacing $\tilde{\Xi}$ with these new basis vectors as columns, we repeat until $\tilde{\Xi}$ stabilises. This does not quite provide our $\Xi_{\rho_1\otimes\rho_2}(\tau)$, as we still need to perform a change of basis so that Equation (17) holds.

In the case of any irrep ρ with kernel containing $\Gamma(N)$ for some $N=\prod_p p^{\nu_p}$, we write $\rho\cong\oplus_i\rho_i$ and $\rho_i\cong\otimes_p\rho_{i;p}$ as before. It then suffices to know the Λ and χ for each irrep $\rho_{i;p}$ appearing in that decomposition. Each such $\rho_{i;p}$ is an irrep in some Weil representation associated to lattices, and so some $\mathbb{X}(\tau)\in\mathcal{M}^!(\rho_{i;p})$ with linearly independent components can be built up from lattice theta functions. For 'small' powers p^ν, Λ,Ξ_1 have been computed for every irrep ρ of SL$(2,\mathbb{Z}_{p^\nu})$, by Timothy Graves in his PhD thesis. This means that the full space $\mathcal{M}^!(\rho)$ can be determined fairly quickly from his tables for any representation of SL$(2,\mathbb{Z}_N)$, provided the prime powers dividing N are not too large (<32).

A minor technicality: it is possible for the tensor product $\otimes_p\rho_{i;p}$ to be even, even though some (necessarily an even number of) $\rho_{i;p}$ may be odd. One way to handle this is to replace any such odd factor $\rho_{i;p}$ with the even irrep $\rho_2^{(2)}\otimes\rho_{i;p}$, as an even number of $\rho_2^{(2)}$'s tensor to 1. For the modular data associated to the extended Haagerup subfactor, all components $\rho_{i;p}$ which arise are even.

These calculations can be a little delicate. We suggest two strong consistency checks. First,

$$\mathcal{A}_2\left(\mathcal{A}_2-\frac{1}{2}I\right) = 0\,, \tag{20}$$

$$\mathcal{A}_3\left(\mathcal{A}_3-\frac{1}{3}I\right)\left(\mathcal{A}_3-\frac{2}{3}I\right) = 0\,, \tag{21}$$

where

$$\mathcal{A}_2 = -\frac{31}{72}\Lambda-\frac{1}{1728}(\Xi_1+[\Lambda,\Xi_1])\,,\quad \mathcal{A}_3 = -\frac{41}{72}\Lambda+\frac{1}{1728}(\Xi_1+[\Lambda,\Xi_1])\,. \tag{22}$$

Given Λ and Ξ_1, construct $\Xi(\tau)$ through (18); then the columns of $\Xi(\tau)$ will freely generate the $\mathbb{C}[J]$-module $\mathcal{M}^!(\rho)$ for some SL$(2,\mathbb{Z})$-representation ρ, iff the corresponding $\mathcal{A}_2,\mathcal{A}_3$ satisfy (20),(21). Incidentally, $e^{2\pi i\mathcal{A}_2}$ is similar to S and $e^{2\pi i\mathcal{A}_3}$ is similar to $\rho\left(\begin{smallmatrix}-1&1\\-1&0\end{smallmatrix}\right)=TS$. This representation ρ is, as always, uniquely determined by its values on $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ (which is $T=e^{2\pi i\Lambda}$) and $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ (which is S). The S-matrix can be estimated numerically by using the recursion (18) to compute the first few terms of the series expansion of $\Xi(\tau)$; then $\Xi(\tau)$ is invertible anywhere in $\mathbb{H}$ except at the countably many elliptic fixed points $\mathrm{SL}(2,\mathbb{Z}).i \cup \mathrm{SL}(2,\mathbb{Z}).\xi_3$, so as long as we avoid those elliptic points we can estimate $S = \Xi(-1/\tau)\,\Xi(\tau)^{-1}$. We have applied both tests to all Λ, Ξ_1 given below.

Since the central charge c is determined only up to mod 8 by the modular tensor category, there always are three $\mathrm{SL}(2,\mathbb{Z})$-representations which have to be considered, namely ρ, $\rho_2^{(3)} \otimes \rho$, and $\rho_3^{(3)} \otimes \rho$, where $\rho_2^{(3)}, \rho_3^{(3)}$ are described in Section 2.2. There is no straightforward relation between the matrices $\Xi(\tau)$ for these three representations. However, Proposition 4.1(2) of [Gan14] gives a shortcut. Suppose we know $\Xi(\tau)$ for ρ; then the columns for $\Xi(\tau)$ for $\rho_2^{(3)} \otimes \rho$ will be linear combinations over $\mathbb{C}$ of the columns of $E_4(\tau)\eta(\tau)^{-8}\Xi(\tau) = q^{\Lambda}(I_d + \cdots)$ and of

$$\eta(\tau)^{-8}\left(E_4(\tau)\Xi(\tau)\Lambda\left(\Lambda - \frac{1}{6}I_d\right) - \left(q\frac{d}{dq} - \frac{1}{6}E_2(\tau)\right)q\frac{d}{dq}\Xi(\tau)\right) = q^{\Lambda}\left(1728\mathcal{A}_3\left(\mathcal{A}_3 - \frac{1}{3}I_d\right)q + \cdots\right).$$

Now $\mathrm{rank}(\mathcal{A}_3(\mathcal{A}_3 - \frac{1}{3}I_d))$ equals the multiplicity of ξ_3^2 as an eigenvalue of TS, and this is the number of vectors that should be chosen from the latter. This method applied to $\rho_2^{(3)} \otimes \rho$ gives $\Xi(\tau)$ for $\rho_3^{(3)} \otimes \rho$.

Obtaining the possible character vectors is now easy combinatorics. Suppose $\mathbb{X}(\tau)$ is a character vector, and write $\mathbb{X}(\tau) = \Xi(\tau)\,\mathbb{Y}(J(\tau))$ for some vector-valued polynomial $\mathbb{Y}(J) \in \mathbb{C}^d[J]$. Write d_j for the degree of the component $\mathbb{Y}_j$, and d_M for the maximum of all d_j. Then $d_M \leq c/24 + \max_j \Lambda_{jj}$. More precisely, if $d_1 = d_M$, then $d_1 = c/24 + \Lambda_{11}$ and $\mathbb{Y}_1$ is monic. If $d_j = d_M$ and $j \neq 1$, then $d_j < c/24 + \Lambda_{jj}$ and the leading coefficient of $\mathbb{Y}_j$ must be a positive integer.

Given some $\mathbb{X}(\tau) \in \mathcal{M}^!(\rho)$, write $\mathbb{X}(\tau) = q^{\lambda}\sum_{n=0}^{\infty}\mathbb{X}_n q^n$ where each entry of $\mathbb{X}_0$ is nonzero. To prove a candidate $\mathbb{X}(\tau)$ is indeed a character vector, we need to prove each $\mathbb{X}_n \in \mathbb{Z}^d$, and that each $\mathbb{X}_n \in \mathbb{R}^d_{\geq 0}$. The first statement is accomplished by:

Lemma 10.2. *Suppose $f(\tau) = q^{\lambda}\sum_{n=0}^{\infty} f_n q^n$ is a (scalar-valued) weakly holomorphic modular function for some subgroup Γ of SL$(2,\mathbb{Z})$, possibly with multiplier $\mu : \Gamma \to \mathbb{C}^{\times}$. Suppose Γ has index m in SL$(2,\mathbb{Z})$, and contains a congruence subgroup. Choose $k \in \mathbb{Z}_{\geq 0}$ so that $\lambda \geq -k/24$ for all j. Suppose the Fourier coefficients f_n are integral for all $n \leq km/24$. Then $f_n \in \mathbb{Z}$ for all n.*

This is Lemma 3(b) of [Gan12], applied to $\eta(\tau)^k f(\tau)$. A useful fact is that the index of $\Gamma(N)$ in $\mathrm{SL}(2,\mathbb{Z})$ is $N^3\prod_{p|N}(1-p^{-2})$. We apply the Lemma by taking $f(\tau)$ to be any component $\mathbb{X}_j(\tau)$ of our vector-valued modular function $\mathbb{X}$, so $\lambda = \lambda_j$ and Γ is the projective kernel of the multiplier ρ.

