Draft version (July 10, 2010), read with caution.

1 Introduction

In this note we prove the following result

Theorem 1.1 *Every subfactor planar algebra $\mathcal{P}$ embeds in the graph planar algebra of its principal graph.*

This theorem has been known to Vaughan Jones since he first defined the graph planar algebra (indeed, he's said "if the theorem hadn't been true, I would have had to change the definition of the graph planar algebra") but a proof has not yet appeared in print. In fact, nearly simultaneously with the publication of this note, Dave Penneys and Vaughan Jones will publish an alternative proof [].

Here, however, it is most convenient to prove a more general theorem. First, we deal with all sorts of semisimple planar algebras, not just the subfactor planar algebras. This includes the case of fusion categories. The paper begins in §**??** with a definition of a G-planar algebra, for G a directed graph with an involution, that encompasses several previously used variants of the graph planar algebra formalism. Second, we work over an arbitrary field (the theorem above is implicitly over $\mathbb{C}$). If a planar algebra $\mathcal{P}$ over a field $\mathfrak{k}$ has a complete rational form (in the sense of [6]), then we can embed it in the graph planar algebra over that same field of its principal graph.

This theorem has already provided impetus for the construction of several subfactor planar algebras. Knowing that we could find the extended Haagerup subfactor planar algebra inside the appropriate graph planar algebra, we were motivated to use a computer to identify the subalgebra. This resulted in the construction, with Stephen Bigelow, Emily Peters and Noah Snyder that appeared in [2]. In fact, this construction was inspired by Emily Peters' thesis work [7] identifying the Haagerup subfactor planar algebra inside the appropriate graph planar algebra.

Soon, we expect obstructions to the existence of subfactor planar algebras on the basis of this theorem. Making certain assumptions about the principal graph we can show that there 'isn't room' inside the graph planar algebra for a nontrivial subfactor planar algebra. We know of one such obstruction, but more powerful ones may be around the corner.

The proof presented here was first discovered by realising that the Turaev-Viro state sum invariant associated to a subfactor planar algebra naturally gave a map into the graph planar algebra of its principal graph. The formal properties (specifically, the gluing formulas) for the Turaev-Viro invariant ensure that this map is a map of graph planar algebras. However, we then realised that it is possible to describe the resulting map directly, and in fact easier to check that it is a map of planar algebras straight from the definition. We include at the end §5 describing the connection between the map described here and the Turaev-Viro map.

TODO: acknowledgements

2 Graph planar algebras

Fix a field $\mathfrak{k}$.

We'll begin by defining a very general class of planar algebras, with an arbitrary set of strand labels and region labels. Each strand label has a pair of corresponding region labels. We encode all this data as a bidirected graph.

Definition 2.1 *A* bidirected graph *is a finite directed graph together with an involution (called duality) of the edges that reverses sources and targets.*

Sometimes we'll abbreviate a pair of dual edges by a single undirected edge. In the following, all of our graphs are bidirected graphs.

Definition 2.2 *Let G be a bidirected graph. A G-planar tangle consists of a disc, with some number of embedded inner discs, all with a marked point on their boundaries, along with strands in the complementary region which are either embedded circles or embedded intervals meeting the boundaries of the discs, with a labelling of the regions between the strands by vertices of G, and a labelling of the strands by a pair consisting of an edge of G between the vertices on either side and its reversal.*

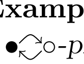
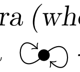
Given a G-planar tangle T we will write T_0 for the cycle in G appearing (always reading counterclockwise, from the marked point) around the outer boundary of T, and T_i for the cycle appearing around the boundary of the i-th inner disc of T.

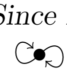
Definition 2.3 *Given a bidirected graph G, a G-*planar algebra *$\mathcal{P}$ consists of*

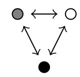
- *a $\mathfrak{k}$-vector space $\mathcal{P}_g$ for each loop g on G, and*
- *a multilinear map $\mathcal{P}(T) : \otimes \mathcal{P}_{T_i} \to \mathcal{P}_{T_0}$ for each G-planar tangle T,*

such that the maps only depend on the G-planar tangle up to isotopy, and the usual associativity constraints for planar algebras hold.

If all the vector spaces $\mathcal{P}_g$, for g a length zero loop (that is, just a vertex), are 1-dimensional, we say $\mathcal{P}$ is *evaluatable* (because every closed diagram can be 'evaluated' as a multiple of the empty diagram). These 1-dimensional vector spaces are canonically identified with $\mathfrak{k}$, sending the empty diagram to 1.

Example 1 *The most familiar example of a shaded planar algebra is in this language a*  *-planar algebra (where the involution switches the edges). An unshaded oriented planar algebra is a*  *-planar algebra (where the involution switches the edges). An unshaded unoriented planar algebra is a*  *-planar algebra (where the involution fixes the edge).*

Example 2 (Kuperberg's spider for $\mathfrak{su}_3$) *Since Kuperberg's spider for $\mathfrak{su}_3$ has a single oriented strand type, it is naturally a*  *-planar algebra. However, since the representation theory of $\mathfrak{su}_3$ is graded by $Z(SU(3)) = \mathbb{Z}/3\mathbb{Z}$, we can in fact*

3-color the regions, and think of it as a  *-planar algebra. This has an obvious generalisation to the representation theory of $\mathfrak{su}_n$, which is $\mathbb{Z}/n\mathbb{Z}$ graded. It is a C_n-planar algebra when only the standard representation is allowed as a strand label, or a K_n-planar algebra when all fundamental representations are allowed.*

Example 3 (Intermediate subfactors) *Given a diagram G of inclusions of II_1 subfactors, the intertwiners of tensor products of the corresponding bimodules are naturally a G-planar algebra.*

If the loop labelled by a directed edge e pointing from the outer region to the inner region is some multiple δ_e of the empty diagram we say $\mathcal{P}$ has e-modulus δ_e. In a shaded planar algebra the "spherical" axiom asks that the moduli are equal, while for a "lopsided" planar algebra we ask that one of the moduli is equal to 1.

