Abstract  This paper gives two applications of Jones’s quadratic tangles tech-
niques to the classification of subfactors with index below 5. In particular, we
eliminate two of the five families of possible principal graphs called “weeds” in
the classification from [13]. The two families we eliminate here each have principal
graph pairs whose first branch point is a triple point, and which continue assymetrically past that.

AMS Classification  46L37; 18D10

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1 Introduction

Jones’ index theorem for subfactors [10] states that the index of a subfactor lies in the
range \{4 \cos^2(\frac{\pi}{n})|n = 3, 4, \ldots \} \cup \{4, \infty \}. All of these values are realized; however, once
you ignore subfactors whose principal graph is \(A_\infty\), the possible indices just above 4
for irreducible subfactors are again quantized. Haagerup started the classification
of subfactors with index ‘only a little larger’ than four in [9]. In that paper, he
showed there were no extremal subfactors (other than \(A_\infty\)) with index in the range
\((4, \frac{5+\sqrt{13}}{2})\). Furthermore, he gave a complete list of possible principal graphs of
extremal subfactors whose index falls in the range \((4, 3 + \sqrt{3})\). (He claims the result up to 3 + \sqrt{3},
but only proves it up to 3 + \sqrt{2}.) Most of the graphs on this list were excluded by Bisch [4] and Asaeda-Yasuda [2], while the remaining 3
graphs were shown to come from (unique) subfactors by Asaeda-Haagerup [1] and
Bigelow-Morrison-Peters-Snyder [3].

Haagerup’s classification stops at index 3 + \sqrt{3} for reasons of computational conven-
ience, and because an exotic subfactor, the Goodman-de la Harpe-Jones subfactor [7], was already known to exist at that depth. However, modern technology makes it
possible to extend the classification of small-index subfactors further.

This paper is the second in a series of papers which attempt to classify subfactors
of index less than 5. In the first paper [13] we gave an initial classification result
analogous to Haagerup’s initial classification. In that paper, we use the term
translation of a principal graph pair to indicate the graph pair obtained by increasing
the supertransitivity by an even integer (the supertransitivity is the number of edges
between the initial vertex and the first vertex of degree more than two). An extension
of a principal graph pair is a graph pair obtained by extending the graphs in any way at greater depths (i.e. adding vertices and edges at the right), even infinitely.

The main result of that paper was the following.

**Theorem 1.1**  The principal graph of any subfactor of index between 4 and 5 is a translate of one of an explicit finite list of graph pairs (which we call the *vines*), or is a translated extension of one of the following graph pairs (which we call the *weeds*).

\[
C = \left( \begin{array}{c}
\text{graph 1} \\
\text{graph 2}
\end{array} \right),
\]

\[
F = \left( \begin{array}{c}
\text{graph 3} \\
\text{graph 4}
\end{array} \right),
\]

\[
B = \left( \begin{array}{c}
\text{graph 5} \\
\text{graph 6}
\end{array} \right),
\]

\[
Q = \left( \begin{array}{c}
\text{graph 7} \\
\text{graph 8}
\end{array} \right),
\]

\[
Q' = \left( \begin{array}{c}
\text{graph 9} \\
\text{graph 10}
\end{array} \right).
\]

The main result of this paper is to show that two of the above weeds do not appear as principal graphs of subfactors.

**Theorem 1.2**  There are no subfactors with principal graphs a translated extension of the pair

\[
C = \left( \begin{array}{c}
\text{graph 1} \\
\text{graph 2}
\end{array} \right).
\]

**Theorem 1.3**  There are no subfactors with principal graphs a translated extension of the pair

\[
F = \left( \begin{array}{c}
\text{graph 3} \\
\text{graph 4}
\end{array} \right).
\]

These are proved by applying a result of Jones' [12] which uses “quadratic tangles” planar algebra techniques.

**Remark.** The techniques of this paper fail to rule out

\[
B = \left( \begin{array}{c}
\text{graph 5} \\
\text{graph 6}
\end{array} \right)
\]

because of the symmetry of its principal graph. See Example 1.8 for a description of why.

**Remark.** The approach of this paper is also viable for ruling out a large subset of the vines described in the first paper [13]. However, it requires a certain amount of work for each vine, along the lines of the calculations we do here. Happily, there is a uniform approach, which works for all vines, based on [5]. A later paper in this series [14] will use that technique to reduce the vines to a finite set of graphs.