Positivity is more delicate, and again follows the methods of [Gan12]. The general argument will be developed elsewhere, and here we will limit the discussion to the following. Assume ρ is a unitary $\mathrm{SL}(2,\mathbb{Z})$-representation, and that $S_{1j} > 0$ for all j. (This is true for the modular data of any unitary modular tensor category.) Assume also that $\lambda_{11} < \lambda_{jj}$ for all $j \neq 1$ (this is true for any character vector). Then for large n, the Rademacher expansion for $\mathbb{X}(\tau)$ implies

$$(\mathbb{X}_n)_j \sim S_{1j}(\mathbb{X}_0)_1 \frac{e^{4\pi\sqrt{n|\lambda_{11}|}}}{\sqrt{2}n^{3/4}}\,.$$

Hence for all sufficiently large n, all coefficients $\mathbb{X}_n$ will be positive, provided $\mathbb{X}_0 \in \mathbb{R}^d_{>0}$. To prove a given $\mathbb{X}$ is truly a character vector, we would need to make this estimate effective. This can be quite involved, and will be treated in generality in future work following the positivity method developed in [Gan12]. We will not address this further in this paper.

10.2. **Specialisation to the double of the even part of EH.** For the modular data of the extended Haagerup, the central charge c will be a multiple of 8 (a positive multiple, if we insist, as we will, that the hypothetical vertex operator algebra be unitary). The corresponding conductor N will be $N = 5 \cdot 13$ if $24|c$ or $N = 3 \cdot 5 \cdot 13$ otherwise. We have the decompositions:

$\rho \cong \rho_{14}^{(13)} \oplus \rho_8^{(5)} \oplus \rho_1^{(1)} \oplus \rho_1^{(1)} \oplus \rho_1^{(1)}$ if $c \equiv 0 \pmod{24}$;

$\rho \cong \left(\rho_2^{(3)} \otimes \rho_{14}^{(13)}\right) \oplus \left(\rho_2^{(3)} \otimes \rho_8^{(5)}\right) \oplus \rho_2^{(3)} \oplus \rho_2^{(3)} \oplus \rho_2^{(3)}$ if $c \equiv 8 \pmod{24}$;

$\rho \cong \left(\rho_3^{(3)} \otimes \rho_{14}^{(13)}\right) \oplus \left(\rho_3^{(3)} \otimes \rho_8^{(5)}\right) \oplus \rho_3^{(3)} \oplus \rho_3^{(3)} \oplus \rho_3^{(3)}$ if $c \equiv 16 \pmod{24}$.

Moreover,

$$\Lambda(\rho_1^{(1)}) = \Xi_1(\rho_1^{(1)}) = (0)\,;$$

$$\Lambda(\rho_2^{(3)}) = (-1/3)\,,\ \Xi_1(\rho_2^{(3)}) = (248)\,;$$

$$\Lambda(\rho_3^{(3)}) = (-2/3)\,,\ \Xi_1(\rho_3^{(3)}) = (496)\,;$$

$$\Lambda(\rho_8^{(5)}) = \mathrm{diag}\left(0, -\frac{4}{5}, -\frac{3}{5}, -\frac{2}{5}, -\frac{6}{5}\right)\,,\ \Xi_1(\rho_8^{(5)}) = \begin{pmatrix} 25 & -57750 & -11550 & -1350 & -819000 \\ -3/2 & -39 & -126 & -7 & 468 \\ -5/3 & -1050 & 248 & -9 & 1950 \\ 5 & 1650 & 264 & 282 & -28600 \\ -1/6 & -7 & -4 & -3 & -12 \end{pmatrix}$$

$$\rho_2^{(3)} \otimes \rho_8^{(5)}\ :\ \Lambda = \mathrm{diag}\left(-\frac{1}{3}, -\frac{2}{15}, -\frac{14}{15}, -\frac{11}{15}, -\frac{8}{15}\right)\,,\ \Xi_1 = \begin{pmatrix} -52 & -30 & -22050 & -6600 & -1680 \\ -100 & 0 & -39200 & 3850 & 1728 \\ -5 & -4 & 56 & 11 & 16 \\ -10 & 2 & 84 & 220 & -108 \\ -25 & 8 & 1200 & -1100 & 32 \end{pmatrix};$$

$$\rho_3^{(3)} \otimes \rho_8^{(5)}\ :\ \Lambda = \mathrm{diag}\left(-\frac{2}{3}, -\frac{7}{15}, -\frac{4}{15}, -\frac{16}{15}, \frac{2}{15}\right)\,,\ \Xi_1 = \begin{pmatrix} -29 & -294 & -60 & -2640 & 3 \\ -375 & 56 & -50 & 3300 & -1 \\ -2025/2 & -686 & 82 & 4312 & -1/2 \\ -25/2 & 14 & 2 & -104 & -1/2 \\ -6125 & 2401 & 100 & -411600 & 3 \end{pmatrix}$$

$$\Lambda(\rho_{14}^{(13)}) = \mathrm{diag}\left(0, -\frac{12}{13}, -\frac{11}{13}, -\frac{10}{13}, -\frac{9}{13}, -\frac{8}{13}, -\frac{7}{13}, -\frac{6}{13}, -\frac{5}{13}, -\frac{4}{13}, -\frac{3}{13}, -\frac{15}{13}, -\frac{14}{13}, 0\right)\,,$$

$$\Xi_1(\rho_{14}^{(13)}) = \begin{pmatrix} 61/3 & 16731 & -36374 & 20748 & 8281 & 1703 & -2145 & 962 & 169 & -117 & 39 & -276822 & 45474 & -29/3 \\ -2/3 & -4 & 35 & -12 & 21 & -30 & 0 & 20 & -7 & 0 & -5 & -33 & 44 & 4/3 \\ 1 & 103 & 42 & 35 & -14 & 45 & -22 & 3 & 6 & 15 & -10 & -84 & -66 & -1 \\ 5 & -1 & 168 & -96 & -182 & -27 & 0 & -34 & 21 & 27 & -8 & -512 & -330 & -3 \\ 13/3 & 203 & 70 & -330 & -18 & 161 & 78 & 14 & -14 & 0 & -14 & -924 & -286 & -8/3 \\ -1 & -805 & 585 & -308 & 365 & 231 & -55 & -30 & 21 & 11 & 10 & 462 & 66 & 1 \\ 5 & 1000 & -1230 & -693 & 168 & -75 & 246 & -60 & 48 & 54 & -10 & -1836 & -3732 & -4 \\ -14/3 & 3276 & 721 & 0 & 637 & -210 & -99 & 200 & 42 & 0 & 39 & 9702 & -3432 & 4/3 \\ -4 & 278 & 522 & 1034 & -658 & 324 & 154 & 114 & 162 & -88 & -2 & 14124 & -9186 & 1 \\ 1 & 3093 & 4299 & 1260 & -1729 & 633 & 468 & 150 & -189 & 21 & 39 & -1554 & -5946 & -1 \\ -14/3 & -11375 & -8980 & 1617 & -1078 & 770 & 0 & 440 & -154 & 99 & -4 & 25872 & -5929 & 7/3 \\ -2/3 & -5 & -10 & 6 & 8 & -2 & -6 & 5 & 0 & -3 & 2 & -12 & -5 & 1/3 \\ -1 & -13 & -9 & 0 & 14 & -9 & -13 & 6 & -6 & -5 & 1 & -6 & -18 & 1 \\ 29/3 & 62985 & -26962 & 15288 & -3094 & 6175 & 858 & -494 & 650 & 117 & 39 & -204516 & 172458 & -7/3 \end{pmatrix}$$

$$\Lambda(\rho_2^{(3)}\otimes\rho_{14}^{(13)}) = \mathrm{diag}\left(-\frac{1}{3}, -\frac{10}{39}, -\frac{7}{39}, -\frac{4}{39}, -\frac{40}{39}, -\frac{37}{39}, -\frac{34}{39}, -\frac{31}{39}, -\frac{28}{39}, -\frac{25}{39}, -\frac{22}{39}, -\frac{19}{39}, -\frac{16}{39}, -\frac{1}{3}\right),$$

