ABSTRACT

We reconsider the $\mathfrak{su}_3$ link homology theory defined by Khovanov in [10] and generalized by Mackaay and Vaz in [16]. With some slight modifications, we describe the theory as a map from the planar algebra of tangles to a planar algebra of complexes of 'cobordisms with seams' (actually, a 'canopolis'), making it local in the sense of Bar-Natan's local $\mathfrak{su}_2$ theory of [1].

We show that this 'seamed cobordism canopolis' decategorifies to give precisely what you'd both hope for and expect: Kuperberg's $\mathfrak{su}_3$ spider defined in [15]. We conjecture an answer to an even more interesting question about the decategorification of the Karoubi envelope of our cobordism theory.

Finally, we describe how the theory is actually completely computable, and give a detailed calculation of the $\mathfrak{su}_3$ homology of the $(2, n)$ torus knots.

Keywords: Categorification, Cobordism, Spider, Jones Polynomial, Khovanov Homology, Quantum Knot Invariants.

Mathematics Subject Classification 2000: 57M25, 57M27, 57Q45

ON KHOVANOV'S COBORDISM THEORY FOR $\mathfrak{su}_3$ KNOT HOMOLOGY.

SCOTT MORRISON and ARI NIEH

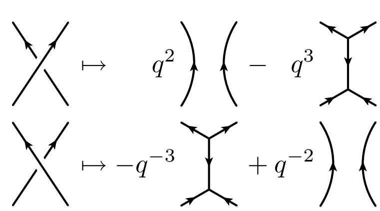
Department of Mathematics
University of California, Berkeley
Berkeley CA 94720
USA

1. Introduction

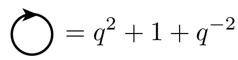
Bar-Natan formulated a highly geometric version of Khovanov homology in [1]. His approach uses the language of planar algebras for the construction of the complex. In particular, it has the pleasant feature of being a local theory, which makes it useful for fast 'divide and conquer' computations [2].

Khovanov constructed a homology theory of links that categorifies the $\mathfrak{su}_3$ quantum knot invariant in [10]. Mackaay and Vaz generalized this theory in [16]. In the spirit of Bar-Natan, we provide a local perspective on this knot homology. Our formulation uses a planar algebra of categories (a 'canopolis') as the setting for the complex.

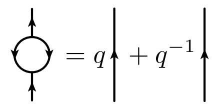
The $\mathfrak{su}_3$ quantum knot invariant is determined by the following formulas, which should be thought of as defining a map of planar algebras:



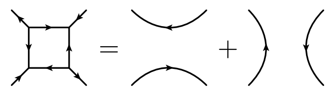
This sends an oriented link diagram to a $\mathbb{Z}[q, q^{-1}]$-linear combination of oriented planar graphs with trivalent vertices ('webs'). We then evaluate these webs using the relations of Kuperberg's $\mathfrak{su}_3$ spider[a] [15]



(1.1)



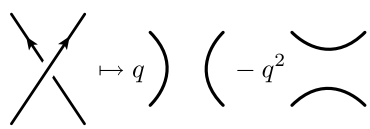
(1.2)



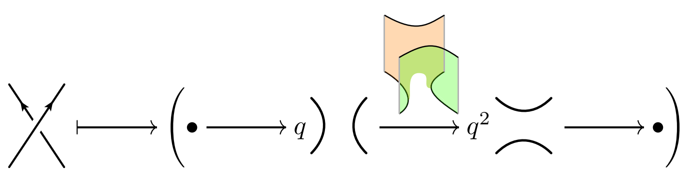
(1.3)

to obtain a polynomial invariant of links.

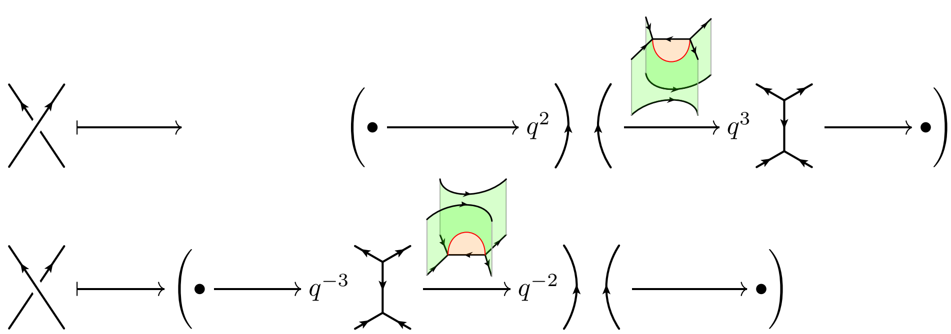
Just as the categorified version of (one variation of) the Kauffman skein relation for the Jones polynomial[b]



becomes the following complex in Khovanov's theory,



we should expect the categorified $\mathfrak{su}_3$ invariant to associate to a crossing some two step complex, with something like a cobordism for the differential. However, since the diagrams in the $\mathfrak{su}_3$ spider have singularities, the category of cobordisms can't suffice; therefore, we'll work with seamed cobordisms (or 'foams') that allow singular seams where three half-planes meet:



[a]Note that these aren't precisely Kuperberg's relations. Following [10], we've replaced q with $-q$. We do this in order to produce relations with positive coefficients, which are thus more readily categorifiable.
[b]This isn't quite the quantum $\mathfrak{su}_2$ skein theory; see [4].

We'll describe this construction in detail, essentially paralleling the work of Khovanov and of Mackaay and Vaz, with some minor differences which we find appealing.[c] For most of the paper, it isn't necessary to have read their work (although §Appendix A which explicitly compares the details of our construction with that of Khovanov and of Mackaay and Vaz assumes this). We emphasize the local nature of our construction, giving automatic proofs of Reidemeister invariance, following Bar-Natan's simplification algorithm, in §4.2. Later, in §6.1, we provide explicit detailed calculations of the $\mathfrak{su}_3$ Khovanov invariant for the $(2, n)$ torus knots.

Our version of this invariant associates to every tangle an up-to-homotopy complex in the canopolis of foams. In §5, we prove 'decategorification' results both for this canopolis and for Bar-Natan's canopolis of cobordisms corresponding to the original Khovanov homology. Roughly speaking, this involves collapsing the categorical structure of the canopolis (taking the split Grothendieck group) while preserving its planar algebra structure. The decategorification of Bar-Natan's canopolis is the Temperley-Lieb planar algebra. Similarly, the decategorification of the canopolis of foams is the Kuperberg's $\mathfrak{su}_3$ spider. As we will see, the $\mathfrak{su}_3$ case requires more complicated techniques, because the morphisms are much harder to classify than the cobordisms in the $\mathfrak{su}_2$ canopolis. Among these techniques is a kind of duality: in §5.4.2 we'll produce isomorphisms $\mathrm{Hom}\,(U \otimes V, W) \cong \mathrm{Hom}\,(U, W \otimes V^*)$ in the canopolis of $\mathfrak{su}_3$ foams, which we think of as meaning that it's secretly a 'spatial algebra' (i.e. a higher dimensional analogue of a planar algebra), not just a canopolis.

Some interesting things happen in the $\mathfrak{su}_3$ theory which have no analogues for $\mathfrak{su}_2$. In particular, there are grading 0 morphisms other than the identity between irreducible diagrams. We'll discuss an example in which the identity morphism can be written as a sum of orthogonal idempotents, and make a conjecture about the decategorification of the Karoubi envelope. (The Karoubi envelope is the category we get by adding in all idempotents as extra objects.) A further conjecture says that the minimal idempotents correspond to the dual canonical basis in the $\mathfrak{su}_3$ spider [11].

The authors would like to thank Rahel Wachs for teaching us how to draw the figures in this paper, and our referee for making many useful suggestions.

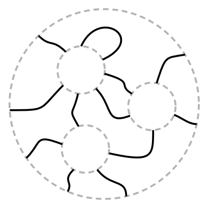
2. Preliminaries

2.1. *Locality, or, "What is a planar algebra?"*

A planar algebra is a gadget specifying how to combine objects in planar ways. They were introduced in [9] to study subfactors, and have since found more general use.

[c] Much of our work was done before the appearance of [16], which perhaps partially excuses our giving a self-contained development of the theory.

In the simplest version, a planar algebra $\mathcal{P}$ associates a vector space $\mathcal{P}_k$ to each natural number k (thought of as a disc in the plane with k points on its boundary) and a linear map $\mathcal{P}(T) : \mathcal{P}_{k_1} \otimes \mathcal{P}_{k_2} \otimes \cdots \otimes \mathcal{P}_{k_r} \to \mathcal{P}_{k_0}$ to each 'spaghetti and meatballs' diagram T, for example

,

with internal discs with $k_1, k_2, \ldots, k_r$ points, and k_0 points on the external disc. These maps (the 'planar operations') must satisfy certain properties: radial spaghetti induce identity maps, and composition of the maps $\mathcal{P}(T)$ is compatible with the obvious composition of spaghetti and meatballs diagrams by gluing one inside the other.

For the exact details, which are somewhat technical, see [9].

Planar algebras also come in more subtle flavors. Firstly, we can introduce a label set, and associate a vector space to each disc with boundary points colored by this label set. (The simplest version discussed above thus has a singleton label set, and the discs are indexed by the number of boundary points.) The planar tangles must now have arcs colored using the color set, and the rules for composition of diagrams require that labels match up. We can also have a oriented label set; the label set has an involution and the arcs carry both an orientation and a label, modulo reversing both. Secondly, we needn't restrict ourselves to vector spaces and linear maps between them; a planar algebra can be defined over an arbitrary monoidal category, associating objects to discs, and morphisms to planar tangles. Thus we might say "$\mathcal{P}$ is a planar algebra over the category $\mathcal{C}$ with label set $\mathfrak{L}$." [d]

A 'canopolis', introduced by Bar-Natan in [1][ef], is simply a planar algebra defined over some category of categories, with monoidal structure given by cartesian product. Thus to each disc, we associate some category of a specified type. A planar tangle then induces a functor from the product of internal disc categories to the outer disc category, thus taking a tuple of internal disc objects to an external disc object, and a tuple of internal disc morphisms to an external disc morphism. It is picturesque to think of the objects living on discs, and the morphisms in cans, whose bottom and top surfaces correspond to the source and target objects. Composition of morphisms is achieved by stacking cans vertically, and the planar operations put cans side by side.