The structure of this paper is as follows. Section 1.1 recalls some key results from [11, 12] about the structure of annular Temperley-Lieb modules and about two equations involving the 'chirality' (a certain rotational eigenvalue) and the 'branch factors' (a certain ratio of two dimensions) of principal graphs. The main observation
in this paper is that since the chirality of a principal graph is always a root of unity, these equations produce an inequality which can be used to eliminate certain weeds. Section 1.2 describes what we can deduce about the dimensions of bimodules from incomplete graph information for potential principal graphs and calculates these dimensions for graphs coming from the weeds $C$ and $F$. In Section 2 we argue that the inequality of Section 1.1 cannot be satisfied for graphs coming from the weeds $C$ and $F$, except for a few exceptions. In Section 3 we eliminate the exceptions, which are all graphs with index below $3 + \sqrt{3}$, by running the odometer described in [13].

Bundled with this tex file are two Mathematica notebooks, Crab.nb and fsm.nb, which contain all relevant calculations for what follows. These make use of a package called FusionAtlas; see [13] for a tutorial on its use.

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1.1 Background on quadratic tangles and annular multiplicities

The main technique of this paper is to apply the formula for the rotational eigenvalue of the annular low-weight vector for $n$-supertransitive subfactors with annular multiplicities *10 given by Jones in [12]. We rapidly recall the language from [11, 12] to make sense of this statement and put it in context.

A subfactor is called $n$-supertransitive if up to the $n$-box space its planar algebra is just Temperley-Lieb. Equivalently, a subfactor is $n$-supertransitive if and only if the principal graph up to depth $n$ is $A_{n+1}$.

Any planar algebra is a module for the annular Temperley-Lieb algebra, and as such decomposes into irreducible modules. The theory of annular Temperley-Lieb modules is laid out in Graham-Lehrer [8] (and in Jones [11], where the idea to apply annular Temperley-Lieb theory to planar algebras appears). Each such module is cyclic, generated by a ‘lowest weight vector’ (that is, a submodule of a planar algebra $P$ is a direct sum of subspaces of $P_k$ closed under action by annular Temperley-Lieb tangles; the weight of a vector in $P_k$ is $k$, and for $n$ the lowest weight appearing in a submodule, the subspace of $P_n$ is one dimensional). Each such lowest weight vector with nonzero weight $n$ has a rotational eigenvalue which is an $n$-th root of unity. (Lowest weight vectors with weight 0 have instead a ‘ring eigenvalue’.)

The annular multiplicities of a planar algebra are the sequence of multiplicities of lowest weight vectors. A theorem of Jones [11] shows that the annular multiplicities are actually determined entirely by the principal graph. Thus we can discuss the annular multiplicities of a principal graph regardless of whether it comes from a subfactor.

The 0th annular multiplicity of a subfactor planar algebra is always 1, corresponding to the empty diagram which generates Temperley-Lieb as an annular Temperley-Lieb
module. If the planar algebra is \( n \)-supertransitive (i.e. the principal graph begins with \( n \) edges before the first branch point or multiple edge), then the next \( n \) annular multiplicities are 0, because the vector spaces \( P_1 \) through \( P_n \) are each no larger than their Temperley-Lieb subalgebra. An \( n \)-supertransitive subfactor of annular multiplicities \( *10 \) means that the first two annular multiplicities after the long string of \( n \) zeroes are 1 and 0.

If an \( n \)-supertransitive principal graph has \( n \)-th annular multiplicity 1, then it begins like \( D_{n+3} \) (i.e. it starts with a ‘triple point’). We define the branch factor, usually written \( r \) (and \( \tilde{r} \) for the branch factor of the dual principal graph), to be the ratio of the dimensions of the two vertices immediately past the branch point (where we take the larger divided by the smaller). If the next annular multiplicity is 0, there are exactly two possibilities:

\[
\begin{array}{c}
\text{and} \\
\end{array}
\]

Consider now a principal graph pair with annular multiplicities \( *10 \), and supertransitivity \( m - 1 \). Haagerup proved in [9], using Ocneanu’s triple point obstruction, that the supertransitivity must be odd, and the principal and dual principal graphs must be different. For convenience, we’ll always order the principal graph pair so the principal graph starts like the first graph above, and the dual principal graph starts like the second graph above.