$$\Xi_1(\rho_2^{(3)}\otimes\rho_{14}^{(13)}) = \begin{pmatrix} -18 & 15 & -24 & 5 & 11960 & 2854 & -5083 & 3212 & 685 & -1190 & 924 & -536 & 62 & 78 \\ -27 & 20 & 16 & -2 & 6320 & -10695 & 4420 & 5533 & 150 & 645 & -880 & -258 & 300 & 101 \\ -98 & 50 & 6 & 4 & 1810 & 17990 & -13770 & 3010 & 842 & 2110 & -1540 & -284 & -175 & -34 \\ 112 & -16 & 15 & -2 & -56192 & -21651 & -15912 & -649 & 3038 & 1473 & 616 & -1082 & -420 & 50 \\ 4 & 0 & 1 & -2 & 0 & 10 & 0 & 7 & -2 & -5 & 0 & -8 & -4 & -1 \\ 2 & -5 & 4 & -3 & 40 & 42 & -17 & -12 & 13 & 10 & 11 & -4 & -2 & 6 \\ -7 & 4 & -4 & -2 & 48 & -35 & 68 & -11 & 22 & 29 & 0 & -13 & -20 & 5 \\ 14 & 10 & 2 & -1 & 195 & -90 & -34 & 118 & 31 & 30 & 44 & 36 & -15 & -10 \\ -7 & 0 & 1 & 2 & -320 & 189 & 136 & 77 & 194 & -135 & -56 & 54 & -28 & 18 \\ -28 & 5 & 6 & 3 & -1040 & 476 & 442 & 198 & -337 & 40 & 154 & 4 & -14 & 20 \\ 21 & -20 & -11 & 2 & -880 & 761 & 52 & 638 & -342 & 375 & -88 & 75 & -12 & 11 \\ -38 & -14 & -6 & -3 & -3404 & 514 & -782 & 1386 & 733 & -18 & 176 & 64 & 80 & -10 \\ -19 & 40 & -8 & -4 & -2720 & 271 & -3536 & -1122 & -1076 & -255 & -88 & 194 & 56 & 60 \\ 56 & 60 & -18 & 7 & -4160 & 10688 & 1768 & -1166 & 2675 & 440 & 693 & -376 & 274 & 20 \end{pmatrix}$$

$$\Lambda(\rho_3^{(3)}\otimes\rho_{14}^{(13)}) = \mathrm{diag}\left(-\frac{2}{3}, -\frac{23}{39}, -\frac{20}{39}, -\frac{17}{39}, -\frac{14}{39}, -\frac{11}{39}, -\frac{8}{39}, -\frac{5}{39}, -\frac{2}{39}, -\frac{38}{39}, -\frac{35}{39}, -\frac{32}{39}, -\frac{29}{39}, -\frac{2}{3}\right),$$

$$\Xi_1(\rho_3^{(3)}\otimes\rho_{14}^{(13)}) = \begin{pmatrix} -39 & 40 & -128 & 72 & 35 & 5 & -10 & 4 & 1 & -760 & 749 & -564 & 100 & 155 \\ -78 & 80 & 98 & -14 & 14 & -20 & 7 & 6 & 0 & 532 & -924 & -351 & 546 & 260 \\ -312 & 234 & 64 & 65 & 0 & 30 & -16 & 3 & 0 & 2184 & -2002 & -480 & -390 & -104 \\ 455 & -92 & 160 & -68 & -98 & -28 & -16 & 0 & 1 & 1862 & 980 & -2240 & -1160 & 182 \\ 975 & 235 & 16 & -254 & 21 & 60 & 10 & 6 & -1 & -6840 & -2058 & -2700 & -750 & -715 \\ -403 & -884 & 523 & -208 & 169 & 76 & -9 & -4 & 1 & 6916 & 3796 & 962 & 156 & 1443 \\ -2119 & 832 & -896 & -351 & 91 & -28 & 30 & -2 & 1 & 13832 & 728 & -3108 & -5174 & 1547 \\ 3042 & 2392 & 460 & -26 & 182 & -44 & -8 & 10 & 1 & 15067 & 16380 & 12960 & -3770 & -2197 \\ -1170 & 169 & 280 & 351 & -182 & 57 & 20 & 5 & 2 & -55328 & -20475 & 15560 & -8385 & 4342 \\ -13 & 2 & 8 & 5 & -7 & 3 & 2 & 1 & -1 & 0 & 21 & 4 & -9 & 13 \\ 13 & -24 & -19 & 6 & -7 & 4 & 1 & 2 & -1 & 76 & -28 & 30 & -10 & 13 \\ -39 & -26 & -16 & -26 & -13 & 2 & -2 & 3 & 1 & 0 & 78 & 28 & 65 & -13 \\ -26 & 92 & -28 & -34 & -7 & 0 & -8 & -2 & -1 & -133 & -56 & 160 & 58 & 91 \\ 118 & 179 & -88 & 57 & -14 & 25 & 4 & -1 & 2 & 304 & 574 & -408 & 389 & 38 \end{pmatrix}$$

In all three cases (namely, $c \equiv 8k \pmod{24}$ for $k = 0, 1, 2$), the full 22-by-22 matrices Λ and Ξ_1 are obtained by $\Lambda = Q^{-1}\left(\Lambda(\rho_{13}) \oplus \Lambda(\rho_5) \oplus \Lambda(\rho_1) \oplus \Lambda(\rho_1) \oplus \Lambda(\rho_1)\right) Q$ and $\Xi_1 = Q^{-1}\left(\Xi_1(\rho_{13}) \oplus \Xi_1(\rho_5) \oplus \Xi_1(\rho_1) \oplus \Xi_1(\rho_1) \oplus \Xi_1(\rho_1)\right) Q$, for Q explicitly given in Section 9, and where $\rho_1 = \rho_2^{(3)\,\otimes k}$, $\rho_5 = \rho_2^{(3)\,\otimes k} \otimes \rho_8^{(5)}$, and $\rho_{13} = \rho_2^{(3)\,\otimes k} \otimes \rho_{14}^{(13)}$.

Let us explain how we found these matrices Λ and Ξ_1. Consider first the A_4 root lattice and its dual A_4^* (we use the standard lattice notation and terminology explained in e.g. [CS99]). The group A_4^*/A_4 has 5 elements, and these have theta series $\theta_{[0]}(\tau) = 1 + 40q + \cdots$, $\theta_{[1]}(\tau) = \theta_{[4]}(\tau) = q^{2/5}(5 + 30q + \cdots)$ and $\theta_{[2]}(\tau) = \theta_{[3]}(\tau) = q^{3/5}(10 + 25q + \cdots)$, where $\theta_{[i]} = \theta_{[5-i]}$ follows because a coset and its negative always have identical theta series. These 3 functions form the components of a vector-valued modular form of weight 2 for $\mathrm{SL}(2,\mathbb{Z})$, for a multiplier equivalent to $\rho_5^{(5)}$. The products $\theta_{[i]}(\tau)\theta_{[j]}(\tau)$ will form a vector-valued modular form of weight 4 for $\mathrm{SL}(2,\mathbb{Z})$, for a multiplier equivalent to the symmetric square of $\rho_5^{(5)}$. That symmetric square is isomorphic to $1 \oplus \rho_8^{(5)}$. We can make them weight 0 by dividing by $\eta(\tau)^8$, but this tensors the multiplier by $\rho_2^{(3)}$. In particular, the first column of the matrix $\Xi(\tau)$ for $\rho_2^{(3)}\otimes\rho_8^{(5)}$ has components $(\theta_{[0]}^2-2\theta_{[1]}\theta_{[2]})\eta^{-8}$, $-\theta_{[2]}^2\eta^{-8}$, $-\theta_{[0]}\theta_{[1]}\eta^{-8}$, $-\theta_{[0]}\theta_{[2]}\eta^{-8}$, $-\theta_{[1]}^2\eta^{-8}$. This generates the full module $\mathcal{M}^!(\rho_2^{(3)}\otimes\rho_8^{(5)})$, using the differential operators ∇_i and $\mathbb{C}[J]$.