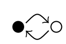
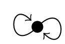
Given a bidirected graph homomorphism $\pi : \Gamma \to G$ we can take any Γ-planar algebra $\mathcal{P}$ and 'collapse along π' to form a G-planar algebra $\pi_* \mathcal{P}$. This is defined by

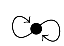
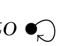
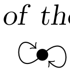
$$(\pi_* \mathcal{P})_g = \bigoplus_{\gamma \in \pi^{-1}(g)} \mathcal{P}^\Gamma_\gamma$$

and

$$(\pi_* \mathcal{P})(S) = \sum_{T \in \pi^{-1}(S)} \mathcal{P}^\Gamma(T).$$

Here we implicitly think of π as a map taking cycles in Γ to cycles in G, and also as a map sending Γ-planar tangles to G-planar tangles by applying π to all strand and region labels.

Example 4 *Collapsing along the obvious homomorphism from*  *to*  *takes a shaded planar algebra and produces an oriented planar algebra, in which all spaces are zero except those where the boundary labels alternate incoming and outgoing, and the action of oriented planar tangles is determined by first filling in the checkerboard shading, and then acting.*

Example 5 *Collapsing along the homomorphism from*  *to*  *takes an oriented planar algebra $\mathcal{P}$ and produces an unoriented planar algebra $\mathcal{Q}$, where the vector space $\mathcal{Q}_k$ is the direct sum of the vector spaces for $\mathcal{P}$ corresponding to the 2^k different loops of length k on*  *.*

TODO: This paragraph needs more explanation, and some thought as to whether it's nonsense or not. Fix a graph homomorphism $\pi : \Gamma \to G$ and a modulus $\delta_e \in \mathfrak{k}$ for each edge of G. Dimension data d for (π, δ) is an assignment $d(v) \in \mathfrak{k}$ for each vertex of Γ such that for each edge $a \to b$ of G we have

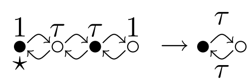
$$d(A)\delta_{a \to b} = \sum_{(A \to B) \in \pi^{-1}(e)} d(B)$$

for each vertex $A \in \pi^{-1}(a)$ of Γ, and

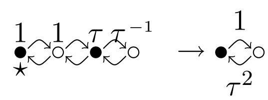
$$d(B)\delta_{a \to b} = \sum_{(A \to B) \in \pi^{-1}(e)} d(A)$$

for each vertex $B \in \pi^{-1}(b)$. If d is positive on every vertex then we say that d is *Perron-Frobenius dimension data* for π. If such d exists, it is uniquely determined up to a normalization on each connected component. If Γ has a base point in each connected component, we always choose the Perron-Frobenius dimension data with $d(\star) = 1$ for each base point $\star$.

Example 6

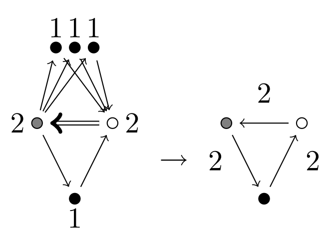


is Perron-Frobenius dimension data for the A_4 digraph with spherical modulus, while

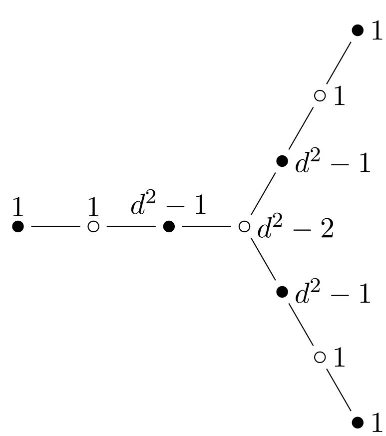


is Perron-Frobenius dimension data for lopsided modulus. In both cases, we've written the dimension data next to vertices on the initial graph, and the modulus of each edge on the final graph.