An improved version of the triple point obstruction was given by Jones in [12] where he also gives the following formulas for \( r \), \( \tilde{r} \) and \( \lambda \), the rotational eigenvalue of the unique weight \( m \) lowest weight vector.

\[
\begin{align*}
    r + 1 & = \frac{\lambda + \lambda^{-1} + 2}{[m][m + 2]} + 2 \\
    \tilde{r} & = \frac{[m + 2]}{[m]} 
\end{align*}
\]  

The formula for \( \tilde{r} \) follows from working out dimensions in the dual principal graph (see Example 1.5), but the formula for \( r \) takes significantly more work.

Since \( \lambda \) must be an \( m \)-th root of unity, we have the following inequalities which do not involve \( \lambda \):

\[
-4 \leq \left( r + \frac{1}{r} - 2 \right) [m][m + 2] - 4 \leq 0. \tag{1.3}
\]

1.2 Relative dimensions of vertices/bimodules

Consider \( \Gamma_0 \) a weed with annular multiplicities \( *10 \). Now consider \( \Gamma \) an \( n \)-translate of an extension of \( \Gamma_0 \). Suppose that this graph comes from a subfactor of index \( (q + q^{-1})^2 \). In order to apply the quadratic tangents inequality from the last section, we need to write \( r \) as a function of \( n \) and \( q \). In general this is impossible, but if we’re lucky and know a lot about the principal graphs, we may determine the dimensions
of the vertices of each graph as functions of \( n, q \) using the following three sets of equations. First, we set the dimension of the leftmost vertex of each graph to be

\[
[n + 1] = \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}}.
\]

Second, if two vertices correspond to bimodules which are dual to each other, they must have the same dimension. Third, for each vertex \( V \) which only connects to vertices which appear in the known segment of our graphs, we have an equation

\[
\dim(V) = [2] \sum \text{dim}(W)
\]

where \([2] = q + q^{-1}\).

**Remark 1.4** We do not assume that \( \Gamma \) is finite depth nor that it is amenable. Thus the dimensions mentioned above need not be the Frobenius-Perron eigenvector for \( \Gamma \), and the index of the subfactor with principal graph \( \Gamma \) need not be the square of the Frobenius-Perron eigenvalue. Nonetheless, we note that the index of any subfactor with principal graph \( \Gamma \) must be greater than or equal to \( \|\Gamma_0\|^2 \) (the square of the Frobenius-Perron eigenvalue).

**Example 1.5** We label the vertices of the graphs

\[
(\xrightarrow{\text{}}), \quad (\xrightarrow{\text{}}), \quad (\xrightarrow{\text{}})
\]

by \( V^i_{j,k} \) where \( i \) is either \( p \) or \( d \) corresponding to the principal or dual principal graph, \( j = 0, 1, \ldots, 5 \) is the depth, and \( k \) is the index of the vertex at that depth counting from the bottom to the top.

We cannot solve explicitly for the dimensions in terms of \( n, q \) for these graphs. Rather, there is a one parameter family of solutions. We set \( \alpha = \dim(V^p_{4,2}) \), and the dimensions of the vertices as functions of \( n, q, \alpha \) are given by:

\[
\begin{align*}
\dim(V^p_{0,1}) &= \frac{q^{-n} (q^{2n+2} - 1)}{q^2 - 1} & \dim(V^p_{1,1}) &= \frac{q^{-n-1} (q^{2n+4} - 1)}{q^2 - 1} \\
\dim(V^d_{2,1}) &= \frac{q^{-n-2} (q^{2n+6} - 1)}{q^2 - 1} & \dim(V^d_{3,1}) &= \frac{q^{-n-3} (q^{2n+8} - 1)}{q^2 - 1} \\
\dim(V^d_{4,1}) &= \frac{q^{-n-4} (\alpha q^{n+4} - \alpha q^n + q^{2n+10} - 1)}{q^2 - 1} & \dim(V^d_{4,2}) &= \alpha \\
\dim(V^d_{5,1}) &= \frac{q^{-n-5} (\alpha q^{n+4} - \alpha q^n + q^{2n+12} - 1)}{q^2 - 1} \\
\dim(V^d_{5,2}) &= \frac{q^{-n-3} (\alpha q^{n+2} - \alpha q^n + q^{2n+8} - 1)}{q^2 - 1}
\end{align*}
\]