[d] A subfactor planar algebra is defined over Vect, and has a 2 element label set. One imposes an additional condition that only discs with an even number of boundary points and with alternating labels have non-trivial vector spaces attached. There is also a positivity condition. See [2, §4].
[e] He called it a 'canopoly', instead, but we're taking the liberty of fixing the name here.
[f] See also [20] for a description of Khovanov-Rozansky homology [12,13] using canopolises.

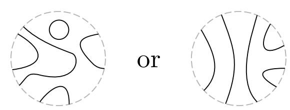
The functoriality of the planar algebra operations ensure that we can build a 'city of cans' (hence the name canopolis) any way we like, obtaining the same result: either constructing several towers of cans by composing morphisms, then combining them horizontally, or constructing each layer by combining the levels of all the towers using the planar operations, and then stacking the levels vertically.

2.2. *The $\mathfrak{su}_2$ cobordism theory*

We will now briefly recall the canopolis defined by Bar-Natan in [1], and used in his local link homology theory.

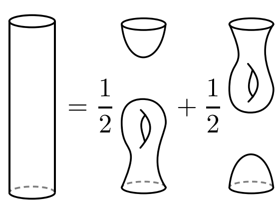
Slightly modifying Bar-Natan's notation, $\mathcal{C}ob\,(\mathfrak{su}_2)$ is our name for his $\mathcal{C}ob^3_{/l}$, the canopolis of cobordisms in cans modulo the $\mathfrak{su}_2$ relations.

The objects of $\mathcal{C}ob\,(\mathfrak{su}_2)$ consist of planar tangle diagrams:



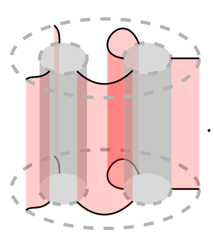
equipped with the obvious planar algebra structure[g].

Let R_0 be any commutative ring in which 2 is invertible. If D_1 and D_2 are diagrams with identical boundary, a morphism between them is a formal R_0-linear combination of cobordisms from D_1 to D_2 modulo the following local relations:



(2.1)

The planar algebra structure on morphisms is given by plugging cans into $T \times [0, 1]$, where T is a spaghetti and meatballs diagram, as in this example:



[g] We may think of this as the free planar algebra with no generators.

We refine the theory by introducing a grading on the canopolis. We equip the objects of $\mathcal{C}ob\,(\mathfrak{su}_2)$ with a formal grading shift, so that they are of the form $q^m D$, where m is an integer[h]. (We will, however, sometimes suppress the grading for simplicity, or conflate diagrams with objects when it is convenient.) We let grading shifts add under planar algebra operations. The degree of a cobordism C from $q^{m_1} D_1$ to $q^{m_2} D_2$ is defined as $\chi(C) - k/2 + m_2 - m_1$, where χ is the Euler characteristic and k is the number of boundary points of D_i. It is not hard to see that degrees are additive under both composition and planar operations.[i] Note also that the local relations are degree-homogeneous, and therefore this grading descends to the quotient.

We can further introduce formal direct sums, and allow matrices of morphisms between direct summands. This is the matrix category construction, applied to each category in our canopolis. We denote the result $\mathbf{Mat}\,(\mathcal{C}ob\,(\mathfrak{su}_2))$.

2.2.1. *The structure of morphisms in $\mathcal{C}ob\,(\mathfrak{su}_2)$*

The structure of this canopolis has been thoroughly analyzed elsewhere, in Bar-Natan's paper [1, §9] and in Gad Naot's [17]. We will need one of their results.

First, note that almost all closed surfaces in $\mathcal{C}ob\,(\mathfrak{su}_2)$ can be evaluated as scalars. In fact, applying the 'neck-cutting' relation (2.1) shows that they are all zero except for the surfaces of genus one and three. We saw above that the torus was equal to 2, but there is no *a priori* way to evaluate the surface of genus three. Therefore, we absorb it into our ground ring, letting $R = R_0[$  $]$.

Proposition 2.1. *For any two diagrams D_1 and D_2, let l be the number of components of $D_1 \cup D_2 \cup (\partial \times [0,1])$. Consider the set of cobordisms $C \in \mathrm{Hom}_{\mathcal{C}ob(\mathfrak{su}_2)}(D_1, D_2)$ such that every component of C is either a disc or a punctured torus. These cobordisms form a basis for $\mathrm{Hom}_{\mathcal{C}ob(\mathfrak{su}_2)}(D_1, D_2)$ over R.*

Note that such cobordisms must have exactly l components, and the boundary of each component is a single component of $D_1 \cup D_2 \cup (\partial \times [0,1])$.

Remark. This classification requires the neck-cutting relation, and only holds when 2 is invertible. (See [17] for details otherwise.)

We call a diagram 'non-elliptic' [j] if it contains no circles. By the previous result and some Euler characteristic calculations, we get:

Corollary 2.2. *Endomorphisms of a non-elliptic diagram are all in non-positive degree.*

Corollary 2.3. *If a nonzero endomorphism of a non-elliptic diagram factors through a different non-elliptic diagram, then it necessarily has negative grading.*

[h] This is Bar-Natan's $D\{m\}$.

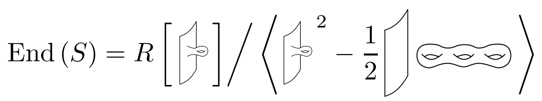
[i] Observe that $\chi(c) - \frac{k}{2}$ and $m_2 - m_1$ are additive separately.

[j] This is the obvious extension of Kuperberg's meaning of 'non-elliptic' in [15] to the $\mathfrak{su}_2$ case.

Remark. It's easy to see that elliptic diagrams have positively graded endomorphisms; for example, a circle which dies and is born again, each time via a disc cobordism, has grading +2.

This classification also yields a description of the 'sheet algebra' for the $\mathfrak{su}_2$ canopolis:

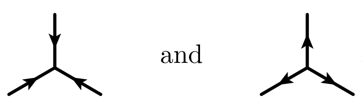
Corollary 2.4. *Let S be the diagram consisting of a single arc. Then*



3. The $\mathfrak{su}_3$ cobordism theory

3.1. *Seamed cobordisms, and the $\mathfrak{su}_3$ theory*

We now describe $\mathcal{C}ob\,(\mathfrak{su}_3)$, the analogous canopolis of 'seamed cobordisms' associated to $\mathfrak{su}_3$. The objects consist of 'webs' – elements of the planar algebra freely generated by the trivalent vertices



(It's a planar algebra whose label set consists of just two labels: 'in' and 'out'.) Let S be a commutative ring in which 2 and 3 are invertible. The set of morphisms between two webs with the same boundary will be an S-module generated by 'seamed cobordisms', also called 'foams'.

The local model for a seamed cobordism is the space $Y \times [0,1]$, the space obtained by gluing together three copies of $[0,1] \times [0,1]$ along $[0,1] \times \{0\}$, with orientations on the three squares, all inducing the same orientation on the common $[0,1] \times \{0\}$, along with a cyclic orientation of the three squares.[k]

Definition 3.1. Given two webs, D_1 and D_2, drawn in a disc, both with boundary ∂, a seamed cobordism from D_1 to D_2 is a 2-dimensional CW-complex [l] F (the 'foam') with

- exactly three 2-cells meeting along each singular 1-cell,
- a cyclic ordering on those three 2-cells,
- orientations on the 2-cells, compatible with the cyclic orderings,

[k]We say that a seamed cobordism C is locally modeled on $Y \times [0,1]$ in the same sense that that a topological n-manifold is modeled on (topological) $\mathbb{R}^n$. We mean that for every point p of C, there is a point p' of Y, neighborhoods $p \in U_p \subset C$ and $p' \in U'_{p'} \subset Y \times [0,1]$ and a bijection $f_p : U_p \to U'_{p'}$. Moreover, the 'transition maps' $f_p^{-1} f_q$ should preserve the local structure specified for $Y \times [0,1]$; in particular, the topological structure and, more importantly, the orientation data.
[l]We don't care about the actual cell decomposition, of course.

- and an identification of the boundary of F with $D_1 \cup D_2 \cup (\partial \times [0,1])$ such that
  - the orientations on the sheets induce the orientations on the edges of D_1, and the opposite orientations on the edges of D_2,
  - and the cyclic orderings around the singular seams agree with the cyclic orderings around a vertex in D_1 or D_2 given by its embedding in the disc; the anticlockwise ordering for 'inwards' vertices, the clockwise ordering for 'outwards' vertices.

We think of such a foam as living inside the 'can' $D^2 \times [0,1]$, even though it is not embedded there; there's just an identification of its boundary with a subset of the surface of the can.

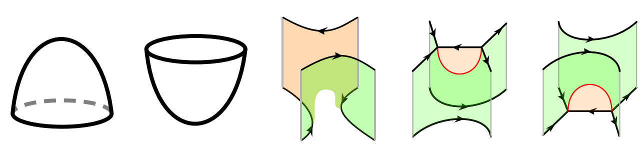
Compositions, both vertical (everyday composition of morphisms in a category) and horizontal (the action of planar tangles on morphisms), are almost trivial to describe. To compose vertically, we stack cans on top of each other, and to compose horizontally using a spaghetti and meatballs diagram T, we glue together $T \times [0,1]$ with the input cans.