A similar method works to find Λ,Ξ_1 for $\rho_{14}^{(13)}$. For this let the lattice be $L=A_3 52[1,\frac{1}{4}]$, which means $\cup_{i=0}^3 L_0+([i],\frac{i}{4})$ for the orthogonal direct sum $L_0=A_3\oplus\sqrt{52}\mathbb{Z}$. Then L^*/L has 13 elements, with theta functions $\psi_{[0]}(\tau)=1q^0+\cdots$, $\psi_{[1]}(\tau)=\psi_{[12]}(\tau)=1q^{2/13}+\cdots$, $\psi_{[2]}(\tau)=\psi_{[11]}(\tau)=5q^{8/13}+\cdots$, $\psi_{[3]}(\tau)=\psi_{[10]}(\tau)=4q^{5/13}+\cdots$, $\psi_{[4]}(\tau)=\psi_{[9]}(\tau)=4q^{6/13}+\cdots$, $\psi_{[5]}(\tau)=\psi_{[8]}(\tau)=10q^{11/13}+\cdots$, $\psi_{[6]}(\tau)=\psi_{[7]}(\tau)=6q^{7/13}+\cdots$. These $\psi_{[i]}$ form a vector-valued modular form of weight 2 for $\mathrm{SL}(2,\mathbb{Z})$ with multiplier $\rho_5^{(13)}$, so the products $\psi_{[i]}\psi_{[j]}$, $i\leq j$, form one of weight 4 whose multiplier is the symmetric square of $\rho_5^{(13)}$, namely $1\oplus\rho_{12}^{(13)}\oplus\rho_{14}^{(13)}$. Then the third column of $\Xi(\rho_2^{(3)}\otimes\rho_{14}^{(13)})$ is the vector-valued modular form with components $2\psi_{1,5}-\psi_{2,3}-\psi_{4,6}$, $\psi_{6,6}-\psi_{2,4}$, $\psi_{0,1}-\psi_{2,6}$, $\psi_{3,5}-\psi_{2,2}$, $\psi_{1,1}-\psi_{4,5}$, $\psi_{0,3}-\psi_{5,6}$, $\psi_{2,5}-\psi_{0,4}$, $\psi_{0,6}-\psi_{1,3}$, $\psi_{0,2}-\psi_{1,4}$, $\psi_{1,6}-\psi_{5,5}$, $\psi_{1,2}-\psi_{3,3}$, $\psi_{0,5}-\psi_{3,4}$, $\psi_{4,4}-\psi_{3,6}$, and $\psi_{1,5}+\psi_{2,3}-2\psi_{4,6}$, where we write $\psi_{i,j}:=\psi_{[i]}\psi_{[j]}\eta^{-8}$.

At $c=8$, we find $c/24+\max_j\Lambda_{jj}=\frac{3}{13}<1$, and so we only need to consider

$$\mathbb{Y}(J(\tau))=(1,Y_2,Y_3,Y_4,0,0,0,0,0,0,0,0,0,Y_{14},Y_{15},Y_{16},0,0,0,Y_{20},Y_{21},Y_{22})^{\mathsf{T}}$$

for $Y_2,Y_3,Y_4,Y_{14},Y_{15},Y_{16},Y_{20},Y_{21},Y_{22}\in\mathbb{N}$.

Writing $\mathbb{X}(\tau)=\Xi(\tau)\,\mathbb{Y}(J(\tau))=q^\lambda\sum_{n=0}^\infty\mathbb{X}_nq^n$, the conditions $\mathbb{X}_n\in\mathbb{R}_{\geq0}^{22}$ merely for $n=0,1$ give the 27 inequalities

$$0\leq 50Y_1+50Y_2+50Y_3-50Y_4$$
$$0\leq 7Y_1-5Y_2-2Y_3+11Y_{14}+2Y_{15}-4Y_{16}-35Y_{20}-20Y_{21}+22Y_{22}$$
$$0\leq 4Y_1-Y_2-3Y_3+7Y_{14}-2Y_{15}-Y_{16}-10Y_{20}+4Y_{21}+2Y_{22}$$
$$0\leq 21Y_1+11Y_2-32Y_3+638Y_{14}+2Y_{15}+11Y_{16}-761Y_{20}+12Y_{21}+342Y_{22}$$
$$0\leq 38Y_1+10Y_2-48Y_3-1386Y_{14}+3Y_{15}-6Y_{16}+514Y_{20}+80Y_{21}+733Y_{22}$$
$$0\leq 27Y_1-101Y_2+74Y_3-5533Y_{14}+2Y_{15}+16Y_{16}-10695Y_{20}+300Y_{21}+150Y_{22}$$
$$0\leq 76Y_1+20Y_2+2Y_3+50Y_4-1848Y_{14}+3Y_{15}+4Y_{16}-6174Y_{20}-162Y_{21}-1555Y_{22}$$
$$0\leq 20Y_1+78Y_3+50Y_4-682Y_{14}-4Y_{15}-14Y_{16}+4514Y_{20}+112Y_{21}+1120Y_{22}$$
$$0\leq 595Y_1-425Y_2-170Y_3-45177Y_{14}-6Y_{15}+56Y_{16}+184289Y_{20}-644Y_{21}-54530Y_{22}$$
$$0\leq 320Y_1-236Y_2-84Y_3+6468Y_{14}+9Y_{15}-48Y_{16}-34736Y_{20}-1120Y_{21}+6998Y_{22}$$
$$0\leq 112Y_1+50Y_2-162Y_3-649Y_{14}-2Y_{15}-15Y_{16}+21651Y_{20}+420Y_{21}-3038Y_{22}$$
$$0\leq 98Y_1+34Y_2-132Y_3-3010Y_{14}-4Y_{15}+6Y_{16}+17990Y_{20}-175Y_{21}+842Y_{22}$$
$$0\leq 532Y_1+192Y_2-724Y_3+110924Y_{14}+5Y_{15}+88Y_{16}-226308Y_{20}+481Y_{21}+45493Y_{22}$$
$$0\leq 632Y_1+245Y_2-877Y_3-200684Y_{14}+16Y_{15}-32Y_{16}+128651Y_{20}+2144Y_{21}+79472Y_{22}$$
$$0\leq 410Y_1-1526Y_2+1116Y_3-504274Y_{14}+5Y_{15}+86Y_{16}-1555890Y_{20}+6480Y_{21}+11658Y_{22}$$
$$0\leq -2Y_1-6Y_2+8Y_3+12Y_{14}+3Y_{15}+4Y_{16}+42Y_{20}-2Y_{21}+13Y_{22}$$
$$0\leq 19Y_1-60Y_2+41Y_3+1122Y_{14}+4Y_{15}-8Y_{16}+271Y_{20}+56Y_{21}-1076Y_{22}$$
$$0\leq 7Y_1-18Y_2+11Y_3-77Y_{14}-2Y_{15}+Y_{16}+189Y_{20}-28Y_{21}+194Y_{22}$$

along with some redundant ones. Some linear programming easily gives upper bounds on all the variables: $Y_2, Y_3 \leq 1, Y_4 \leq 3, Y_{15}, Y_{16} \leq 2$, and $Y_{14}, Y_{20}, Y_{21}, Y_{22} = 0$. We then easily enumerate all solutions, obtaining 13 possible character vectors. Of these, 9 have components which are identically zero, which is not allowed. The remaining four have vacuum components as given below.

$$\begin{aligned}
\mathbb{Y}_1 &= (1,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0)^{\intercal} \\
q^{1/3}\mathbb{X}_1(\tau)_{\omega_0} &= 1 + 12q + 73q^2 + 346q^3 + 1390q^4 + 4956q^5 + 16715q^6 + 52982q^7 + \cdots \\
\mathbb{Y}_2 &= (1,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0)^{\intercal} \\
q^{1/3}\mathbb{X}_2(\tau)_{\omega_0} &= 1 + 3q + 22q^2 + 86q^3 + 461q^4 + 1992q^5 + 8343q^6 + 30997q^7 + \cdots \\
\mathbb{Y}_3 &= (1,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0)^{\intercal} \\
q^{1/3}\mathbb{X}_3(\tau)_{\omega_0} &= 1 + 13q + 83q^2 + 372q^3 + 1460q^4 + 5112q^5 + 17053q^6 + 53651q^7 + \cdots \\
\mathbb{Y}_4 &= (1,0,0,0,0,0,0,0,0,0,0,0,0,2,0,0,0,0,0,0,0)^{\intercal} \\
q^{1/3}\mathbb{X}_4(\tau)_{\omega_0} &= 1 + 4q + 32q^2 + 112q^3 + 531q^4 + 2148q^5 + 8681q^6 + 31666q^7 + \cdots
\end{aligned}$$

All four of these possible character vectors will have four components equal to $\theta_{[2]}^2\eta^{-8}, \theta_{[0]}\theta_{[1]}\eta^{-8}, \theta_{[0]}\theta_{[2]}\eta^{-8}, \theta_{[1]}^2\eta^{-8}$, i.e. identical with components of the character vector of the lattice VOA for $A_4 \oplus A_4$. This is highly suggestive: the extended Haagerup VOA (at $c = 8$) should contain some orbifold of the $A_4 \oplus A_4$ lattice VOA. That subVOA would also have $c = 8$, which means the (hypothetical) extended Haagerup VOA would be a finite extension of that lattice orbifold. Something similar happens for the (still hypothetical) $c = 8$ Haagerup VOA, but there the lattice orbifold VOA (which is $\mathcal{V}_L^+$ for $L = A_3 52[1, \frac{1}{4}]$) only has $c = 4$. So in this sense the extended Haagerup VOA is more accessible than the Haagerup VOA. Curiously, this the same lattice $A_3 52[1, \frac{1}{4}]$ makes an appearance both in the Haagerup and extended Haagerup.