**Example 1.6** We can solve for the dimensions in \( n, q \) for the graphs

\[
(\xrightarrow{\text{}}), \quad (\xrightarrow{\text{}}), \quad (\xrightarrow{\text{}})
\]

(which is an extension of the previous example) because we have one additional equation: \([2]V^d_{5,1} = V^d_{4,1} \). The dimensions of the vertices through depth 5, as functions of \( n, q, \alpha \), are given by:

\[
\begin{align*}
\dim(V^p_{0,1}) &= \frac{q^{-n} (q^{2n+2} - 1)}{q^2 - 1} & \dim(V^p_{1,1}) &= \frac{q^{-n-1} (q^{2n+4} - 1)}{q^2 - 1} \\
\end{align*}
\]
Thus, the branch factor for this principal graph as a function of \( n \) and \( q \) is

\[
r(n, q) = \frac{(q^4 + q^2 + 1)(q^{2n+12} - 1)}{q^2(q^{2n}(2q^{12} + 2q^{10} + q^8) - q^4 - 2q^2 - 2)}.
\]

**Example 1.7** We can solve for the dimensions just past the branch points for the graphs

\[
\begin{align*}
\dim(V^p_{2,1}) &= \dim(V^d_{2,1}) = \frac{q^{-n-2}(q^{2n+6} - 1)}{q^2 - 1}, \\
\dim(V^p_{3,1}) &= \dim(V^d_{3,1}) = \frac{q^{-n-3}(q^{2n+8} - 1)}{q^2 - 1}, \\
\dim(V^p_{4,1}) &= \frac{q^{-n-2}(q^{2n+12} - 1)}{q^4 - 1}, \\
\dim(V^p_{5,1}) &= \frac{q^{-n-3}(q^{2n+12} - 1)}{(q^2 - 1)(q^2 + 1)^2}, \\
\dim(V^p_{6,2}) &= \frac{q^{-n-5}(q^{2n}(q^{16} - q^{12} - q^{10}) + q^6 + q^4 - 1)}{(q^2 - 1)(q^2 + 1)^2}.
\end{align*}
\]

Thus, the branch factor for this principal graph as a function of \( n \) and \( q \) is

\[
r(n, q) = \frac{q^{-4n}(-1 - q^2(1 + q^2)(2 + q^2)(1 + q^4) + q^{2(5+n)}(1 + 3q^2 + 3q^4 + 3q^6 + 2q^8 + q^{10}))}{(1 + q^2)^3(-23q^2 + 3q^4 + 2q^6)}
\]

**Example 1.8** For the graphs

\[
B = \begin{pmatrix}
\text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\
\text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---}
\end{pmatrix},
\]

the branch factor \( r(n, q) \) is equal to one. This is because in the principal graph, at depth six, we have a duality between two identical vertices on two identical branches, which implies that the dimensions are the same on both branches.

Therefore, **Inequality 1.3** (and indeed Equation 1.1 with \( \lambda = -1 \)) always holds for translations and extensions of these graphs, and the techniques of this paper cannot eliminate this weed.

## 2 Index inequalities

**Proposition 2.1** Any subfactor with principal graphs a translated extension of the pair

\[
\begin{pmatrix}
\text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\
\text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---}
\end{pmatrix},
\]

must have index at most \( 3 + \sqrt{3} \).
Proof. Suppose a subfactor exists with principal graphs an extension of the pair translated by \( n \in 2\mathbb{Z}_{\geq 0} \), and let \((q + q^{-1})^2\) be the index. Plugging the branch factor

\[
r(n, q) = \frac{(q^4 + q^2 + 1)(q^{2n+12} - 1)}{q^2(2q^{12} + 2q^{10} + q^8) - q^4 - 2q^2 - 2}
\]
calculated in Example 1.6 into Inequality (1.3) (with \( m = n + 4 \)), we get the following inequality:

\[
q^{-2n-10} \left( q^{n+5} - 1 \right)^2 \left( q^{n+5} + 1 \right)^2 \times (q^{n+10} - q^{n+8} - q^{n+6} - q^{n+4} - q^6 - q^4 - q^2 + 1) \times (q^{n+10} - q^{n+8} - q^{n+6} - q^{n+4} + q^6 + q^4 + q^2 - 1) \times (q - 1)^{-2}(q + 1)^{-2} (q^2 - q + 1)^{-1} (q^2 + q + 1)^{-1} \times (q^{2n+8} + 2q^{2n+10} + 2q^{2n+12} - q^4 - 2q^2 - 2)^{-1} \leq 0.
\]