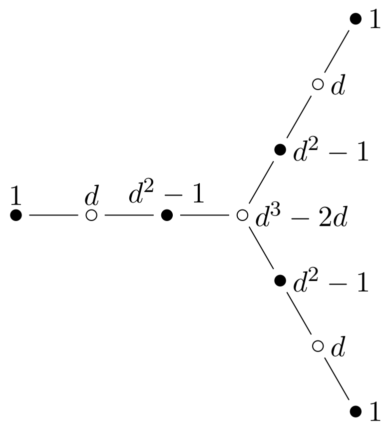
Example 7 *Perron-Frobenius dimension data for a particular graph homomorphism (see Example 13 below for motivation). Here each arrow has a corresponding dual arrow, which we've omitted for clarity. There are two parallel arrows from the white vertex to the gray vertex.*



Example 8 *Let $d = \sqrt{\frac{5+\sqrt{13}}{2}}$. Lopsided dimension data for the Haagerup graph with moduluses 1 and d^2 exists over the field $\mathbb{Q}(\sqrt{13}) = \mathbb{Q}(d^2)$,*



while for spherical dimension data with modulus d we need to extend the field to include d:



Definition 2.4 *The* trivial planar algebra $\mathcal{G}(\Gamma, d)$ *on a bidirected graph Γ with dimension data d is a Γ-planar algebra. The vector spaces $\mathcal{G}(\Gamma)_\gamma$ are all just $\mathfrak{k}$. For any planar tangle T, choose an isotopy representative so all strands attach only along the bottom halves of the discs, with each disc's starred region on the right. Since all our vector spaces are 1-dimensional, the multilinear map associated to a planar tangle is just a number, given by*

$$\mathcal{G}(\Gamma)(T) = \prod_{\substack{\text{critical} \\ \text{points } c}} d(c^+)^{\text{sign}(c)}.$$

Here c^+ is the vertex of Γ appearing above the critical point c.

One checks that this satisfies the axioms of a planar algebra in the usual way, see e.g. [4]. Note however that this definition is not the usual one, even in the "shaded" case: we've rearranged the critical point coefficients slightly. This rearrangement is essential, however, if we expect to work over an arbitrary field: the definition in [4] takes square roots of dimensions.

Definition 2.5 *The graph planar algebra $\mathcal{G}(\pi)$ for a homomorphism $\pi : \Gamma \to G$ of bidirected graphs with base points is the G-planar algebra obtained by collapsing the trivial planar algebra for Γ with Perron-Frobenius dimension data along π:*

$$\mathcal{G}(\pi) = \pi_* \mathcal{G}(\Gamma, d_{PF}).$$

Unravelling the definition, we have for g a cycle in G

$$\mathcal{G}(\pi)_g = \{\text{based directed cycles on } \Gamma \text{ over } g\}^*$$

and

$$\mathcal{G}(\pi)(T)\left(\bigotimes f_i\right) = \sum_{\substack{\Gamma\text{-labels} \\ \ell \text{ of } T}} \left(\prod_{\substack{\text{critical} \\ \text{points } c}} d(c^+(\ell))^{\text{sign}(c)} \right) \left(\prod_i f_i(\ell) \right).$$

Here a Γ-label of T is a lifting of the specified labelling of T (recall, regions by vertices of G, strands by edges by G) to a labelling by Γ, $d(c^+(\ell))$ is the Perron-Frobenius dimension of the vertex of Γ appearing above the critical point c in the labelling ℓ, and $f_i(\ell)$ is the functional f_i evaluated on the cycle in Γ given by reading the labelling ℓ around the boundary of the i-th inner disc. We've chosen to write the vector spaces as functionals on directed cycles, rather than formal linear combinations, for notational convenience below.

3 Pivotal 2-categories and planar algebras

[not exactly sure but sprinkle semisimple and degenerate below]

In this section we recall the now familiar translation between pivotal categories and unshaded planar algebras, and give the generalized translation between pivotal 2-categories and G-planar algebras as described above. This section serves a double purpose: the embedding map will be easier to describe if we allow ourselves to move freely between the two descriptions, and we want to make clear that the embedding theorem is relevant for all (pseudo-unitary) pivotal 2-categories.

We assume all our categories are linear, so all morphism spaces are finite dimensional $\mathfrak{k}$-vector spaces, and moreover that they are additive, so all direct sums of objects exist. (If a linear category is not additive, we can always embed it in the matrix category which has direct sums.) When discussing 2-categories, linear means that the 2-morphism spaces are vector spaces, and that direct sums of 1-morphisms exist.

All our 2-categories are strictly associative. We always write compositions of 1-morphisms using $\otimes$, so tensor categories are simply 2-categories with a single 0-morphism. The axioms for pivotal categories from [1] generalize immediately to give a definition of pivotal 2-categories (we interpret pivotal categories as 2-categories, then relax the requirement that there is only one 0-morphism). Throughout, we assume all our pivotal categories are strictly pivotal (that is, the double dual functor is the identity, not just naturally isomorphic to the identity).

Fix a pivotal 2-category $\mathcal{C}$, and suppose $\mathcal{X}$ is a finite set of dominant simple 1-morphisms (so every simple 1-morphism is a summand of some tensor product of elements of $\mathcal{X}$) which is closed under duals. We can consider $\mathcal{X}$ as the edges of a bidirected graph. Associated to $\mathcal{C}$ and $\mathcal{X}$ is a $\mathcal{X}$-planar algebra.

Definition 3.1 *The $\mathcal{X}$-planar algebra $\mathcal{P}(\mathcal{C}, \mathcal{X})$ has vector spaces*

$$\mathcal{P}(\mathcal{C}, \mathcal{X})_k = \mathrm{Hom}_{\mathcal{C}}(k, \mathbf{1}).$$

Here we interpret a directed cycle k on $\mathcal{X}$ as the corresponding tensor product of 1-morphisms in $\mathcal{C}$. The multilinear maps for tangles are given by choosing an isotopy representative in which all strands attach only along the bottom halves of the discs, then interpreting the diagram as an element of the 2-category by replacing critical points in strands with the evaluation and coevaluation maps in the usual manner.