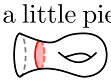
As before, to put a grading on our canopolis, we endow diagrams with formal grading shifts written as factors of q. The degree of a cobordism C from $q^{m_1} D_1$ to $q^{m_2} D_2$ is defined as

$$\deg C = 2\chi(C) - \sharp\partial + \frac{\sharp V}{2} + m_2 - m_1, \tag{3.1}$$

where $\sharp\partial$ is the number of boundary points of D_i and $\sharp V$ is the total number of trivalent vertices in D_1 and D_2. We leave it to the reader to check that this is additive under canopolis operations.

It is not hard to verify that this canopolis of $\mathfrak{su}_3$ foams is generated (as a canopolis!) by the morphisms cup, cap, saddle, zip, and unzip (after [19]):



As a little piece of nomenclature, we'll introduce the cobordism we call a 'choking torus', . Whenever you see this, you should assume the cyclic ordering at the seam is 'bulk/handle/disc'.

3.2. *Local relations*

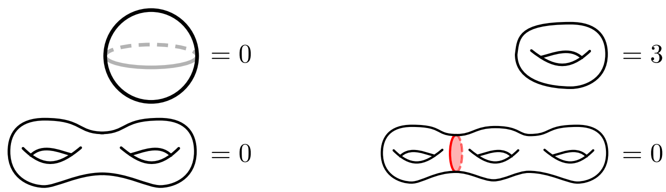
We now introduce local relations on the modules of seamed cobordisms. These are motivated in two ways:

(1) We expect that the canopolis of seamed cobordisms should have isomorphisms reflecting the relations appearing in the $\mathfrak{su}_3$ spider.

(2) We intend to construct an invariant of tangles, valued in complexes of seamed cobordisms.

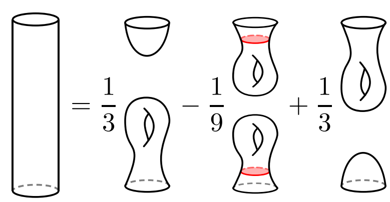
We'll see both of these motivations validated, in sections §3.4 and §4.1 respectively.

- 'Closed foam' relations:



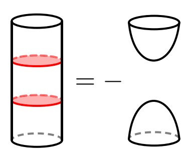
(3.2)

- The 'neck cutting' relation:



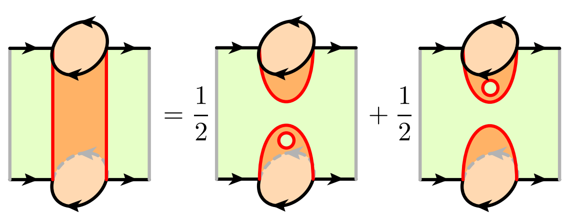
(3.3)

- The 'airlock' relation:



(3.4)

- The 'tube' relation

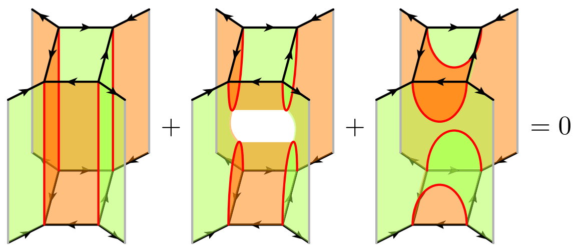


(3.5)

The small[m] circles here indicate the two sheets coming together; they're a composition, zip followed by unzip.

[m]Green, if you read the online version of this paper.

- The 'three rocket' relation:



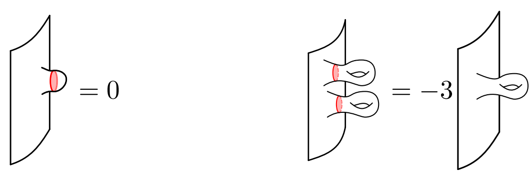
(3.6)

- The 'seam-swap' relation: reversing the cyclic order of the three 2-cells attached to a closed singular seam is equivalent to multiplication by -1.

The relations have appeared in other forms before, in [16] and [10]. See in particular Figure 19 of [10] for our Equation (3.4), and in the proof of Proposition 9 of the same for our Equation (3.6). For the other relations, you should read §Appendix A, and then check that following that translation the relations in Equations (3.3) and (3.5) also appeared in [10]. For an example of why we *impose* all of these relations, rather than impose some and derive others, see §3.3.1.

As consequences of the above relations, it is not hard to derive the following:

- The sheet relations:

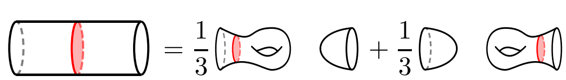


(3.7)

(3.8)

The 'blister' relation follows directly from seam-swapping. The 'choking torus multiplication' relation on the first line follows from applying neck-cutting in reverse. The equations in the last line follow from neck cutting, and the closed foam relations.

- The 'bamboo' relation:



(3.9)

which follows from neck-cutting one side of the bamboo, then reducing terms via airlocks and blisters.

As before, we introduce formal direct sums of the objects and matrices of morphisms, yielding a canopolis we call $\mathbf{Mat}\,(\mathcal{C}ob\,(\mathfrak{su}_3))$.

3.3. *Consistency*

The purpose of this section is two-fold. First, we want to provide a set of assumptions, plausibly desirable in any categorification of the $\mathfrak{su}_3$ planar algebra, which allows us to to derive the relations described in the previous section. Second, we prove the following result:

Theorem 3.2. *The local relations of §3.2 are consistent, in the sense that*

$$\mathrm{Hom}_{\mathcal{C}ob(\mathfrak{su}_3)}(\emptyset, \emptyset) \neq 0.$$

These two goals are related. In the process of justifying the local relations, we will divide them into two classes: the 'evaluation relations', and the 'local kernel' relations. The evaluation relations are the 'closed foam' relations, 'seam swapping', 'neck cutting' and 'airlock'. The 'local kernel' relations are 'tube' and 'rocket'. We begin by showing the evaluation relations follow from some appealing assumptions. We then show that these relations, living up to their name, suffice to evaluate any closed foam. Further, in §3.3.2 we'll show they're consistent; denoting the canopolis in which we only impose the evaluation relations by $\mathcal{C}ob\,(\mathfrak{su}_3)^{\mathrm{ev}}$, we have

Lemma 3.3.

$$\mathrm{Hom}_{\mathcal{C}ob(\mathfrak{su}_3)^{ev}}(\emptyset, \emptyset) = S.$$

It's then time to introduce the local kernel relations. The canopolis $\mathcal{C}ob\,(\mathfrak{su}_3)^{\mathrm{ev}}$ is an unsatisfactory one, in the sense that it is 'degenerate' or has a 'local kernel': non-zero foams with boundary, all of whose completions to a closed foam are zero. In a slightly different guise, Khovanov proved the following lemma in [10]:

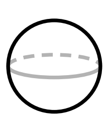
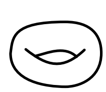
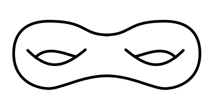
Lemma 3.4. *The tube relation and rocket relation are in the local kernel (justifying the name 'local kernel relations').*

We'll show in §3.3.3 that

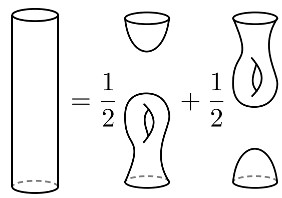
Lemma 3.5. *The local kernel is generated, as a canopolis ideal, by the tube and rocket relations.*

We thus impose the local kernel as additional relations, and together Lemmas 3.3 and 3.5 imply Theorem 3.2.

Note that this distinction between 'evaluation' and 'local kernel' relations is a new feature of the $\mathfrak{su}_3$ theory. The relations required for evaluation in the $\mathfrak{su}_2$ theory, namely

 $= 0$  $= 3$  $= 0$

and

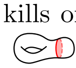
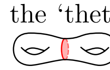


produce a canopolis which is already non-degenerate in the sense above, so there is no need to add 'local kernel' relations to produce a satisfactory local theory.

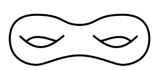
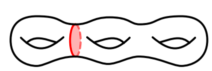
3.3.1. *Explaining the relations*

We now set out some plausible assumptions one might make about any categorification of the $\mathfrak{su}_3$ spider. (Perhaps these assumptions might be useful to someone categorifying something else, as well!)

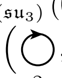
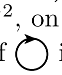
Firstly, we'll ask, without much motivation, for the grading rule given previously; the grading of a morphism is given by twice its Euler characteristic, as in Equation (3.1).

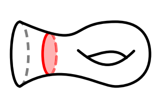
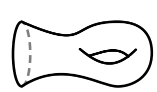
We'll just have to pull the 'seam-swapping' relation described earlier out of a hat.[n] This relation kills off certain closed foams, amongst them the 'theta' foam, the 'blistered torus'  (in fact, any foam with a blister) and .

We'll then put in by hand a few relations motivated by the desire that $\mathrm{Hom}_{\mathcal{C}ob(\mathfrak{su}_3)}(\emptyset, \emptyset)$, the space of closed foams, as a graded S-module, be just S generated by the empty foam. Later, we'll see that the relations we've imposed do in fact imply this. First of all, we force the sphere to be zero (it's in positive degree) and the torus to be some element of S. We'll assume, in fact, that the torus is invertible. Briefly, we'll write t for this value, but very shortly discover that $t = 3$. Further, various closed foams with negative degrees are forced to be zero, such as

 and .

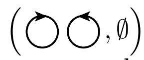
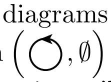
(However, see §A.2 for a discussion of the variation in which we just ask that $\mathrm{Hom}_{\mathcal{C}ob(\mathfrak{su}_3)}(\emptyset, \emptyset)_{>0} = 0$ and $\mathrm{Hom}_{\mathcal{C}ob(\mathfrak{su}_3)}(\emptyset, \emptyset)_0$ is 1-dimensional.)