We now employ Lemma 10.2 to ensure integrality of the Fourier coefficients.

Lemma 10.3. *The vector valued modular forms* $\mathbb{Y}_1, \mathbb{Y}_2, \mathbb{Y}_3, \mathbb{Y}_4$ *are integral.*

Proof. We begin by showing that the columns of $\Xi_n(\rho_{13})$ and $\Xi_n(\rho_5)$ with $\Lambda_{jj} \geq -1/3$ are themselves integral. (These are the only relevant columns, as all other entries of the $\mathbb{Y}_i$ are automatically zero.) To see this, we apply the Lemma to the vector-valued modular form $Q\Xi(\tau)Q^{-1}e_i$ for $i \in \{1, 2, 3, 4, 14, 15, 16, 20, 21, 22\}$. For $i \in \{1, 2, 3, 4, 14\}$, the projective kernel is $\pm\Gamma(13)$ with index 1092, while for $i \in \{15, 16\}$ the projective kernel $\pm\Gamma(5)$ has index 60, and for $i \in \{20, 21, 22\}$ the index is 1. Thus it suffices to check out as far as $8 \cdot 1092/24 = 364$.

Next, note that $\mathbb{Y}_1 - \mathbb{Y}_i$ is supported on ρ_{13}: more precisely, each of the differences $\mathbb{X}_1(\tau) - \mathbb{X}_i(\tau)$ lies in the $\mathbb{Z}$-span of the third and fourth columns of $\Xi(\rho_2^{(3)} \otimes \rho_{14}^{(13)})$, so this is covered by the previous paragraph.

Finally, we need to see that $\mathbb{X}_1(\tau)$ is integral. We observe that the inverse of Q is almost integral:

$$Q^{-1} = \begin{pmatrix}
\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
-\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
-\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 & 0 & -\frac{1}{6} & \frac{5}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 & 0 & \frac{5}{6} & -\frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 & 0 & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

We see from the locations of denominators in Q^{-1} (and our earlier observation about the integrality of the matrices $\Xi_n(\rho)$) that it is only the first six entries of $\mathbb{X}_1(\tau)$ which might not be integral. Consider first $\mathbb{X}_1(\tau)_{\alpha_1}$, the 4th component (which also equals the fifth and sixth components). Note $Q\mathbb{Y}_1 = e_1 - e_3 + e_{15} + e_{22}$, and compute

$$\begin{aligned}(Q^{-1}\Xi_n Q\mathbb{Y}_1)_4 = \frac{1}{6}(&-(\Xi_n)_{15,1} + (\Xi_n)_{15,3} - (\Xi_n)_{15,15} - (\Xi_n)_{15,22} \\ &-(\Xi_n)_{20,1} + (\Xi_n)_{20,3} - (\Xi_n)_{20,15} - (\Xi_n)_{20,22} \\ &-(\Xi_n)_{21,1} + (\Xi_n)_{21,3} - (\Xi_n)_{21,15} - (\Xi_n)_{21,22} \\ &+(\Xi_n)_{22,1} - (\Xi_n)_{22,3} + (\Xi_n)_{22,15} + (\Xi_n)_{22,22})\end{aligned}$$

and observe that most of these vanish as Ξ_n is block diagonal, obtaining

$$(\mathbb{X}_1)_{\alpha_1} = (Q^{-1}\Xi_n Q\mathbb{Y}_1)_4 = \frac{1}{6}(-(\Xi_n)_{15,15} + (\Xi_n)_{22,22}).$$

These are the coefficients of a (scalar) modular function for $\Gamma = \pm\Gamma(5)$, so we can apply Lemma 10.2 with $m = 60$, $k = 8$. After checking explicitly that the first 20 values of $(Q^{-1}\Xi_n Q\mathbb{Y}_1)_4$ are integral, this ensures that $\mathbb{X}_1(\tau)_{\alpha_1}$ is integral.

Now consider $\mathbb{X}_1(\tau)_{\omega_0}$, the 1st component. We need to show that $(Q^{-1}\Xi_n Q\mathbb{Y}_1)_1$ is integral. To do this, we take advantage of the fact that $(Q^{-1})_1 + (Q^{-1})_4 \pmod 1 = \frac{2}{3}e_1 + \frac{2}{3}e_{14} + \frac{1}{3}e_{22}$, and that we have already shown $\Xi_n Q\mathbb{Y}_1$ and $(Q^{-1}\Xi_n Q\mathbb{Y}_1)_4$ are integral. We then see

$$\left(\frac{2}{3}e_1 + \frac{2}{3}e_{14} + \frac{1}{3}e_{22}\right)\Xi_n Q\mathbb{Y}_1 = \frac{2}{3}\left((\Xi_n)_{1,1} - (\Xi_n)_{1,3} + (\Xi_n)_{14,1} - (\Xi_n)_{14,3}\right) + \frac{1}{3}(\Xi_n)_{22,22}$$

is a modular function for $\Gamma = \pm\Gamma(13)$. Again, Lemma 10.2 with $m = 1092$, $k = 8$ allows us to check the first 364 coefficients to ensure that $\mathbb{X}_1(\tau)_{\omega_0}$ is integral.

Finally $(Q^{-1})_1 - (Q^{-1})_2$ and $(Q^{-1})_1 - (Q^{-1})_3$ are integral and supported in entries $1, 14$, and the corresponding columns of $\Xi(\tau)$ are integral, so $\mathbb{X}_1(\tau)_{\omega_i}$ are all integral. $\square$

Multiplying any character vector at $c = 8$ by $J(\tau)^{1/3}$ resp. $J(\tau)^{2/3}$ will give a character vector at $c = 16$ resp. $c = 24$. But there should be many more as c grows, and knowing other candidates could be important if all 4 candidates at $c = 8$ fail to be realised by a vertex operator algebra. At $c = 16$ we find $c/24+\max_j \Lambda_{jj} = \frac{4}{5} < 1$, so we consider

$$\mathbb{Y}(J(\tau)) = (1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, 0, 0, 0, 0, Y_{14}, Y_{15}, Y_{16}, Y_{17}, 0, Y_{19}, Y_{20}, Y_{21}, Y_{22})^{\mathsf{T}} .$$