Going in the other direction, given a $\mathcal{X}$-planar algebra $\mathcal{P}$ we can define a pivotal 2-category $\mathcal{C}(\mathcal{P})$. We make an intermediate definition first.

Definition 3.2 *The pivotal 2-category $\check{\mathcal{C}}(\mathcal{P})$ has*

objects *vertices of $\mathcal{X}$,*

1-morphisms $\mathrm{Hom}_{\check{\mathcal{C}}(\mathcal{P})}(a,b) = \{paths\ on\ \mathcal{X}\ from\ a\ to\ b\}$,

2-morphisms $\mathrm{Hom}_{\check{\mathcal{C}}(\mathcal{P})}(\lambda,\mu) = \mathcal{P}_{\bar{\mu}\lambda}$, *where* $\bar{\mu}$ *denotes the involute of* μ, *and* $\bar{\mu}\lambda$ *is the concatenation of* $\bar{\mu}$ *and* λ

where composition, tensor products and duality are all given by the obvious corresponding $\mathcal{X}$*-planar tangles.*

Definition 3.3 *The pivotal* 2*-category* $\mathcal{C}(\mathcal{P})$ *is the idempotent completion of* $\check{\mathcal{C}}(\mathcal{P})$*.*

If $P \in \mathcal{P}_{\bar{\lambda}\lambda}$ and $Q \in \mathcal{P}_{\bar{\mu}\mu}$ are idempotents (with respect to the multiplication tangles) in a $\mathcal{X}$-planar algebra $\mathcal{P}$, with λ and μ paths in $\mathcal{X}$ with the same start and end points, we have the notion of the Hom-space between them, the subspace of $\mathcal{P}_{\bar{\lambda}\mu}$ given by

$$\mathrm{Hom}_{\mathcal{P}}(P,Q) = \left\{ f \in \mathcal{P}_{\bar{\lambda}\mu} \mid fP = f = Qf \right\}.$$

(In fact P and Q are 1-morphisms in the associated 2-category $\mathcal{C}(\mathcal{P})$, and this just translates the Hom-space of the idempotent completion back to the planar algebraic description.)

A [semisimple] 2-category $\mathcal{C}$ is pseudo-unitary if the Perron-Frobenius dimensions of objects and the quantum dimensions coincide. Note that if $\mathcal{C}$ is unitary then it is automatically semisimple, non-degenerate and pseudo-unitary.

Example 9 ($\mathrm{Rep}\,\mathfrak{g}$) *Fix a complex semisimple Lie algebra* $\mathfrak{g}$*. As* $\otimes$*-categories,* $\mathrm{Rep}\,\mathfrak{g}$*,* $\mathrm{Rep}\,U\mathfrak{g}$ *and* $\mathrm{Rep}\,U_q\mathfrak{q}$ *are all equivalent. In each case, the irreducible representations with highest weight a fundamental weight* $\otimes$*-generate the category. Let* χ *be the graph with a single vertex and an edge for each fundamental representation of* $\mathfrak{g}$ *(and the representation theoretic duals). From* $\mathrm{Rep}\,\mathfrak{g}$ *we can construct a* χ*-planar algebra, which we call* $\mathrm{FundRep}\,\mathfrak{g}$*.*

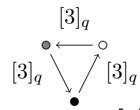
Example 10 ($\mathrm{Rep}\,\mathfrak{g}$ graded by $Z(G)$) *Since the* $\otimes$*-product in* $\mathrm{Rep}\,\mathfrak{g}$ *is graded by the group* $Z(G)$*, the centre of the corresponding simply connected compact Lie group (equivalently, the group of weight vectors modulo root vectors), we can think of* $\mathrm{Rep}\,\mathfrak{g}$ *as a* 2*-category with objects* $Z(G)$*. (Each* 1*-morphism space is just a copy of* $\mathrm{Rep}\,\mathfrak{g}$*, and composition of* 1*-morphisms is* $\otimes$*-product.) Now, we can take the graph* χ_Z *with vertices* $Z(G)$*, and an edge from* a *to* b *for each fundamental weight* λ *such that* $a + \lambda = b$ *in* $Z(G)$*. Again, we can construct a* χ_Z*-planar algebra, which we call* $\mathrm{FundRep}^Z\,\mathfrak{g}$*.*

Definition 3.4 *A quantum subgroup of* $\mathfrak{g}$ *is a* χ*-planar algebra* $\mathcal{P}$ *in the sense of Example 9, along with a map* $\mathrm{FundRep}\,U_q\mathfrak{g} \to \mathcal{P}$*.*

Definition 3.5 *A graded quantum subgroup of* $\mathfrak{g}$ *is a* χ_Z*-planar algebra* $\mathcal{P}$ *in the sense of Example 10, along with a map* $\mathrm{FundRep}^Z\,U_q\mathfrak{g} \to \mathcal{P}$*.*

Remark. Collapsing along the graph homomorphism $\chi_Z \to \chi$ allows us to forget the grading on a quantum subgroup of $\mathfrak{g}$.