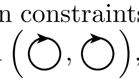
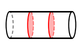
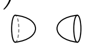
Next, we'll ask that $\mathrm{Hom}_{\mathcal{C}ob(\mathfrak{su}_3)}\left(\right.$  $\left., \emptyset\right)$ is a free module of rank 3, and in fact with graded dimension $q^2 + 1 + q^{-2}$, on the basis that we expect this graded dimension to agree with the evaluation of  in the $\mathfrak{su}_3$ spider. Since the cobordisms

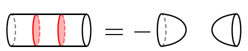
  and  (3.10)

[n]Note though, that it's the $n = 3$ special case of the idea described in [10, §6] that if the 'k-sheets' of an $\mathfrak{su}_n$ foam were to be labeled by elements of the cohomology ring of $\mathrm{Gr}(k \subset n)$, then the relations around a seam should be the kernel of the map $\bigotimes_i H^\bullet(\mathrm{Gr}(k_i \subset n)) \to H^\bullet\left(\mathrm{Flag}(k_1 \subset k_1 + k_2 \subset \cdots \subset \left(\sum_i k_i\right))\right)$ induced by the 'take orthogonal complements' map at the geometric level.

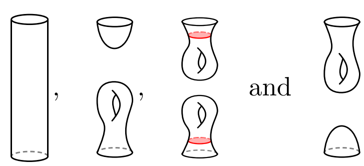
lie in this morphism space, with gradings 2, 0 and -2 respectively, we'll further ask that in fact the morphism space is freely generated by these three cobordisms. (Unsurprisingly, we'll ask the same thing for .) Remember there are two variations of the middle cobordism above, differing in the cyclic ordering of the sheets at the seam; the two cyclic orderings only differ by a sign, however, by the seam-swapping relation.

Further, we'll ask that Hom  $\cong$ Hom , with the isomorphism given by isotopy. This behavior will follow from any good notion of duality in a categorification; moreover, it certainly happens in the $\mathfrak{su}_2$ canopolis, and we'll see the appropriate generalization to arbitrary diagrams in §3.3.3. Even more, we'll ask that the obvious map Hom  $\otimes$ Hom  $\to$ Hom , given by disjoint union, is actually an isomorphism; again, we'll later see that this is generally true.

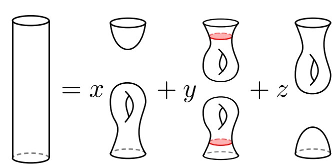
With these relatively benign constraints, we can get a long way! Firstly, looking at the degree 4 piece of Hom , we see it's 1 dimensional, and so the 'airlock'  must be proportional to . We'll declare[o] that



Next, looking at the degree 0 piece, we see a 3 dimensional space. Writing down 4 obvious cobordisms here,

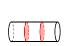
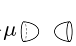


we see there must be some relation amongst them (this will turn out to be neck cutting, of course), which we'll suppose is of the form



We can determine the coefficients here by considering various closures.

Adding a punctured torus at the top and a disc at the bottom gives us $t = xt^2$, and vice versa gives us $t = zt^2$, so $x = z = \frac{1}{t}$. Adding a 'choking torus' at top and bottom gives $-t^2 = yt^4$, so $y = -\frac{1}{t^2}$. Finally, gluing top to bottom gives $t = \frac{1}{t}t - \frac{1}{t^2}(-t^2) + \frac{1}{t}t = 3$. We've at this stage recovered the neck cutting relation!

[o]We could try an arbitrary constant here,  $= -\mu$ , say. The argument above would continue much the same, except that we wouldn't be able to find an analogue of the tube and rocket relations in the local kernel.

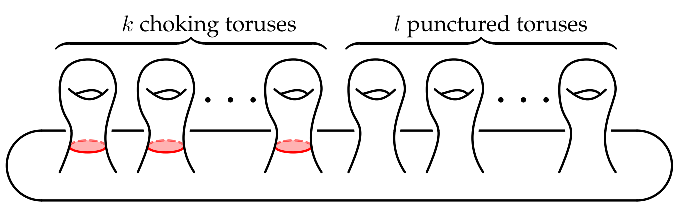
3.3.2. *Consistency of the evaluation relations*

Proof. [Proof of Lemma 3.3.] In $\mathcal{C}ob\,(\mathfrak{su}_2)$, all closed foams are equivalent to scalars. This is not as immediately apparent in $\mathcal{C}ob\,(\mathfrak{su}_3)$, but it's in fact true even in $\mathcal{C}ob\,(\mathfrak{su}_3)^{\text{ev}}$; that is, even when we only impose the evaluation relations. We describe an algorithm for evaluating closed foams and prove that it's well-defined with respect to the evaluation relations. This is perhaps a somewhat unsatisfying proof of consistency, but it's the only method we can see available, in our setup.

The first step, in which we do nearly all the work (exactly following Khovanov's method from [10]), is to perform neck cutting on each sheet incident at each seam (all of which are circles). Thus if there are k seams in a closed foam, we perform neck cutting $3k$ times, resulting in 3^{3k} terms. The compensation for creating so many terms is that each term is now relatively simple, being a disjoint union of two different types of small closed foams.

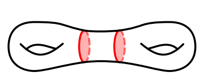
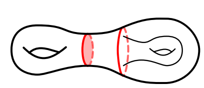
The first type, arising from a seam in the original closed foam, consists simply of a seam, with three of the elements appearing in Equation (3.10) attached.

The second type, arising from a sheet in the original closed foam, consists of a closed foam in which the only seams appears as part of some 'choking torus'. Notice that all of these choking toruses are of the same type; the cyclic order around the seam is 'bulk-handle-disc', simply because this is the cyclic order appearing in the neck cutting relation. These surfaces are thus parameterized by two numbers; the number of choking toruses, and the number of punctured toruses. We'll write such a surface as $\Sigma_{k,l}$:

$\Sigma_{k,l} =$ 

.

The second step of the algorithm is to evaluate all of these small closed foams. In the first type, we quickly see by the seam swapping relation that nearly all are zero. In particular, unless the three different sheets carry different surfaces, the closed foam must be zero. There are thus only two non-zero possibilities, depending on the cyclic order around the seam. We can either have 'disc/choking torus/punctured torus' or 'disc/punctured torus/choking torus':

 and  (3.11)

We now apply the seam-swapping if we find ourselves in the second case, then evaluate the first closed foam (via 'airlock') as -9.

We evaluate nearly every case of the second type of closed foam, $\Sigma_{k,l}$ by making use of Equation (3.8). Specifically, if $k \geq 1$ and $l \geq 1$, or simply $l \geq 2$, we see

$\Sigma_{k,l} = 0$. If $k \geq 3$, Equations (3.7) and (3.8) together imply $\Sigma_{k,l} = 0$. This leaves four cases, shown in Figure 3.1, each of which we already know how to evaluate directly.

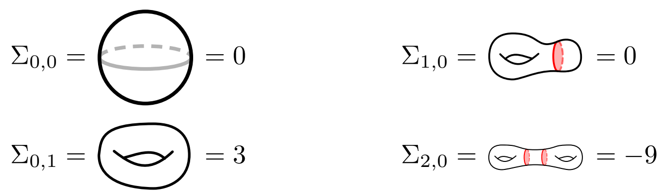
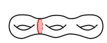


Fig. 3.1. The irreducible examples of $\Sigma_{k,l}$, modulo neck cutting.

The algorithm described so far evaluates any closed foam as a scalar. We now check that the evaluation relations are consistent, by showing that the evaluation algorithm produces the same result on either side of each relation, when applied to some large closed foam. This check requires a few cases, each of which is almost trivial.

The first, and most trivial, cases are the closed foam relations. It's easy to see that applying the above algorithm to any of the four closed foams in Equation (3.2) above simply gives the specified evaluation. This is completely trivial in 3 cases, and a short calculation for  (because there we do some 'unnecessary' neck cutting).

The seam-swapping relation is also relatively trivial. If we change the cyclic order at a seam, the evaluation algorithm only differs in that the two surfaces in Equation (3.11) are interchanged, resulting in an extra sign (actually, these two surfaces actually occur three times each, corresponding to the three cyclic permutations around the seam, but each pair is interchanged).

Slightly more interesting is the airlock relation. Here we simply need to check that when we cut both seams in an airlock, modulo the specified closed foam evaluations, we obtain exactly the other side of the airlock relation.

Most interesting is the neck cutting relation. There are three distinct ways we can apply the neck cutting relation; parallel to a seam, not parallel but still separating the sheet into two pieces, and non-separating. The first is easy; the evaluation algorithm produces the same result, simply because neck cutting twice along parallel circles is the same as neck cutting once (modulo evaluating the 9 resulting closed foams). If we apply neck cutting separating a sheet into two pieces, it's obviously the same as applying a corresponding neck cutting to one of the second type of small closed foams resulting from the evaluation algorithm. Thus we need to check that the evaluation algorithm produces the same results on $\Sigma_{k_1+k_2,l_1+l_2}$

and on

$$\frac{1}{3}\Sigma_{k_1,l_1+1}\Sigma_{k_2,l_2} - \frac{1}{9}\Sigma_{k_1+1,l_1}\Sigma_{k_2+1,l_2} + \frac{1}{3}\Sigma_{k_1,l_1}\Sigma_{k_2,l_2+1}.$$

This check involves quite a few cases; when $k_1 + k_2, l_1 + l_2 \geq 1$ (which splits into two subcases, $k_1, l_1 \geq 1$, and $k_1, l_2 \geq 1$), when $l_1 + l_2 \geq 2$, when $k_1 + k_2 \geq 3$, and the 'small' cases when none of these hold. Each case is pretty much immediate, however.