Again the conditions $\mathbb{X}_n \in \mathbb{R}^{22}_{\geq 0}$ for $n = 0, 1$ suffice to obtain finitely many cases; we obtain inequalities

$$\begin{aligned}
0 &\leq 13Y_1 - 13Y_2 - Y_{14} - 5Y_{15} + 8Y_{16} - 21Y_{17} + 2Y_{19} + 3Y_{20} - 9Y_{21} - Y_{22} \\
0 &\leq 2119Y_1 - 1547Y_2 - 572Y_3 + 2Y_{14} + 351Y_{15} - 896Y_{16} - 728Y_{17} + 832Y_{19} - 28Y_{20} - 5174Y_{21} + Y_{22} \\
0 &\leq 975Y_1 - 715Y_2 - 260Y_3 + 6Y_{14} - 254Y_{15} - 16Y_{16} - 2058Y_{17} - 235Y_{19} - 60Y_{20} + 750Y_{21} + Y_{22} \\
0 &\leq 39Y_1 + 13Y_2 - 52Y_3 - 3Y_{14} + 26Y_{15} - 16Y_{16} - 78Y_{17} - 26Y_{19} + 2Y_{20} + 65Y_{21} + Y_{22} \\
0 &\leq 339Y_1 + 103Y_2 + 25Y_3 + 175Y_4 + 175Y_5 + 175Y_6 + 98Y_7 + 20Y_8 + 880Y_9 - 4Y_{14} \\
&\qquad + 28Y_{15} + 32Y_{16} + 266Y_{17} - 212Y_{19} - 30Y_{20} - 452Y_{21} - 2Y_{22} \\
0 &\leq 103Y_1 + 27Y_2 + 337Y_3 + 175Y_4 + 175Y_5 + 175Y_6 + 98Y_7 + 20Y_8 + 880Y_9 - 2Y_{14} \\
&\qquad - 86Y_{15} - 144Y_{16} - 882Y_{17} + 146Y_{19} + 20Y_{20} + 326Y_{21} + 2Y_{22} \\
0 &\leq 175Y_1 + 175Y_2 + 175Y_3 + 817Y_4 - 175Y_5 - 175Y_6 - 98Y_7 - 20Y_8 - 880Y_9 \\
0 &\leq 175Y_1 + 175Y_2 + 175Y_3 - 175Y_4 + 817Y_5 - 175Y_6 - 98Y_7 - 20Y_8 - 880Y_9 \\
0 &\leq 175Y_1 + 175Y_2 + 175Y_3 - 175Y_4 - 175Y_5 + 817Y_6 - 98Y_7 - 20Y_8 - 880Y_9 \\
0 &\leq 375Y_1 + 375Y_2 + 375Y_3 - 375Y_4 - 375Y_5 - 375Y_6 + 56Y_7 - 50Y_8 + 3300Y_9 \\
0 &\leq 2025Y_1 + 2025Y_2 + 2025Y_3 - 2025Y_4 - 2025Y_5 - 2025Y_6 - 1372Y_7 + 164Y_8 + 8624Y_9 \\
0 &\leq 227375Y_1 + 227375Y_2 + 227375Y_3 - 227375Y_4 - 227375Y_5 - 227375Y_6 + 46648Y_7 + 850Y_8 - 44107350Y_9 \\
0 &\leq 6084Y_1 - 4394Y_2 - 1690Y_3 - 6Y_{14} - 374Y_{15} + 1702Y_{16} - 49168Y_{17} + 1564Y_{19} + 88Y_{20} - 4582Y_{21} - 2Y_{22} \\
0 &\leq 115765Y_1 - 83993Y_2 - 31772Y_3 + 8Y_{14} + 7956Y_{15} - 28112Y_{16} - 79196Y_{17} \\
&\qquad + 34684Y_{19} - 308Y_{20} - 367952Y_{21} + Y_{22} \\
0 &\leq 67704Y_1 - 49114Y_2 - 18590Y_3 + 22Y_{14} - 6878Y_{15} - 906Y_{16} - 309708Y_{17} \\
&\qquad - 12288Y_{19} - 780Y_{20} + 70830Y_{21} + 2Y_{22} \\
0 &\leq 3042Y_1 - 2197Y_2 - 845Y_3 + 10Y_{14} - 26Y_{15} - 460Y_{16} + 16380Y_{17} - 2392Y_{19} + 44Y_{20} + 3770Y_{21} - Y_{22} \\
0 &\leq 455Y_1 + 182Y_2 - 637Y_3 - 68Y_{15} - 160Y_{16} + 980Y_{17} + 92Y_{19} + 28Y_{20} + 1160Y_{21} - Y_{22} \\
0 &\leq 312Y_1 + 104Y_2 - 416Y_3 - 3Y_{14} - 65Y_{15} + 64Y_{16} + 2002Y_{17} + 234Y_{19} + 30Y_{20} - 390Y_{21} \\
0 &\leq 13Y_1 + 13Y_2 - 26Y_3 + 2Y_{14} + 6Y_{15} + 19Y_{16} - 28Y_{17} + 24Y_{19} - 4Y_{20} + 10Y_{21} + Y_{22} \\
0 &\leq 8450Y_1 + 3211Y_2 - 11661Y_3 - 20Y_{14} + 1768Y_{15} - 1320Y_{16} - 52780Y_{17} \\
&\qquad - 3484Y_{19} + 44Y_{20} + 19604Y_{21} + 2Y_{22} \\
0 &\leq 78Y_1 - 260Y_2 + 182Y_3 - 6Y_{14} + 14Y_{15} + 98Y_{16} + 924Y_{17} + 80Y_{19} - 20Y_{20} + 546Y_{21} \\
0 &\leq 403Y_1 - 1443Y_2 + 1040Y_3 + 4Y_{14} + 208Y_{15} + 523Y_{16} - 3796Y_{17} - 884Y_{19} + 76Y_{20} + 156Y_{21} + Y_{22} \\
0 &\leq 26Y_1 - 91Y_2 + 65Y_3 + 2Y_{14} + 34Y_{15} - 28Y_{16} + 56Y_{17} + 92Y_{19} + 58Y_{21} - Y_{22} \\
0 &\leq 1170Y_1 - 4342Y_2 + 3172Y_3 - 5Y_{14} - 351Y_{15} + 280Y_{16} + 20475Y_{17} + 169Y_{19} + 57Y_{20} - 8385Y_{21} + 2Y_{22}
\end{aligned}$$

with 179,459 solutions. All appear to have positive integral Fourier coefficients for many (and probably all) terms. This time, Lemma 10.2 would require checking about twice as many coefficients for integrality as was necessary for $c = 8$. Although this is probably possible, enough effort is involved that we have not done this.

At $c = 24$ we find $c/24 + \max_j \Lambda_{jj} = 1$, so we consider

$$\mathbb{Y}(J(\tau)) = (J(\tau) + Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Y_{10}, Y_{11}, Y_{12}, Y_{13}, Y_{14}, Y_{15}, Y_{16}, Y_{17}, Y_{18}, Y_{19}, Y_{20}, Y_{21}, Y_{22})^{\mathsf{T}},$$

but this time just with $Y_i \in \mathbb{C}$. The requirement that the first component $\Xi(\tau)\,\mathbb{Y}(J(\tau))$ has the strictly lowest leading exponent forces $Y_{10} = -\frac{1}{6}, Y_{18} = -\frac{2}{3}$, and $Y_{21} = -1$. Now, sadly, the conditions $\mathbb{X}_n \in \mathbb{R}^{22}_{\geq 0}$ do not appear to cut out a bounded region, no matter how high an n we consider. (In particular, $Y_1, \ldots, Y_6$ are unbounded.) However the conditions $\mathbb{X}'_{n;j} \geq 0$ for $n = 0, 1$ do cut out a bounded region. We cannot enumerate the points however (the naive upper bound we have on its volume in the Y_i coordinate system is around 10^{43}), and the collection of solutions may shrink further as we consider $\mathbb{X}'_{n;j} \geq 0$ for larger n. Nevertheless, it is possible to find new individual solutions, for example

$$\mathbb{Y}(J(\tau)) = \left(J(\tau) + \frac{519}{2}, -\frac{23}{6}, \frac{83}{6}, \frac{625}{6}, \frac{625}{6}, \frac{625}{6}, \frac{1}{2}, \frac{19}{3}, 12, -\frac{1}{6}, 1, 16, -\frac{10}{3}, \frac{77}{3}, -4, 5, \frac{302}{3}, -\frac{2}{3}, \frac{1}{3}, 10, -1, 49\right)^{\mathsf{T}}$$

which gives non-negative integral $\mathbb{X}'_{n;j}$ at least up to $n = 50$.

Appendix A. Some consequences of Lemma 3.1

We record here some additional consequences of Lemma 3.1, which although unneeded for the present argument, may prove useful to others.