Example 11 *A graded quantum subgroup of $\mathfrak{su}_2$ is just a spherical shaded planar algebra with modulus $[2]_q$. If $[2]_q < 2$, then there is an ADE classification. A quantum subgroup is an oriented unshaded planar algebra with modulus $[2]_q$, and if $[2]_q < 2$ there is an ADET classification. See [5] for details.*

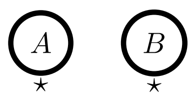
Example 12 *A graded quantum subgroup of $\mathfrak{su}_3$ is a* 

-planar algebra which contains (a quotient of) Kuperberg's $U_q\mathfrak{su}_3$ spider. See [3] for the related notion of an A_2-planar algebra.

4 The embedding map

[Not sure where these go:]

Lemma 4.1



$$= \frac{1}{d_e}$$

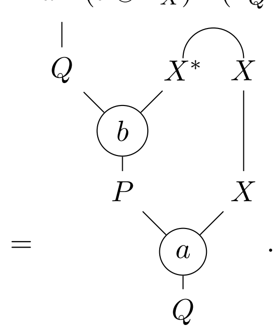
Proof The morphisms A and B are each endomorphisms of the simple object e, and so are just $\mathfrak{k}$ multiples of the identity. □

Given idempotents ${}_AP_B$, ${}_AQ_C$ and ${}_BX_C$ in a G-planar algebra $\mathcal{P}$ (here the subscripts A, B and C denote vertices of G, indicating the left and right shadings of the idempotents; equivalently, interpreting the idempotents as 1-morphisms in the corresponding 2-category, they are the sources and targets) there is a pairing

$$(-,-)_{P,Q,X} : \mathrm{Hom}_{\mathcal{P}}(Q, P\otimes X) \otimes \mathrm{Hom}_{\mathcal{P}}(P, Q\otimes X^*) \to \mathrm{End}(Q)$$

given by

$$(a,b) = a \circ (b \otimes \mathbf{1}_X) \circ (\mathbf{1}_Q \otimes \mathrm{coev}_X)$$



If Q is simple, $\mathrm{End}(Q)$ is 1-dimensional, canonically identified with $\mathfrak{k}$. In this case the pairing $(-,-)_{P,Q,X}$ is non-degenerate as long as $\mathcal{P}$ itself is non-degenerate.

Definition 4.2 *The* principal graph $\Gamma(\mathcal{P})$ *of a nondegenerate G-planar algebra is a bidirected graph fibered over G, which has*

vertices *a minimal idempotent in* $\mathcal{P}$ *from each isomorphism class,*

edges *from* P *to* Q *over* $X \in G$ *corresponding to a basis of* $\mathrm{Hom}_{\mathcal{C}}(Q, P \otimes X)$ *and*

involution *sending a basis element of* $\mathrm{Hom}_{\mathcal{C}}(Q, P \otimes X)$ *to the dual* $\mathfrak{k}$*-basis element with respect to the pairing* $(-,-)_{P,Q,X}$ *in* $\mathrm{Hom}_{\mathcal{P}}(P, Q \otimes X^*)$.

TODO: explain why it's always possible to choose bases so the involution works like this

We're explicitly choosing bases for Hom-spaces in this definition of the principal graph; a more usual definition would just have finite sets of edges with the same cardinality. We will make use of these extra choices in our construction of the embedding map. In the case that Γ is simply-laced (that is $\mathrm{Hom}_{\mathcal{C}}(Q, P \otimes X)$ is always 1-dimensional) choosing a basis is just a choice of normalization: note that these choices nevertheless affect the embedding map below.

The principal graph splits into components according to the region label at the marked point of the idempotents. Each of these components is connected (if Q is an idempotent in $\mathcal{P}_{\bar{\lambda}\circ\lambda}$ then there is a path from the tensor identity to Q in the principal graph that descends to λ), and moreover has a distinguished vertex given by the tensor identity, so $\Gamma(\mathcal{P})$ comes with Perron-Frobenius data.

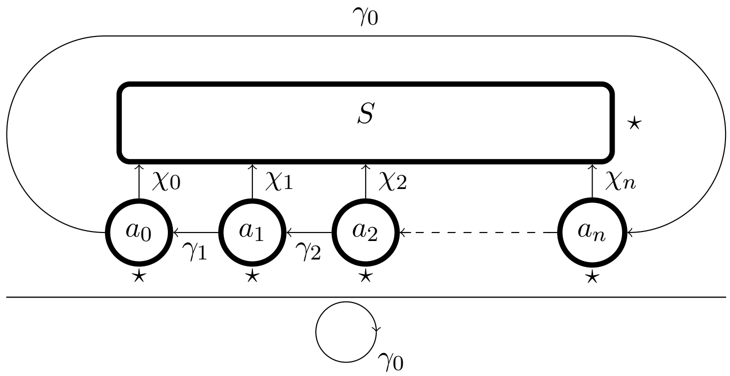
Example 13 *The representation theory of* $U_q\mathfrak{su}_3$ *at* q *a* 12*th root of unity is modular with the usual pivotal structure, but not modular with the unimodal pivotal structure. In particular, there are three invertible objects with twist factor* 1*, those with highest weights* 00*,* 30 *and* 03*. The modular quotient (equivalently, the Müger deequivariantization by the subcategory of invertible objects) is a graded quantum subgroup of* $\mathfrak{su}_3$ *in the sense of Example 12. The graph of Example 7 is the principal graph.*