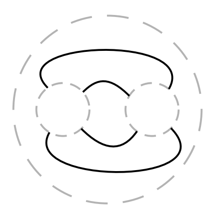
Finally, for a 'non-separating' neck cutting relation we need to check that the evaluation algorithm produces the same results on $\Sigma_{k,l}$ (l here must be at least 1) and

$$\frac{2}{3}\Sigma_{k,l} - \frac{1}{9}\Sigma_{k+2,l-1}. \tag{3.12}$$

If $k \geq 1$, each closed foam appearing here evaluates to 0. If $k = 0$, everything is zero unless $l = 1$, in which case the expression in Equation (3.12) is $\frac{2}{3}3 - \frac{1}{9}(-9) = 3 = \Sigma_{0,1}$. □

3.3.3. *The local kernel*

For a given disc boundary ∂ in a planar algebra $\mathcal{P}$, the 'pairing tangle' has two internal discs, labeled by ∂ and ∂^*, with an empty external circle, and the obvious spaghetti:



We'll denote the result of inserting $x \in \mathcal{P}_\partial$ and $y \in \mathcal{P}_{\partial^*}$ simply by $\langle x, y\rangle$.

Definition 3.6. In a spherical[p] planar algebra $\mathcal{P}$, the 'local kernel' (or maybe the 'kernel of the partition function') is the set of elements $x \in \mathcal{P}_\partial$ such that the pairing of x with any $y \in \mathcal{P}_{\partial^*}$ is zero.

Remark. We need the adjective spherical here in order to give such a snappy definition. In a possibly non-spherical planar algebra, you'd want to say it's the set $x \in \mathcal{P}_\partial$ such that for every planar tangle T, with no labels on the outer boundary and k internal discs, the first of which has label ∂, and for every $k-1$ appropriate elements of $\mathcal{P}$, say $x_2, \ldots, x_k$, the composition $T(x, x_2, \ldots, x_k)$ is zero.

Definition 3.7. In a spherical canopolis $\mathcal{C}$, the 'local kernel' is the set of morphisms $(x : A \to B) \in \mathcal{C}_\partial$ such that for every $(y : C \to D) \in \mathcal{C}_{\partial^*}$ and for any morphisms $z : \emptyset \to \langle A, C\rangle$ and $w : \langle B, D\rangle \to \emptyset$, the composition $w \circ \langle x, y\rangle \circ z$ is zero.

[p] A planar algebra is spherical if two planar tangles with no points on the external disc which only differ by pulling an edge 'around the back' of the disc always act in the same way.

It's obvious that in both cases, the local kernel is an ideal. One can always quotient by the local kernel.

Definition 3.8. A planar algebra or canopolis is 'nondegenerate' if the local kernel is zero.

Lemma 3.9. *Given any two webs A and B with common boundary ∂^*, there is an isomorphism of $\mathbb{Z}$-modules*

$$G : \mathrm{Hom}_{\mathcal{C}ob(\mathfrak{su}_3)}(A, B) \to \mathrm{Hom}_{\mathcal{C}ob(\mathfrak{su}_3)}(\emptyset, \langle A^*, B \rangle)$$

induced by an invertible sequence of canopolis operations. (Here, A^* denotes A with its orientation reversed, so that it has boundary ∂.) In particular, this isomorphism preserves membership in canopolis ideals.

Proof. There is an obvious homeomorphism $h : A \cup B \cup (\partial \times [0,1]) \to \langle A^*, B \rangle$. We define $G(F)$ to be F with its boundary identification map i replaced by $h \circ i$. This yields an isomorphism of the morphism spaces.

To see that this isomorphism is induced by canopolis operations, note that $A \cup B \cup (\partial \times [0,1])$ and $\emptyset \cup \langle A, B \rangle \cup (\emptyset \times [0,1])$ are naturally isotopic in the cylinder $D^2 \cup D^2 \cup (S^1 \times [0,1])$ (which is, of course, just a 2-sphere). One may envision this isotopy as 'pulling A to the ceiling'. Pick a nice isotopy and let M denote its trace in $\left(D^2 \cup D^2 \cup (S^1 \times [0,1])\right) \times [0,1]$. Because M comes with an induced 2-dimensional CW structure, it can be decomposed as a sequence M_* of canopolis operations taking a foam in $\mathrm{Hom}(A, B)$ (the inner can) to a foam in $\mathrm{Hom}(\emptyset, \langle A^*, B \rangle)$ (the outer can). Since h is induced by the isotopy, $M_* = G$. □

Remark. This isomorphism does not preserve gradings of morphisms; see Lemma 5.10 for a statement involving gradings.

Corollary 3.10. *In a spherical canopolis, the local kernel is generated as a canopolis ideal by the set of morphisms $x : \emptyset \to B$ such that for every $y : B \to \emptyset$, the composition $x \circ y$ is zero.*

With these definitions made, it's time to prove Lemma 3.5.

In this section, we'll write $T = \frac{1}{2}T_\downarrow + \frac{1}{2}T_\uparrow - T_z$ for the difference of the foams appearing in the tube relation, and $R = R_x + R_y + R_z$ for the sum of the foams appearing in the rocket relation. (That is, the tube and rocket relations are $T = 0$ and$R = 0$.) We'll write I for the canopolis ideal generated by T and R.

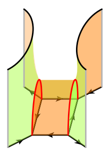
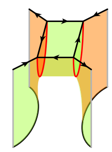
Proof. [Proof of Lemma 3.5.] Let $\sum c_\alpha F_\alpha$ be an element of the local kernel of $\mathcal{C}ob(\mathfrak{su}_3)$; that is, a linear combination of foams $F_\alpha \in \mathrm{Hom}(A, B)$ such that every closure is zero. By Lemma 3.9, we may assume that A is empty, and B has empty boundary. We proceed by induction on the complexity of B.

If B is empty, then each F_α is equivalent to a scalar, so trivially $\sum c_\alpha F_\alpha = 0 \in I$. If B is nonempty, then an Euler characteristic argument shows that B contains a square, bigon, or circle.

Suppose B contains a square. We compose with an 'identity rocket' over the square, writing $F_\alpha = R_z \circ F_\alpha$. Then

$$R_z \circ F_\alpha = R \circ F_\alpha - R_x \circ F_\alpha - R_y \circ F_\alpha.$$

By definition $R \circ F_\alpha \in I$. We expand $R_x \circ F_\alpha$ as $R_x^{\text{upper}} \circ R_x^{\text{lower}} \circ F_\alpha$, where

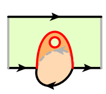
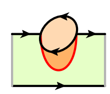
$R_x^{\text{lower}} =$  and $R_x^{\text{upper}} =$ .

Now $\sum R_x^{\text{lower}} \circ c_\alpha F_\alpha = R_x^{\text{lower}} \circ (\sum c_\alpha F_\alpha)$, and since $\sum c_\alpha F_\alpha$ is in the local kernel, so is $R_x^{\text{lower}} \circ (\sum c_\alpha F_\alpha)$. Also, $R_x^{\text{lower}} \circ (\sum c_\alpha F_\alpha)$ has a simpler target than B, and is therefore in I by our inductive hypothesis. Hence $R_x \circ \sum c_\alpha F_\alpha \in I$, and by the same argument, $R_y \circ \sum c_\alpha F_\alpha \in I$. Therefore $\sum c_\alpha F_\alpha \in I$.

The argument when B contains a bigon is similar. We express

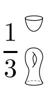
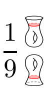
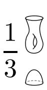
$$F_\alpha = T_z \circ F_\alpha = \frac{1}{2} T_\downarrow \circ F_\alpha + \frac{1}{2} T_\uparrow \circ F_\alpha - T \circ F_\alpha.$$

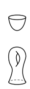
By definition $T \circ F_\alpha \in I$. We write $T_\downarrow \circ F_\alpha = T_\downarrow^{\text{upper}} \circ T_\downarrow^{\text{lower}} \circ F_\alpha$, where

$T_\downarrow^{\text{lower}} =$  and $T_\downarrow^{\text{upper}} =$ .

$T_\downarrow^{\text{lower}} \circ \sum c_\alpha F_\alpha$ is in the local kernel and has simpler target, and is therefore in I. As above, it follows simply that $T_\downarrow \circ \sum c_\alpha F_\alpha$ and $T_\uparrow \circ \sum c_\alpha F_\alpha$ are in I, and therefore so is $\sum c_\alpha F_\alpha$.

Lastly, suppose B contains a circle. Then by the neck-cutting relation,

$F_\alpha =$  $\circ F_\alpha = \frac{1}{3}$  $\circ F_\alpha - \frac{1}{9}$  $\circ F_\alpha + \frac{1}{3}$  $\circ F_\alpha$.

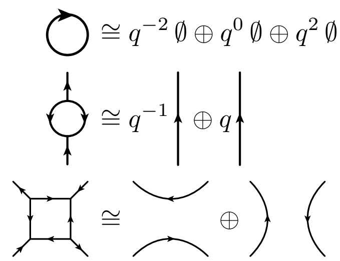
 $\circ \sum c_\alpha F_\alpha$ is an element of the local kernel with simpler target, so by induction, it is in I. So  $\circ \sum c_\alpha F_\alpha \in I$. This argument works for the other two terms in the above equation, and therefore $\sum c_\alpha F_\alpha \in I$. □

3.4. *Isomorphisms*

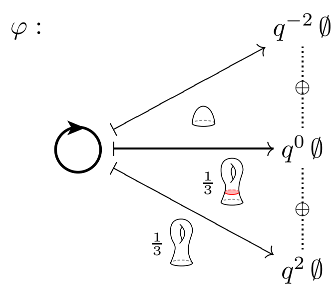
In this section, we discover what all those local relations in $\mathcal{C}ob(\mathfrak{su}_3)$ are really for: they imply certain isomorphisms between objects in the category $\mathbf{Mat}(\mathcal{C}ob(\mathfrak{su}_3))$. These isomorphisms should be thought of as categorifications of relations appearing in the $\mathfrak{su}_3$ spider.