All but the two longest factors in the numerator above (namely the factors on the second and third lines) are positive for all \( q > 1 \). By Remark 1.4, after computing the graph norm, we see that any translated extension of the pair must satisfy \( q > 1.4533 \), so \( q^{10} - q^8 - q^6 - q^4 > 0 \), and

\[
q^n (q^{10} - q^8 - q^6 - q^4) + q^6 + q^4 + q^2 - 1 \geq 0.
\]

We conclude that Inequality (1.3) is satisfied if and only if

\[
q^n (q^{10} - q^8 - q^6 - q^4) - q^6 - q^4 - q^2 + 1 \leq 0. \tag{2.1}
\]

Note that the left hand side only increases as \( n \) increases, so we examine the case \( n = 0 \). The largest root of

\[
q^{10} - q^8 - 2q^6 - 2q^4 - q^2 + 1
\]
is the positive \( q \) such that \((q + q^{-1})^2 = 3 + \sqrt{3}\). Hence the index must be less than or equal to \( 3 + \sqrt{3} \). \( \square \)

Remark. At this point, we could appeal to Haagerup’s classification to index \( 3 + \sqrt{3} \) to completely rule out all of these graphs. Since the published proof of his classification only covered the range up to index \( 3 + \sqrt{2} \), for the sake of completeness we eliminate these graphs in \( \mathcal{S} \).

Proposition 2.2. Any subfactor with principal graphs a translated extension of the pair

\[
(\text{\includegraphics[width=1.2in]{image1.png}})
\]

must either

(1) have principal graphs translated by 0 and have rotational eigenvalue \( \lambda \) and index \((q + q^{-1})^2\) where \( \lambda \) and \( q \) are either:

<table>
<thead>
<tr>
<th>( q )</th>
<th>minimal polynomial for ( q )</th>
<th>( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0106...</td>
<td>( x^{12} - 3x^{10} - 3x^8 - 4x^6 - 3x^4 - 3x^2 + 1 )</td>
<td>1</td>
</tr>
<tr>
<td>1.8449...</td>
<td>( x^{36} + x^{34} + 2x^{32} - 17x^{30} - 46x^{28} - 91x^{26} - 144x^{24} - 197x^{22} - 233x^{20} - 246x^{18} - 233x^{16} - 91x^{14} - 144x^{12} - 91x^{10} - 46x^8 - 17x^6 - 2x^4 + x^2 + 1 )</td>
<td>( \pm i )</td>
</tr>
</tbody>
</table>
(2) have principal graphs translated by 2 and have rotational eigenvalue \( \lambda \) and index 
\((q + q^{-1})^2\) where \( \lambda \) and \( q \) are either:

<table>
<thead>
<tr>
<th>( q )</th>
<th>minimal polynomial for ( q )</th>
<th>( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.6341...</td>
<td>( x^{10} - x^{14} - 2x^{12} - 5x^{10} - 2x^8 - 5x^6 - 2x^4 - x^2 + 1 )</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>
| 1.6069... | \( x^{52} - x^{48} - 4x^{46} - 6x^{44} - 19x^{42} - 38x^{40} - 67x^{38} 
- 98x^{36} - 139x^{34} - 178x^{32} - 218x^{30} - 238x^{28} 
- 246x^{26} - 238x^{24} - 218x^{22} - 178x^{20} - 139x^{18} 
- 98x^{16} - 67x^{14} - 38x^{12} - 19x^{10} - 6x^8 - 4x^6 - x^4 + 1 \) | \( \exp(\pm \pi i/3) \) |

Proof First note that the \( q \) from any translated extension of this pair must be at 
least 1.5932 by Remark 1.4. Proceeding as in Proposition 2.1, the branch factor as 
a function of \( n \) and \( q \) is given by

\[
r(n, q) = \frac{q^{2n} (q^{20} + 3q^{18} + 2q^{16} + 2q^{14} + 2q^{12} + q^{10}) - q^{10} - 2q^8 - 2q^6 - 2q^4 - 3q^2 - 1}{q^{2n} (q^{20} + 2q^{18} + 3q^{16} + 3q^{14} + 3q^{12} + q^{10}) - q^{10} - 3q^8 - 3q^6 - 3q^4 - 2q^2 - 1}.
\]