Example 14 *The modular quotient of unimodal* $U_q\mathfrak{su}_3$ *at* q *a* 14*th root of unity can be realised as a quantum subgroup of* $\mathfrak{su}_3$*, but not as a graded quantum subgroup. This is because the subcategory of invertible objects is not concentrated in the trivial component of the* $Z(G)$*-grading. Its principal graph is* ***TODO:*** *.*

The full classification of quantum subgroups of $\mathfrak{su}_3$ *with* q *a root unity has been announced by Ocneanu.*

Consider now a loop γ in $\Gamma(\mathcal{P})$ for $\mathcal{P}$ a G-planar algebra, with image χ in G under $\Gamma(\mathcal{P}) \to G$. This gives a sequence of idempotents and maps between them in $\mathcal{P}$. We'll write γ_i for the i-th vertex of γ, which is a minimal idempotent in $\mathcal{P}$. Write χ_i for the edge of χ from the vertex i to the vertex $i+1$ in G (recall the edges of G are the strand labels for $\mathcal{P}$). (Notice that γ_i is a vertex of $\Gamma(\mathcal{P})$ while χ_i is an edge of G.) For notational compactness in diagrams, we'll often introduce a second symbol to refer to the edges of γ. Thus when we say 'let γ be a loop with edges a_i' we mean that a_i is the edge between the vertex i and vertex $i+1$ in γ, which is a basis element for $\mathrm{Hom}_{\mathcal{P}}(P_{i+1}, P_i \otimes \chi_i)$.

Definition 4.3 *The graph planar algebra embedding map* $\mathcal{E} : \mathcal{P} \hookrightarrow \mathcal{G}(\Gamma(\mathcal{P}))$ *for a non-degenerate evaluatable* G*-planar algebra* $\mathcal{P}$ *is given by*

$\mathcal{E}(S)(\gamma) =$ 

,

where $S \in \mathcal{P}_\chi$*, with* χ *a loop in* G *of length* $n+1$ *and* γ *is a loop in* $\Gamma(\mathcal{P})$ *which descends to* χ *with edges* a_i*. The loop labelled by* γ_0 *appearing in the denominator is the quantum dimension of the idempotent* γ_0*.*

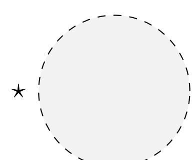
Remark. At this point, we only require that $\mathcal{P}$ be non-degenerate so that $\dim_q \gamma_0 \neq 0$. We'll need non-degeneracy for Theorem 4.4 and Theorem 4.10, however.

Theorem 4.4 *The map* $\mathcal{E}$ *is a map of planar algebras.*

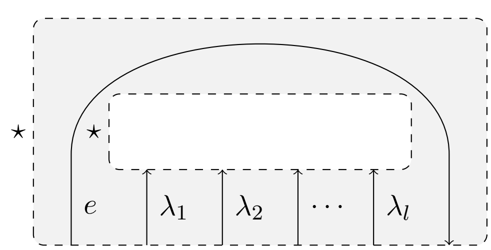
Proof We first observe that every planar tangle can be written as composition of a collection of the four elementary tangles described below. Lemmas 4.6 through 4.9 then establish that $\mathcal{E}$ commutes with each of these elementary tangles.

Lemma 4.5 *Every* G*-planar tangle can be expressed as a composition of tangles from the following four classes:*

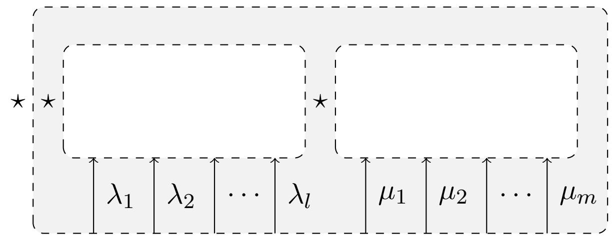
emptiness: *For any vertex* g *of* G*,*

$\emptyset_g =$  $: \mathfrak{k} \to \mathcal{P}_g$

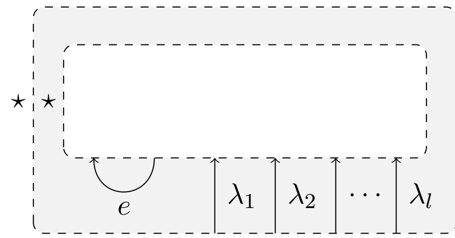
rainbow: *For any edge* e *of* G *and loop* λ *in* G *with basepoint at the end of* e *(i.e. such that* $e\lambda\bar{e}$ *is a loop),*

$R_{e,\lambda} =$  $: \mathcal{P}_\lambda \to \mathcal{P}_{e\lambda\bar{e}}$

tensor product: *For any two loops λ and μ in G with the same basepoint (i.e. such that $\lambda\mu$ is a loop),*

$\otimes_{\lambda,\mu} =$  $: \mathcal{P}_\lambda \otimes \mathcal{P}_\mu \to \mathcal{P}_{\lambda\mu}$

cap: *For any edge e of G and loop λ in G with basepoint the start of e (i.e. such that $e\bar{e}\lambda$ is a loop),*

$C_{e,\lambda} =$  $: \mathcal{P}_{e\bar{e}\lambda} \to \mathcal{P}_\lambda$

Proof Rotation tangles can be built out of rainbows and caps. Conjugating the given cap tangle by a rotation allows a cap at an arbitrary pair of boundary points. Conjugating the rainbow tangle by a rotation allows inserting a cup between any pair of boundary points. Caps and cups together generate the category of annular planar tangles. Applying caps at the boundary of the tensor product tangle produces arbitrary quadratic compositions, and compositions of these and annular planar tangles produce all planar tangles. □