Thus we set out to prove:

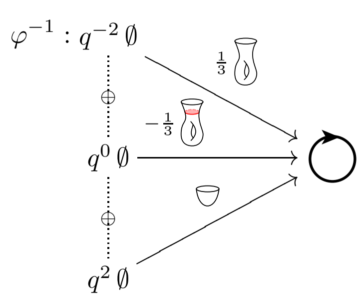
Theorem 3.11. *There are isomorphisms*



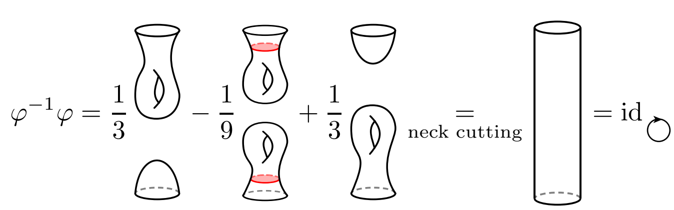
Proof. Let's define $\varphi :$ ○ $\to q^{-2}\,\emptyset \oplus q^0\,\emptyset \oplus q^2\,\emptyset$ and $\varphi^{-1} : q^{-2}\,\emptyset \oplus q^0\,\emptyset \oplus q^2\,\emptyset \to$ ○ by



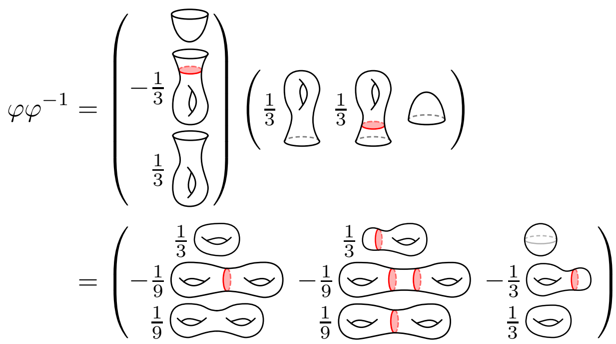
and



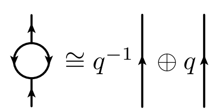
and then perform the routine verification that these are indeed inverses:

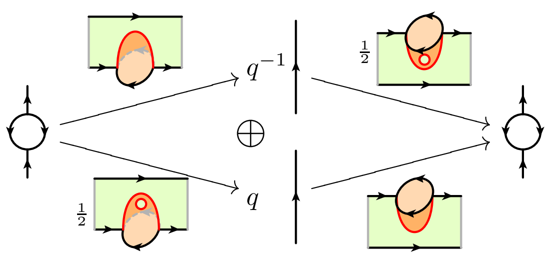


and

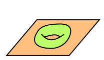
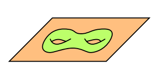


$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{id}_{q^{-2}\,\emptyset \oplus q^{0}\,\emptyset \oplus q^{2}\,\emptyset}$$

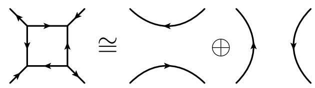
Next we need to define the isomorphism . It's given by

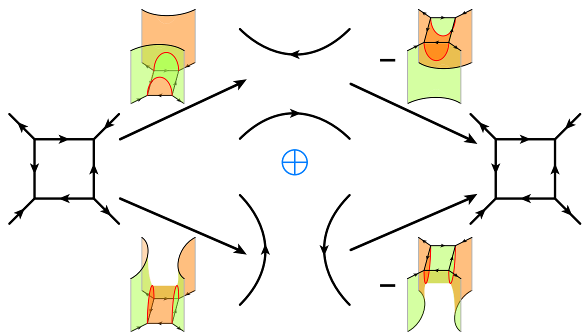


This follows straightforwardly from the relation in Equation (3.5), along with the 'bagel' and 'double bagel' relations:

 $= 2$  $= 0,$ (3.13)

(the 'bagel' here is the union of a torus and the part of the equatorial plane outside the torus; it has two circular seams) which are easy consequences of the 'bamboo' relation appearing in Equation (3.9).

Finally, the isomorphism  is described by the diagram



Verifying that these maps are mutual inverses requires the blister, airlock and rocket relations. □

4. The knot homology map

In this section we will describe the construction of the $\mathfrak{su}_3$ knot homology theory. This description will, of course, be essentially equivalent to the previous constructions in [16,10], but we will emphasize certain differences. In particular, the knot homology theory will be explicitly local, described as a morphism of planar algebras.

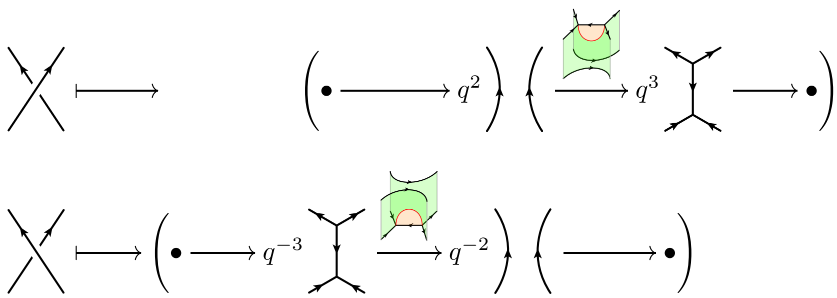
The strength of this locality is that it allows us to perform 'divide and conquer' calculations. We'll explain that Bar-Natan's [2] 'complex simplification algorithm' can be applied in the $\mathfrak{su}_3$ case. This allows us to calculate the invariant of a knot by calculating the invariant for subtangles, simplifying these, then gluing together the simplified complexes by the appropriate planar operations. In §6.1, we'll apply these ideas to compute the $\mathfrak{su}_3$ Khovanov homology of the $(2, n)$ torus knots.

The complex simplification algorithm also allows us to give 'automatic' proofs of Reidemeister invariance; we just simplify the complexes associated to either side of the Reidemeister move, and observe the resulting complexes are the same.

We wish to associate to every oriented tangle a complex in $\mathbf{Mat}\left(\mathcal{C}ob\left(\mathfrak{su}_3\right)\right)$. Oriented tangles form a planar algebra generated by the positive and negative crossings modulo relations given by the Reidemeister moves.

In any canopolis, the complexes again form a planar algebra. Moreover, complexes together with chain maps between them form a canopolis. Bar-Natan proves this for $\mathcal{C}ob\left(\mathfrak{su}_2\right)$ in Theorem 2 of [1], but his argument is completely general. There's also a discussion of the planar algebra structure on complexes in [4].

It thus suffices to define the knot homology map on the positive and negative crossing:



Here, the relative horizontal alignments of the complexes denote homological height; both of the two-strand diagrams are at homological height zero. Further, notice that, with the given grading shifts on the objects, the differentials are grading zero maps. Since degrees are additive under tensor products, this is true for the differentials in the complex for any tangle.

Verifying that this map is a well-defined morphism of planar algebras amounts to checking Reidemeister invariance, which we do in §4.2. Verifying that it's a map of canopolises (from tangle cobordisms to chain maps) remains to be done; we provide some evidence that this is true (on the nose, no sign ambiguities) in §4.3.

4.1. *The simplification algorithm*

The following lemma from [2] is our fundamental tool for simplifying complexes up to homotopy.

Lemma 4.1 (Gaussian elimination for complexes). *Consider the complex*

$$A \xrightarrow{\begin{pmatrix} \bullet \\ \alpha \end{pmatrix}} \begin{matrix} B \\ \bigoplus \\ C \end{matrix} \xrightarrow{\begin{pmatrix} \varphi & \lambda \\ \mu & \nu \end{pmatrix}} \begin{matrix} D \\ \bigoplus \\ E \end{matrix} \xrightarrow{\begin{pmatrix} \bullet & \epsilon \end{pmatrix}} F \tag{4.1}$$

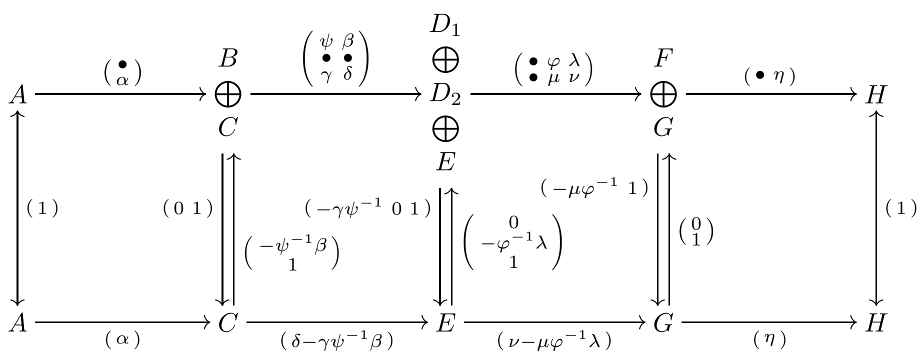
in any additive category, where $\varphi : B \xrightarrow{\cong} D$ is an isomorphism, and all other morphisms are arbitrary (subject to $d^2 = 0$, of course). Then there is a homotopy equivalence with a much simpler complex, 'stripping off' φ.

$$\begin{array}{ccccccc}
A & \xrightarrow{\begin{pmatrix} \bullet \\ \alpha \end{pmatrix}} & \begin{matrix} B \\ \bigoplus \\ C \end{matrix} & \xrightarrow{\begin{pmatrix} \varphi & \lambda \\ \mu & \nu \end{pmatrix}} & \begin{matrix} D \\ \bigoplus \\ E \end{matrix} & \xrightarrow{\begin{pmatrix} \bullet & \epsilon \end{pmatrix}} & F \\
\updownarrow {\scriptstyle (1)} & & {\scriptstyle (0\ 1)} \downarrow \uparrow {\scriptstyle \begin{pmatrix} -\varphi^{-1}\lambda \\ 1 \end{pmatrix}} & & {\scriptstyle (-\mu\varphi^{-1}\ 1)} \downarrow \uparrow {\scriptstyle \begin{pmatrix} 0 \\ 1 \end{pmatrix}} & & \updownarrow {\scriptstyle (1)} \\
A & \xrightarrow{(\alpha)} & C & \xrightarrow{(\nu - \mu\varphi^{-1}\lambda)} & E & \xrightarrow{(\epsilon)} & F
\end{array}$$

Remark. Note that the homotopy equivalence is also a simple homotopy equivalence; we're just stripping off a contractible direct summand. *Remark.* This is simply Lemma 4.2 in [2] (see also Figure 2 there), this time explicitly keeping track of the chain maps. Notice also that in a graded category, if the differentials are all in degree 0, so are the homotopy equivalences which we construct here. In particular, this applies to the homotopy equivalences associated to Reidemeister moves we construct in §4.2.