Corollary A.1. *If the full Galois orbit of some $x \in \Phi$ has cardinality $k \leq 6$, then the root of unity T_{xx} has order dividing some number in the set $\mathcal{N}_k$, where*

$$\begin{aligned}
\mathcal{N}_1 &= \{2^3 \cdot 3\} \\
\mathcal{N}_2 &= \{2^3 \cdot 3 \cdot 5, 2^4 \cdot 3\} \\
\mathcal{N}_3 &= \{2^3 \cdot 3^2, 2^3 \cdot 3 \cdot 7\} \\
\mathcal{N}_4 &= \{2^5 \cdot 3, 2^4 \cdot 3 \cdot 5\} \\
\mathcal{N}_5 &= \{2^3 \cdot 3 \cdot 11\} \\
\mathcal{N}_6 &= \{2^4 \cdot 3^2, 2^4 \cdot 3 \cdot 7, 2^3 \cdot 3^2 \cdot 5, 2^3 \cdot 3 \cdot 5 \cdot 7, 2^3 \cdot 3 \cdot 13\}.
\end{aligned}$$

Proof. Clearly the formula for $k(N_x)$ in Lemma 3.1 is increasing with respect to the factorization of N_x. Moreover N_x can not be divisible by any prime p larger than 13, as otherwise $k(N_x) \geq (p-1)/2 > 6$. Thus we just need to check small exponents in $N_x = 2^{\mu_2} 3^{\mu_3} 5^{\mu_5} 7^{\mu_7} 11^{\mu_{11}} 13^{\mu_{13}}$. ☐

The Mathematica notebook `ConductorsForOrbitsSize.nb` available with the `arXiv` sources of this article readily computes $\mathcal{N}_k$ for values of k up to several hundred.

Corollary A.2. *Let k_x be the size of the full Galois orbit of an object x and N_x be the order of T_{xx}. Then for any $\delta > 0$ we have*

$$N_x \leq C_\delta k_x^{1+\delta}$$

where

$$C_\delta = 24 \prod_{p \in P_\delta} p \left(\frac{2}{p-1} \right)^{1+\delta}$$

where the product is taken over the finite set

$$P_\delta = \left\{ 3 < p : p \text{ is prime and } p \left(\frac{2}{p-1} \right)^{1+\delta} > 1 \right\}.$$

(The set P_δ is certainly finite as all such primes are less than $\max\{7, 1+2\left(\frac{11}{5}\right)^{1/\delta}\}$.)

Proof. Write $N_x = \prod_p p^{\mu_p}$ as before. We have

$$\frac{N_x}{k_x^{1+\delta}} = \prod_{p|N_x} R_p$$

where

$$R_2 = \begin{cases} 2 & \text{if } \mu_2 = 1 \\ 4 & \text{if } \mu_2 = 2 \\ 2^{3+\delta(3-\mu_2)} & \text{if } \mu_2 \geq 3 \end{cases}$$

and

$$R_p = p^{1+\delta(1-\mu_p)} \left(\frac{2}{p-1} \right)^{1+\delta}.$$

Thus in the worst case $\mu_2 = 3$ and $\mu_p = 1$ for each other $p|N_x$. When $\mu_p = 1$, R_p simplifies to $p\left(\frac{2}{p-1}\right)^{1+\delta}$. We then have

$$\begin{aligned} \frac{N_x}{k_x^{1+\delta}} &= R_2 R_3 \prod_{3<p|N_x} R_p \\ &\leq 24 \prod_{p \in P_\delta} p \left(\frac{2}{p-1} \right)^{1+\delta} \\ &= C_\delta. \end{aligned}$$

□

The rank of a modular tensor category is the sum of the sizes of the Galois orbits of objects, while the exponent is the least common multiple of the orders of the eigenvalues of T, so while we have close-to-linear bounds on the conductor on each orbit, it is still possible to have exponential growth of $\operatorname{ord}(T)$ relative to the rank, as for $\operatorname{Rep} DS_n$.

Incidentally, for all odd primes the smallest irrep of $\mathrm{SL}(2, \mathbb{Z}_{p^\nu})$ with conductor p^ν for $\nu \geq 2$ has dimension $(p^2-1)p^{\nu-2}$. The smallest irrep with conductor 2^ν for $\nu \geq 4$ has dimension $3 \cdot 2^{\nu-4}$.

Appendix B. Explicit matrices for some irreps of $SL(2,\mathbb{Z})$

The representations we are interested in both lie in the *principal series* of $\mathrm{SL}(2,\mathbb{Z}_p)$. In particular, write B for the (Borel) subgroup of upper-triangular matrices $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right)$. Each irrep λ of $\mathbb{Z}_p^\times \cong \mathbb{Z}_{p-1}$ extends to B by $\lambda\left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right) = \lambda(a)$. Denote by $\rho^{(p);\lambda}$ the induced representation $\mathrm{Ind}_B^{SL(2,\mathbb{Z}_p)}\lambda$ – it will be $p+1$-dimensional. Then $\rho^{(p);\lambda} \cong \rho^{(p);\bar\lambda}$ is irreducible iff $\lambda^2 \neq 1$. By contrast, $\rho^{(p);1}$ is the direct sum of 1 and an irrep called the Steinberg representation, while $\rho^{(p);\lambda}$ for the order-2 λ is the direct sum of two $(p+1)/2$-dimensional irreps. Coset representatives for $\mathrm{SL}(2,p)/B$ are $\left(\begin{smallmatrix} 1 & 0 \\ j & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1\end{smallmatrix}\right)$, and using this it is easy to work out not merely the characters of $\rho^{(p);\lambda}$, but explicit matrices as well.

The modular data of the centre of the extended Haagerup has two building blocks: the conductor-5 irrep $\rho_8^{(5)}$ and the conductor-13 irrep $\rho_{14}^{(13)}$. The irrep $\rho_8^{(5)}$ is the Steinberg representation for $\mathrm{SL}(2,\mathbb{Z}_5)$, while the other irrep, $\rho_{14}^{(13)}$, is $\rho^{(13);\lambda}$ for the (unique up to complex conjugate) order-3 λ. It is thus easy to work out explicit matrix realisations. First, $\rho_8^{(5)}$ is generated by matrices $T_8^{(5)} = \mathrm{diag}(1,\xi_5,\xi_5^2,\xi_5^3,\xi_5^4)$ and

$$S_8^{(5)} := \frac{1}{5}\begin{pmatrix} -1 & -6 & -6 & -6 & -6 \\ -1 & -2c-c' & 2c' & 2c & -c-2c' \\ -1 & 2c' & -c-2c' & -2c-c' & 2c \\ -1 & 2c & -2c-c' & -c-2c' & 2c' \\ -1 & -c-2c' & 2c & 2c' & -2c-c' \end{pmatrix}$$

where we write $c = 2\cos(2\pi/5)$ and $c' = 2\cos(4\pi/5)$.

Likewise, $\rho_{14}^{(13)}$ is generated by 14×14 matrices $S_{14}^{(13)}$ and $T_{14}^{(13)}$. Label their rows/columns by $0,1,2,\ldots,12,0'$ in that order. Then $\left(T_{14}^{(13)}\right)_{00} = \left(T_{14}^{(13)}\right)_{0'0'} = 1$ and $\left(T_{14}^{(13)}\right)_{ll} = \xi_{13}^l$ for each $1 \le l \le 12$. Write $c_j = 2\cos(2\pi j/13)$. Define vectors $\varepsilon = (1,1,-1,-1,1,-1)$ and $\varepsilon' = (1,1,1,-1,1,1)$ and quantities $s(l) = (c_l - c_{2l} - c_{3l} + c_{5l})/13$, $s'(l) = (1+3c_l+3c_{5l})/13$, $t(l) = (2-c_l+c_{2l}-c_{4l})/13$, and $t'(l) = (2c_{2l}-c_{3l}-c_{5l})/13$. Then

$$S_{00} = s(1)\,,\ S_{00'} = s(2)\,,\ S_{0'0} = -s(4)\,,\ S_{0'0'} = -s(1)\,,$$

$$S_{l^2,0} = \varepsilon_l\, s(l)\,,\ S_{0,l^2} = \varepsilon_l\, s'(l)\,,\ S_{2l^2,0} = \varepsilon'_l\, s(4l)\,,\ S_{0,2l^2} = \varepsilon'_l\, s'(4l)\,,$$

$$S_{l^2,0'} = \varepsilon_l\, s(2l)\,,\ S_{0',l^2} = -\varepsilon_l\, s'(4l)\,,\ S_{2l^2,0'} = \varepsilon'_l\, s(5l)\,,\ S_{0',2l^2} = -\varepsilon'_l\, s'(3l)\,,$$

$$S_{l^2,m^2} = \varepsilon_l\,\varepsilon_m\, t(lm)\,,\ S_{l^2,2m^2} = S_{2m^2,l^2} = \varepsilon_l\,\varepsilon'_m\, t'(lm)\,,\ S_{2l^2,2m^2} = \varepsilon'_l\,\varepsilon'_m\, t(2lm)\,,$$

where subscripts in S_{l^2,m^2} etc are taken mod 13, and l,m run over all numbers $1,2,\ldots,6$. The parameters l,m parametrise the quadratic residues and nonresidues mod 13, which behave slightly differently.