Plugging in \( r(n, q) \) to Equation (1.3) we get the following inequality:

\[
q^{-2n-4} (q^{n+5} - 1)^2 (q^{n+5} + 1)^2 (q - 1)^{-2} (q + 1)^{-2} \times 
\left( q^{2n} (q^{16} - q^{14} - q^{12} - q^{10}) + q^n (-2q^{14} - 3q^{12} + 3q^4 + 2q^2) + q^6 + q^4 + q^2 - 1 \right) \times 
\left( q^{2n} (q^{16} - q^{14} - q^{12} - q^{10}) + q^n (2q^{14} + 3q^{12} - 3q^4 - 2q^2) + q^6 + q^4 + q^2 - 1 \right) \times 
\left( q^{2n} (q^{20} + 2q^{18} + 3q^{16} + 3q^{14} + 3q^{12} + q^{10}) - q^{10} - 3q^8 - 3q^6 - 3q^4 - 2q^2 - 1 \right)^{-1} \times 
\left( q^{2n} (q^{30} + 3q^{18} + 2q^{16} + 2q^{14} + 2q^{12} + q^{10}) - q^{10} - 2q^8 - 2q^6 - 2q^4 - 3q^2 - 1 \right)^{-1} \leq 0.
\]

By similar analysis as above, this inequality is satisfied if and only if

\[
q^{2n} (q^{16} - q^{14} - q^{12} - q^{10}) + q^n (-2q^{14} - 3q^{12} + 3q^4 + 2q^2) + q^6 + q^4 + q^2 - 1 \leq 0.
\]

Let \( p(n, q) \) denote the left hand side. If \( n \geq 4 \) and \( q > 1 \), then

\[
p(n, q) \geq q^{2n} (q^{16} - q^{14} - q^{12} - q^{10}) + q^n (-2q^{14} - 3q^{12}) \geq q^{2n} (-2q^{10} - 3q^8 + q^{16} - q^{14} - q^{12} - q^{10}) = q^{2n+8} (q^8 - q^6 - q^4 - 3q^2 - 3).
\]

The largest root of

\[
q^8 - q^6 - q^4 - 3q^2 - 3
\]

is less than 1.5082 < 1.5932, so there can be no subfactors with an \( n \)-translated 
extension of this pair of principal graphs for \( n \geq 4 \).

Now suppose we have a subfactor with principal graphs an extension of this pair of 
principal graphs. Then \( \lambda \in \{\pm 1, \pm i\} \) and \( \lambda + \lambda^{-1} \in \{-2, 0, 2\} \). Solving Equation \( 1.1 \) for \( q \) when \( \lambda = -1 \) shows that \( q \) must be approximately 1.3123..., with 
minimal polynomial \( x^8 - x^6 - x^4 - x^2 + 1 \). This \( q \) is smaller than 1.5932 so we can ignore 
this case. Solving Equation \( 1.1 \) for \( q \) when \( \lambda \in \{1, \pm i\} \) gives the first table in the 
statement.

Finally, suppose we have a subfactor with principal graphs a 2-translated extension 
of this pair of principal graphs. Then \( \lambda \in \{\pm 1, \exp(\pm 2\pi i/3), \exp(\pm \pi i/3)\} \) and
\[ \lambda + \lambda^{-1} \in \{-2, -1, 1, 2\}. \] Solving Equation (1.1) for \( q \) when \( \lambda \in \{-1, \exp(\pm 2\pi i/3)\} \) gives the cases

\[
\begin{array}{c|c|c}
q & \text{minimal polynomial for } q & \lambda \\
1.3453... & x^{16} - x^{14} - 2x^{10} - 2x^6 - x^2 + 1 & -1 \\
1.5203... & x^{52} - x^{48} - 4x^{46} - 4x^{44} - 9x^{42} - 14x^{40} - 21x^{38} \\
& -24x^{36} - 29x^{34} - 36x^{32} - 42x^{30} - 44x^{28} - 42x^{26} \\
& -44x^{24} - 42x^{22} - 36x^{20} - 29x^{18} - 24x^{16} - 21x^{14} \\
& -14x^{12} - 9x^{10} - 4x^8 - 4x^6 - x^4 + 1 & \exp(\pm 2\pi i/3)
\end{array}
\]

which we ignore as \( q \) is too small. Solving Equation (1.1) for \( q \) when \( \lambda \in \{1, \exp(\pm \pi i/3)\} \) gives the second table above. \( \square \)

**Proof of Theorem 1.3** For each the four allowed values of \( q \) in Proposition 2.2, the index of the possible subfactor, \((q + q^{-1})^2\), is not cyclotomic. By [1] this excludes the possibility of a subfactor.