We'll also state here the result of applying Gaussian elimination twice, on two adjacent but non-composable isomorphisms. Having these chain homotopy equivalences handy will tidy up the calculations for the Reidemeister 2 and 3 chain maps.

Lemma 4.2 (Double Gaussian elimination). *When ψ and φ are isomorphisms, there's a homotopy equivalence of complexes:*



Proof. Apply Lemma 4.1 on the isomorphism ψ. Notice that the isomorphism φ survives unchanged in the resulting complex, and apply the lemma again. □

Remark. Convince yourself that it doesn't matter in which order we cancel the isomorphisms!

We can now state the simplification algorithm for complexes in $\mathbf{Mat}\left(\mathcal{C}ob\left(\mathfrak{su}_3\right)\right)$, analogous to Bar-Natan's algorithm [2] for $\mathfrak{su}_2$:

- If an object in a complex contains a closed loop, bigon, or square, then we replace it with the other side of the corresponding isomorphism in Theorem 3.11. (You might call this step 'delooping', 'debubbling', and 'desquaring'.) This increases the number of objects in the complex, but decreases the number of possible distinct objects, so informally we expect it to make the appearance of isomorphisms more likely.
- If an isomorphism appears as a matrix entry anywhere in the complex, we cancel it using Lemma 4.1.

In practice in $\mathbf{Mat}\left(\mathcal{C}ob\left(\mathfrak{su}_2\right)\right)$ this algorithm provides by far the most efficient algorithm for evaluating the Khovanov homology of a knot. This algorithm, implemented (not-so-efficiently) by Bar-Natan and (efficiently!) by Green [7] proceeds by breaking the knot into subtangles, applying the simplification algorithm above to the corresponding complexes, then gluing two simplified complexes together via the appropriate planar operation, simplifying again, and so on. Sadly, there isn't yet such a program for the $\mathfrak{su}_3$ case. A significant motivation for writing this program would be to entirely mechanise the isotopy invariance proofs of the next section. Indeed, a well written program would also automate checking movie moves.

4.2. *Isotopy invariance*

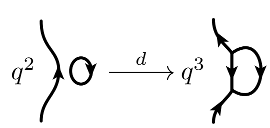
For each Reidemeister move, we will produce the complex associated to the tangle on either side, and apply the simplification algorithm described above (when appropriate, also making use of Lemma 4.2). There's plenty of computational work required, but it's important to notice that no further insight is required. Unusually for mathematics, this is a good thing; it shows that our tools, namely the simplification algorithm, are sufficiently well developed. Actually, below we give an alternative proof of the third Reidemeister move, making use of the insight behind the 'categorified Kauffman trick', even if it isn't strictly necessary.

We'll produce explicit chain maps between either side of each Reidemeister move; a gift to whomever wants to check that the $\mathfrak{su}_3$ theory is functorial!

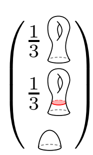
Moreover, because we use the simplification algorithm, we'll see that the two sides of each Reidemeister move aren't just homotopic, they're simply homotopic.[q]

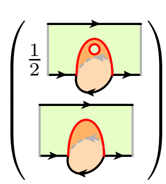
4.2.1. *Reidemeister 1*

The complex associated to  is



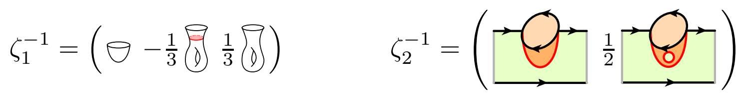
with d simply a zip map. Delooping at homological height 1, and removing the bigon at height 2, using the isomorphisms

$\zeta_1 =$ 

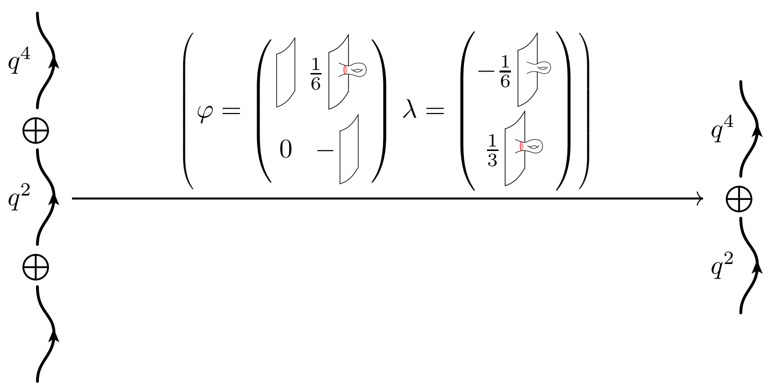
$\zeta_2 =$ 

[q]This will presumably allow an extension of the work of Juan Ariel Ortiz-Navarro and Chris Truman [18] on volume forms on Khovanov homology to the $\mathfrak{su}_3$ case.

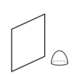
with inverses



we obtain the complex

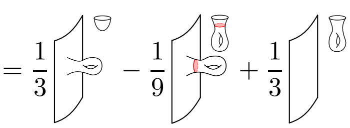


The differential here is the composition $\zeta_2 d \zeta_1^{-1}$, and we've named some components, getting ready to apply Lemma 4.1. Stripping off the isomorphism φ, according to that lemma, we see that the complex is homotopy equivalent to the desired complex: a single strand, in grading zero. The simplifying homotopy equivalence is

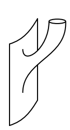
$$s_1 = \begin{pmatrix} 0 & 0 & \mathbf{1} \end{pmatrix} \circ \zeta_1 =$$ 

$$s_2 = 0$$

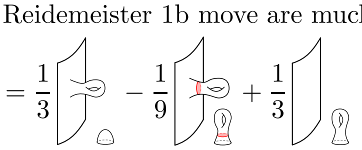
with inverse

$$s_1^{-1} = \zeta_1^{-1} \circ \begin{pmatrix} -\varphi^{-1}\lambda \\ \mathbf{1} \end{pmatrix} =$$ 

$$s_2^{-1} = 0.$$

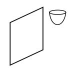
Notice here that $s_1^{-1} =$ , by the neck cutting relation. This agrees with the homotopy equivalence proposed in [16].

The calculations for the Reidemeister 1b move are much the same. We obtain

$$s_1 =$$ 

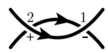
$$s_2 = 0$$

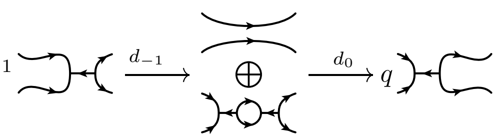
with inverse

$s_1^{-1} =$ 

$s_2^{-1} = 0.$

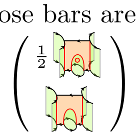
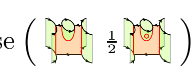
4.2.2. *Reidemeister 2a*

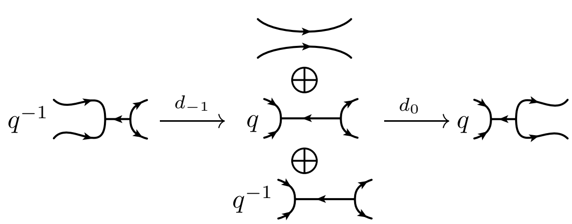
The complex associated to  is



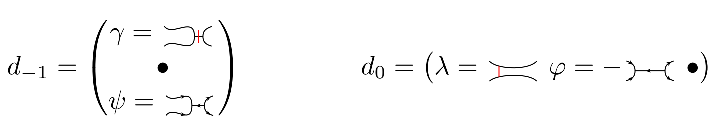
with differentials

(In this and the next section, we'll use the above shorthand for simple foams; a bar connecting two edges denotes a zip, and a bar transverse to an edge denotes an unzip. If you're reading this in colour, those bars are red.)

Applying the debubbling isomorphism  (with inverse ) to the direct summand with a bigon, we obtain the complex

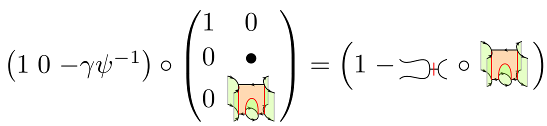


where

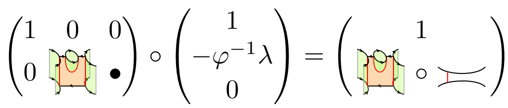


Here we've named the entries of the differentials in the manner indicated in Lemma 4.2. Applying that lemma gives us chain equivalences with the desired one object complex. The chain equivalences we're after are compositions of the chain equivalences from Lemma 4.2 with the debubbling isomorphism or its inverse.