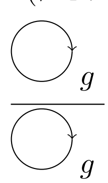
Lemma 4.6 *The map $\mathcal{E}$ intertwines the emptiness tangle.*

Proof For any vertex $g \in G$

$$\begin{aligned}\emptyset(\mathcal{E}(1 \in \mathfrak{k}))(g) &= \emptyset(1 \in \mathfrak{k})(g) \\ &= 1_G(g) \\ &= 1\end{aligned}$$

while

$$\mathcal{E}(\emptyset(1))(g) = \mathcal{E}(\emptyset \in \mathcal{P}_\emptyset)(g)$$

$=$ 

$$= 1$$

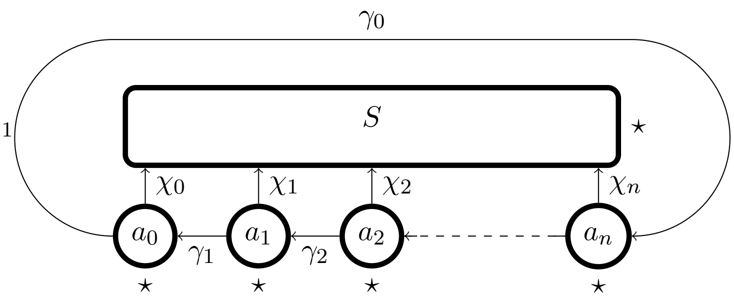
and so $\emptyset \circ \mathcal{E} = \mathcal{E} \circ \emptyset$ on $\mathfrak{k} = \mathcal{P}_\emptyset$. □

Lemma 4.7 *The map $\mathcal{E}$ intertwines the rainbow tangles.*

Proof Consider $S \in \mathcal{P}_\chi$ and γ a loop in Γ with edges a_i descending to the loop χ in G, and e an edge in Γ from a vertex η to γ_0 which descends to the edge ε in G. We have

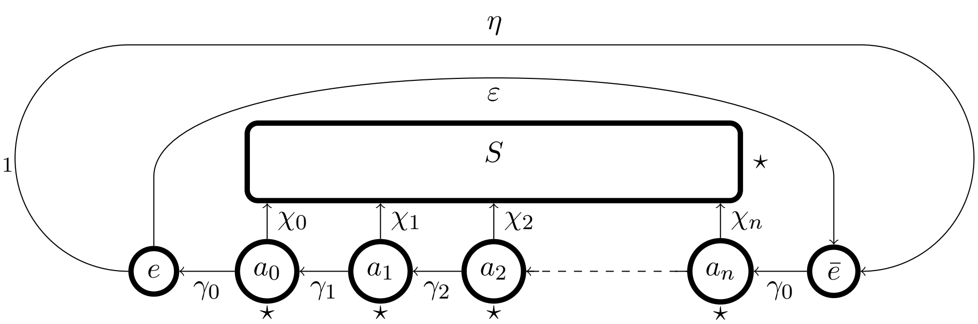
$R(\mathcal{E}(S))(e\gamma\bar{e}) = d(\eta)^{-1}\mathcal{E}(S)(\gamma)$

$= d(\eta)^{-1} d(\gamma_0)^{-1}$



while

$\mathcal{E}(R(S))(e\gamma\bar{e}) = d(\eta)^{-1}$



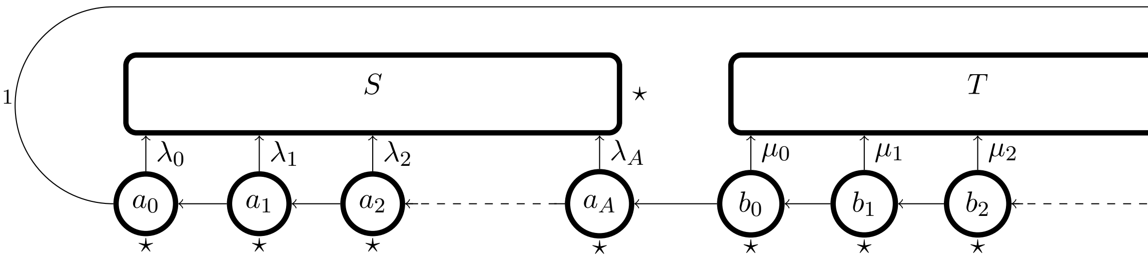
TODO: and ... bubble bursting!