Curiously, the doubles of the even parts of both the Haagerup and Asaeda–Haagerup subfactors are likewise built from the principal series, for $p=13$ and $p=17$ respectively, specifically from one of the $(p+1)/2$-dimensional irreps in $\rho^{(p);\lambda}$ for the order-2 λ.

It would be interesting to investigate the possibility of fitting the modular data of the extended Haagerup into an infinite sequence. This would be somewhat analogous to doing it for the Haagerup. The latter was done in [EG11], but what made that possible was that there was already an infinite family to which the Haagerup hypothetically belonged [Izu01], and the first several subfactors in that sequence were already known to exist [EG11]. Doing this for the extended Haagerup would be a much greater challenge, but a very interesting one!

In particular, we learnt above that the $\mathrm{SL}(2,\mathbb{Z})$-representation for the extended Haagerup is isomorphic to $\rho_{13}\oplus\rho_5\oplus 1\oplus 1\oplus 1$, where ρ_5 is the Steinberg representation of $\mathrm{SL}(2,\mathbb{Z}_5)$ and ρ_{13} lies in the principal series of $\mathrm{SL}(2,\mathbb{Z}_{13})$. So we may look for modular data isomorphic to $\rho_p\oplus\rho_r\oplus n\,1$ for some $n\in\mathbb{Z}_{\geq 0}$ and some primes r,p, where again ρ_r is Steinberg and ρ_p lies in the principal series (and perhaps corresponds to the unique $\lambda\in\mathbb{Z}_p^\times$ of maximal odd order). In the expression for S in Theorem 9.1, there are six awkward submatrices, namely U,V,W,A,B,C. But thanks to Lemma 4.1, W resp. A,B,C can be read off from $\rho_r\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ resp. $\rho_p\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, and as we explained in this appendix those matrices are readily computed. Moreover unitarity of S forces $n=r-2$. So only the symmetric n-by-n matrix U and the n-by-$(p-1)$ matrix V need to be identified. They are directly obtained from $\rho_r\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ and $\rho_p\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ by the change-of-basis matrix we call Q and, as we see in Section 9, Q takes a very simple form for the extended Haagerup. We haven't pursued this any further.

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References

[AH99] Marta Asaeda and Uffe Haagerup. Exotic subfactors of finite depth with Jones indices $(5+\sqrt{13})/2$ and $(5+\sqrt{17})/2$. *Comm. Math. Phys.*, 202(1):1–63, 1999. arXiv:math.OA/9803044 MR1686551 DOI:10.1007/s002200050574.

[ASD71] A. O. L. Atkin and H. P. F. Swinnerton-Dyer. Modular forms on noncongruence subgroups. In *Combinatorics (Proc. Sympos. Pure Math., Vol. XIX, Univ. California, Los Angeles, Calif., 1968)*, pages 1–25. Amer. Math. Soc., Providence, R.I., 1971.

[Ban03] P. Bantay. The kernel of the modular representation and the Galois action in RCFT. *Comm. Math. Phys.*, 233(3):423–438, 2003. MR1962117 DOI:10.1007/s00220-002-0760-x arXiv:math/0102149.

[BG07] Peter Bantay and Terry Gannon. Vector-valued modular functions for the modular group and the hypergeometric equation. *Commun. Number Theory Phys.*, 1(4):651–680, 2007. MR2412268 DOI:10.4310/CNTP.2007.v1.n4.a2 arXiv:0705.2467.

[BMPS12] Stephen Bigelow, Scott Morrison, Emily Peters, and Noah Snyder. Constructing the extended Haagerup planar algebra. *Acta Math.*, 209(1):29–82, 2012. arXiv:0909.4099 MR2979509 DOI:10.1007/s11511-012-0081-7.

[BNRW13] Paul Bruillard, Siu-Hung Ng, Eric C. Rowell, and Zhenghan Wang. Rank-finiteness for modular categories. *J. Amer. Math. Soc.*, 2013. arXiv:1310.7050 DOI:10.1090/jams/842.

[BNRW15] Paul Bruillard, Siu-Hung Ng, Eric C. Rowell, and Zhenghan Wang. On classification of modular categories by rank, 2015. arXiv:1507.05139.

[CG99] Antoine. Coste and Terry Gannon. Congruence subgroups and rational conformal field theory, 1999. arXiv:math/9909080.

[CS99] J. H. Conway and N. J. A. Sloane. *Sphere Packings, Lattices and Groups.* Springer-Verlag, New York, 1999. MR1662447.

[DLN12] Chongying Dong, Xingjun Lin, and Siu-Hung Ng. Congruence property in conformal field theory, 2012. arXiv:1201.6644.

[EG11] David E. Evans and Terry Gannon. The exoticness and realisability of twisted Haagerup-Izumi modular data. *Comm. Math. Phys.*, 307(2):463–512, 2011. arXiv:1006.1326 MR2837122 DOI:10.1007/s00220-011-1329-3.

[EG14] David E. Evans and Terry Gannon. Near-group fusion categories and their doubles. *Adv. Math.*, 255:586–640, 2014. arXiv:1208.1500 MR3167494 DOI:10.1016/j.aim.2013.12.014.

[Gan12] Terry Gannon. Much ado about Mathieu, 2012. arXiv:1211.5531.

[Gan14] Terry Gannon. The theory of vector-valued modular forms for the modular group. In W Kohnen and R Weissauer, editors, *Conformal Field Theory, Automorphic Forms and Related Topics*, pages 247–286. Springer Verlag, 2014. arXiv:1310.4458.

[GIS15] Pinhas Grossman, Masaki Izumi, and Noah Snyder. The Asaeda-Haagerup fusion categories, 2015. arXiv:1501.07324.

[Haa94] Uffe Haagerup. Principal graphs of subfactors in the index range $4 < [M : N] < 3+\sqrt{2}$. In *Subfactors (Kyuzeso, 1993)*, pages 1–38. World Sci. Publ., River Edge, NJ, 1994. MR1317352.

[Hua05] Yi-Zhi Huang. Vertex operator algebras, the Verlinde conjecture, and modular tensor categories. *Proc. Natl. Acad. Sci. USA*, 102(15):5352–5356 (electronic), 2005. MR2140309 DOI:10.1073/pnas.0409901102 arXiv:math/0412261.

[Izu01] Masaki Izumi. The structure of sectors associated with Longo-Rehren inclusions. II. Examples. *Rev. Math. Phys.*, 13(5):603–674, 2001. MR1832764 DOI:10.1142/S0129055X01000818.

[Izu15] Masaki Izumi. A Cuntz algebra approach to the classification of near-group categories, 2015. arXiv:1512.04288.

[LL04] James Lepowsky and Haisheng Li. *Introduction to vertex operator algebras and their representations*, volume 227 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2004. MR2023933 DOI:10.1007/978-0-8176-8186-9.

[MS12] Scott Morrison and Noah Snyder. Non-cyclotomic fusion categories. *Trans. Amer. Math. Soc.*, 364(9):4713–4733, 2012. arXiv:1002.0168 MR2922607 DOI:10.1090/S0002-9947-2012-05498-5.

[MW14] Scott Morrison and Kevin Walker. The centre of the extended Haagerup subfactor has 22 simple objects, 2014. arXiv:1404.3955.

[NS10] Siu-Hung Ng and Peter Schauenburg. Congruence subgroups and generalized Frobenius-Schur indicators. *Comm. Math. Phys.*, 300(1):1–46, 2010. arXiv:0806.2493 DOI:10.1007/s00220-010-1096-6 MR2725181.

[Ost03] Viktor Ostrik. Fusion categories of rank 2. *Math. Res. Lett.*, 10(2-3):177–183, 2003. arXiv:math.QA/0203255 MR1981895.

[Ost13] Victor Ostrik. Pivotal fusion categories of rank 3, 2013. (with an Appendix written jointly with Dmitri Nikshych), arXiv:1309.4822.

[Zhu96] Yongchang Zhu. Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, 9(1):237–302, 1996. MR1317233 DOI:10.1090/S0894-0347-96-00182-8.