We can explicitly check this by calculating the discriminant of each index, then finding a prime \( p \) which does not divide the discriminant, such that the minimal polynomial of the index does not have uniform degree irreducible factors mod \( p \). We exhibit the appropriate data in the following table.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \lambda )</th>
<th>( p )</th>
<th>degrees of factors mod ( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>5</td>
<td>1, 2, 3</td>
</tr>
<tr>
<td>0</td>
<td>( \pm i )</td>
<td>7</td>
<td>1, 1, 2, 2, 4, 8</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2, 2, 4</td>
</tr>
<tr>
<td>2</td>
<td>( \exp(\pm \pi i/3) )</td>
<td>5</td>
<td>1, 2, 3, 3, 6, 8</td>
</tr>
</tbody>
</table>

\( \square \)

3 Running the odometer

**Proposition 3.1** Any subfactor with principal graphs a translated extension of the pair

\[
\begin{array}{c}
\end{array}
\]

with index less than \( 3 + \sqrt{3} \) is in fact a translate of one of the following graphs

1. \[
\begin{array}{c}
\end{array}
\]

2. \[
\begin{array}{c}
\end{array}
\]

3. \[
\begin{array}{c}
\end{array}
\]

**Proof** We run the odometer, as in [3], and find that it terminates after two steps. The four weeds considered are shown in Figure 1. Only the weed labelled 2 satisfies the associativity test, giving case 2 above. We next consider all the graphs obtained by extending one graph of a weed, staying below index \( 3 + \sqrt{3} \) and satisfying the associativity test. The weeds at depth +0 and depth +2 each produce exactly one such graph, giving cases 1 and 3 above. \( \square \)
Figure 1: Running the odometer for Proposition 3.1

**Proposition 3.2** There are no subfactors with principal graphs a translation of the following pairs:

1. \[ (\quad, \quad) \]
2. \[ (\quad, \quad) \]
3. \[ (\quad, \quad) \]

**Proof** Recall from above that for a subfactor with principal graphs a translation by \( n \) of one of the above pairs and index \((q + q^{-1})^2\), we must have that \( n, q \) satisfy Inequality 2.1 (which we recall for the reader's convenience):

\[
q^n \left(q^{10} - q^8 - q^6 - q^4\right) - q^6 - q^4 - q^2 + 1 \leq 0.
\]

For all three cases, \( q > 1.4817 \) by Remark 1.4, so once again

\[
q^{10} - q^8 - q^6 - q^4 > 0,
\]

and the left hand side of Inequality 2.1 only increases as \( n \) increases. Setting \( n = 2 \), we have that the largest root of

\[
q^{12} - q^{10} - q^8 - 2q^6 - q^4 - q^2 + 1
\]

is smaller than \( 1.45 < 1.4817 \), so this expression is always positive. Thus there cannot be subfactors with principal graphs a translation by \( n \geq 2 \) of any of the above pairs.

Finally, to check that these three possibilities cannot occur as principal graphs with translation \( n = 0 \), we note that for each case, the dimension of the lower vertex at depth 4 is not an algebraic integer. The appropriate information is contained in the table below:

<table>
<thead>
<tr>
<th>graph</th>
<th>minimal polynomial of dimension of vertex</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 5x^3 - 16x^2 - 15x + 1 )</td>
</tr>
<tr>
<td>2</td>
<td>( 3x^5 - 19x^4 + 25x^3 + 18x^2 - 25x - 13 )</td>
</tr>
<tr>
<td>3</td>
<td>( 2x^2 - 6x - 9 )</td>
</tr>
</tbody>
</table>
Proof of Theorem 1.2 The result is now an immediate consequence of Propositions 2.1, 3.1 and 3.2.

References


[13] Scott Morrison, Noah Snyder, Towards the classification of subfactor planar algebras with index at most 5, draft available at tft.net/index5-part1


This paper is available online at arXiv:????? and at http://tqft.net/index5-part2