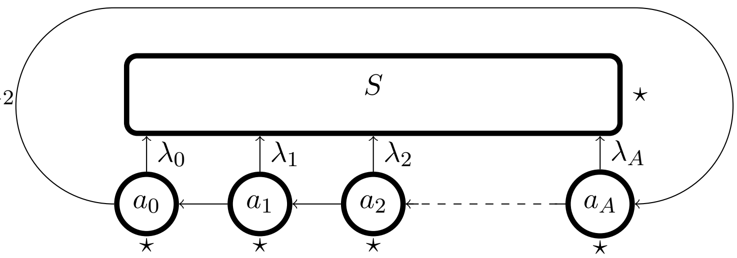
Both $\mathcal{E}(R(S))(g)$ and $R(\mathcal{E}(S))(g)$ are zero for any other g not of the form $e\gamma\bar{e}$, so we conclude that $\mathcal{E} \circ R = R \circ \mathcal{E}$ on $\mathcal{P}_\chi$. $\square$

Lemma 4.8 *The map $\mathcal{E}$ intertwines the tensor tangles.*

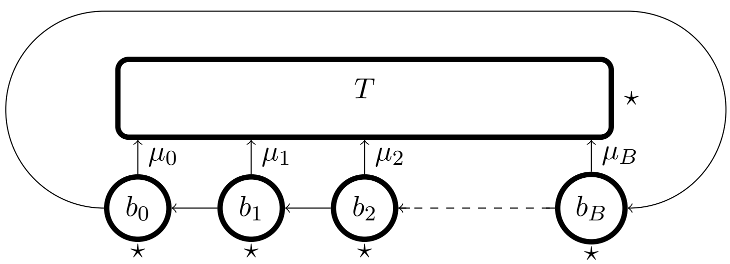
Proof Take $S \in \mathcal{P}_\lambda$, $T \in \mathcal{P}_\mu$ and α a loop with edges a_i descending to λ and β a loop with edges b_i descending to μ, and suppose $\alpha_0 = \beta_0$. We calculate [put coefficients in!]

$\mathcal{E}(S \otimes T)(\alpha\beta) =$

$d(\alpha_0)^{-1}$ 

$= d(\alpha_0)^{-2}$ 

$\times$

$\times$ 

$= \mathcal{E}(S)(\alpha)\mathcal{E}(T)(\beta)$

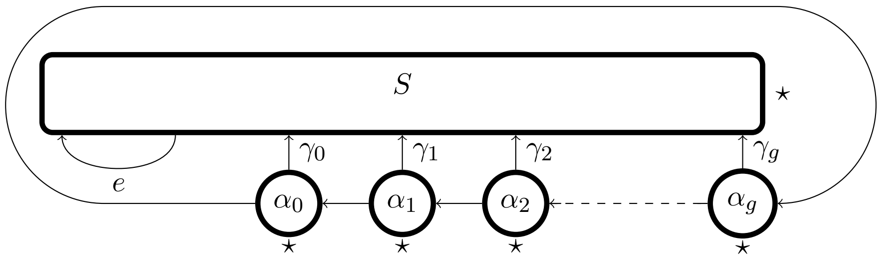
$= \otimes(\mathcal{E}(S), \mathcal{E}(T))(\alpha \circ \beta)$

TODO: Explain the use of Schur's lemma here! □

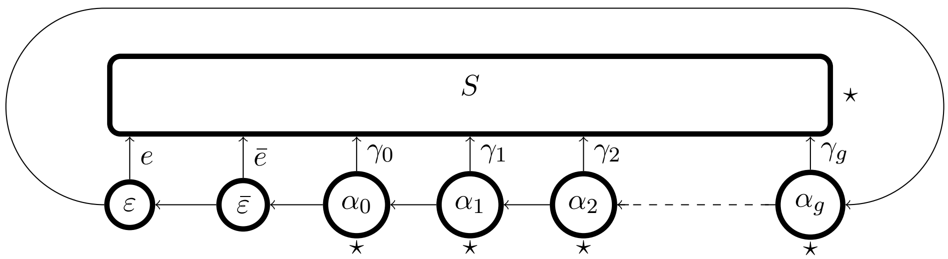
Lemma 4.9 *The map $\mathcal{E}$ intertwines the cap tangles.*

Proof Consider $S \in \mathcal{P}_{e\bar{e}\lambda}$ and γ a cycle in Γ descending to λ, and check that

$\mathcal{E}(C(S))(\gamma) =$



$= \sum_{\varepsilon \mapsto e}$



$$= \sum_{\varepsilon \mapsto e} ??\mathcal{E}(S)(\varepsilon\bar{\varepsilon}\gamma)$$

$$= C(\mathcal{E}(S))(\gamma)$$

□

TODO:

- show the calculation for each case, respectively trivial, schur's lemma, schur's lemma, I=H for two parallel strands

□

Theorem 4.10 *The graph planar algebra embedding map is injective.*

Proof By the hypothesis of nondegeneracy, $\mathcal{P}$ has no non-trivial planar ideals. (See for example [2, Proposition 3.5].) Since $\mathcal{E}$ takes the empty diagram to something manifestly non-zero (the constant 1 function on length 0 cycles), the kernel must be trivial. □

TODO: Optional section:

5 Turaev-Viro state sums

Given a unitary pivotal category, we can construct a TQFT invariant of k-manifolds, for $k = 0, 1, 2, 3$ via the usual construction. The original Turaev-Viro construction of this invariant is via a state sum formula, and a suitable generalization of this formula reproduces the result above (at least in the case that our category $\mathcal{C}$ is unitary).

Given a unitary pivotal 2-category $\mathcal{C}$, we can associate a vector space $A(\Sigma)$ to each closed 2-manifold Σ and a number $A(M)$ to each closed 3-manifold M. Moreover, there are relative versions of these invariants. A 3-manifold M with boundary Σ gives a vector in $A(\Sigma)$. A 2-manifold Σ with boundary ...

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This paper is available online at arXiv:?????, and at http://tqft.net/GPA.