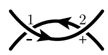
Thus the R2a ‘untuck’ chain map is

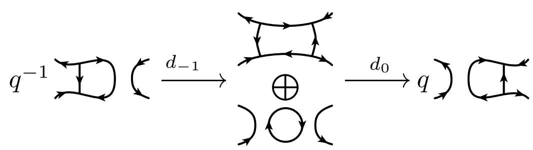


and the ‘tuck’ map is



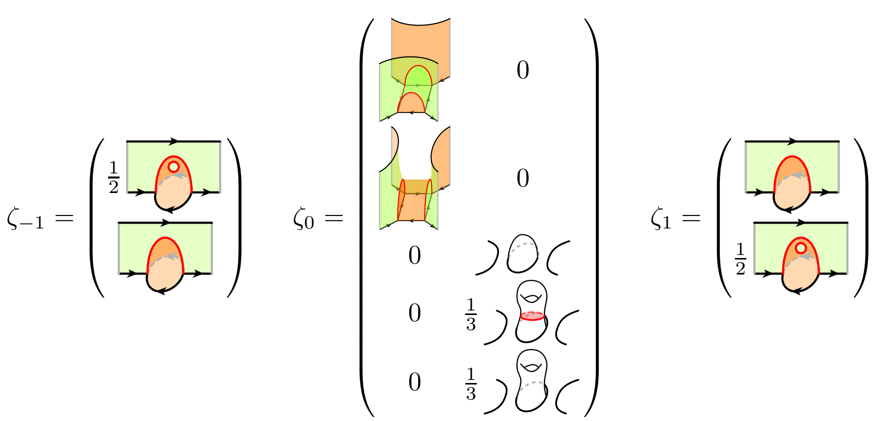
4.2.3. *Reidemeister 2b*

The complex associated to  is

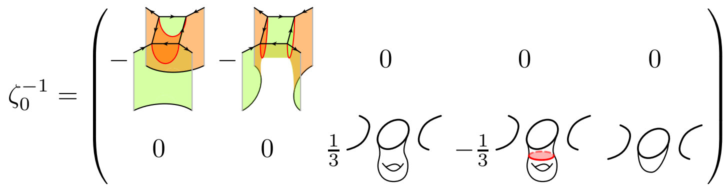


with differentials

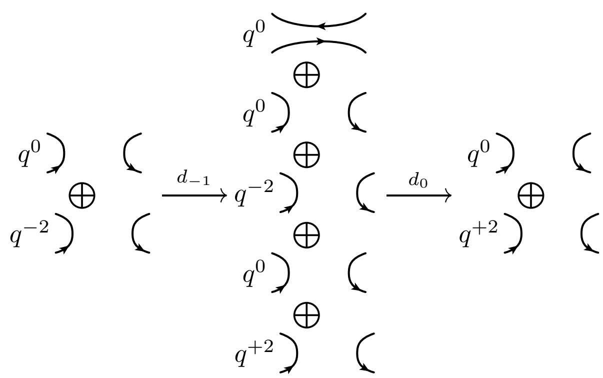
We now apply simplifying isomorphisms at each step (some identity sheets have been omitted in these diagrams):



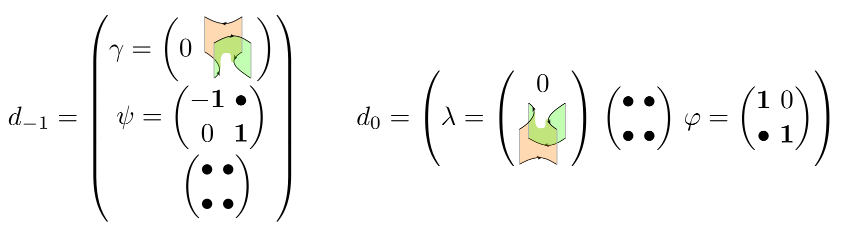
with inverses (which we’ll need later)



We thus obtain the complex

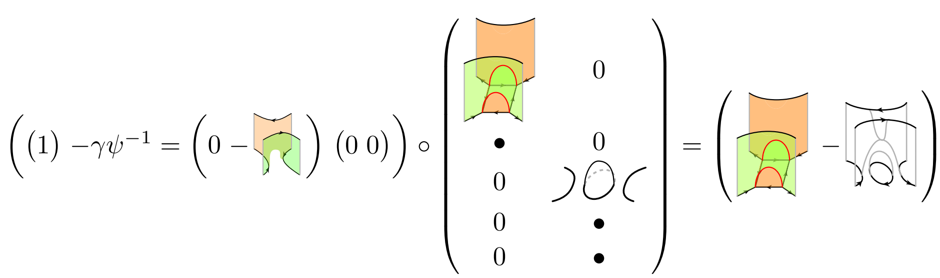


where

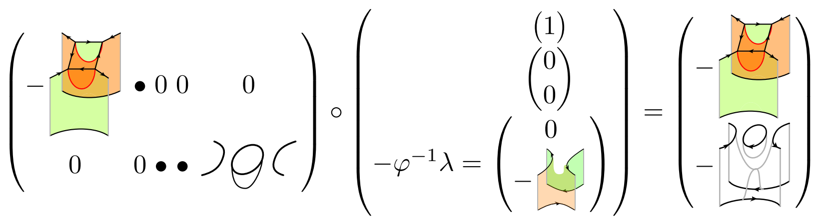


Quite a bit of cobordism arithmetic is hidden in this last step. For example, in calculating the coefficient of the saddle appearing γ, we used the 'bagel = 2' relation. As in the R2a moves above, we've named entries as in Lemma 4.2, and simply written `\bullet` for many matrix entries, because they won't matter in the computations to follow.

Thus the R2b 'untuck' chain map is



and the 'tuck' map is



4.2.4. *Reidemeister 3*

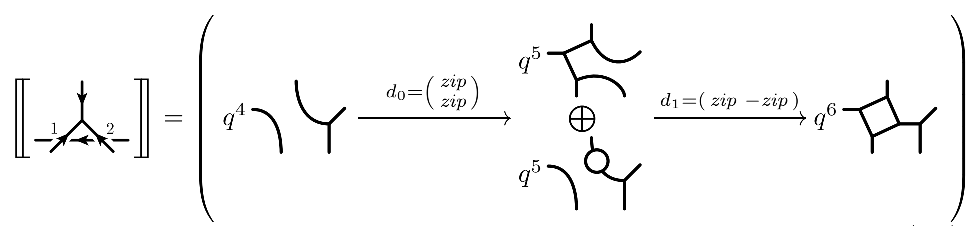
There are two almost equally appealing approaches to the third Reidemeister move. The first is to realize that the simplification algorithm is just as good as it is back in the $\mathfrak{su}_2$ setting:

Proof. [Proof modulo actually doing all the cobordism arithmetic!]Apply the simplification algorithm to the complex associated to either side of a particular variation of the third Reidemeister move, and observe that the results are identical. Thus the two complexes are homotopy equivalent. □

Remark. There's obviously some work to do here, calculating all the maps, identifying isomorphisms, writing down the homotopy equivalences provided by Lemma 4.1, and so on. The point is that this is all entirely algorithmic; it's an automatic proof, with no insight required.

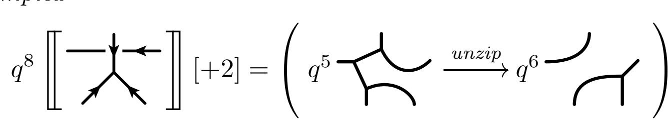
The second method is more conceptual; it allows no real savings in the calculations, but emphasizes that invariance under the third Reidemeister move is a consequence of the 'naturality' of the braiding in the category of complexes, described in the next two lemmas. We'll show most of the details.

Lemma 4.3. *Applying the simplification algorithm to the complex*



(4.2)

gives the complex



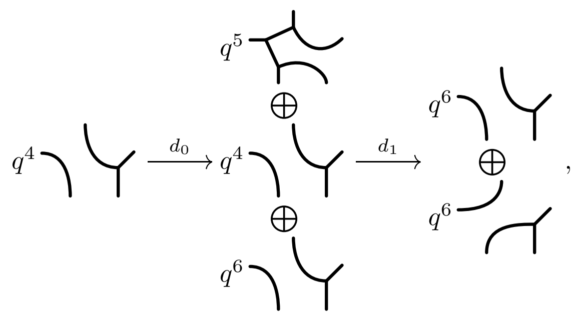
and the simplifying map is

$$s_0 = \begin{pmatrix} 0 \end{pmatrix} \qquad s_1 = \begin{pmatrix} 1 & -z \circ d \end{pmatrix} \qquad s_2 = \begin{pmatrix} r \end{pmatrix}.$$

Here d is the debubbling map, z is a zip map, and r is one of the 'half barrel' cobordisms in the 'rocket isomorphism'. You can work out exactly where all these maps are taking place simply by considering their source and target objects.

Remark. If you follow closely, you'll see we order the crossings so the first crossing is on the right, the second crossing is on the left. Without this, you might not like some of the signs appearing in the proof.

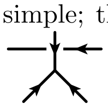
Proof. We begin with the complex in Equation (4.2) which, upon applying the simplifying isomorphisms from §3.4, becomes



with differentials

$$d_0 = \begin{pmatrix} \gamma = z \\ \psi = \mathbf{1} \\ \bullet \end{pmatrix}$$

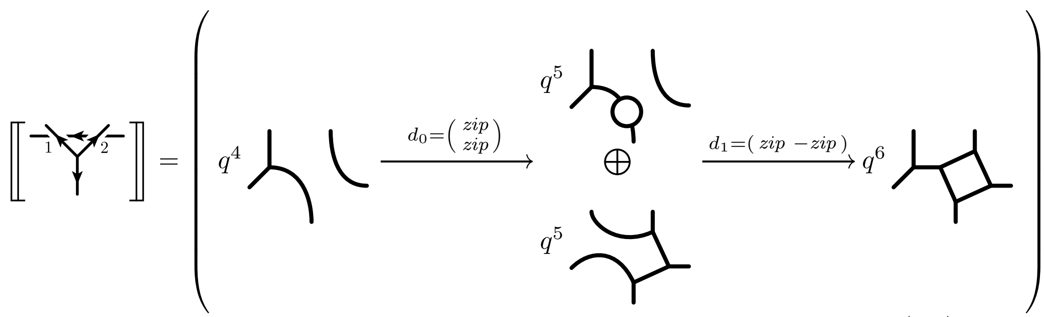
$$d_1 = \begin{pmatrix} \lambda = u \bullet & \phi = \mathbf{1} \\ \nu = u \bullet & \mu = 0 \end{pmatrix},$$

where z indicates a 'zip' map in the appropriate location, and u an 'unzip' map. Here we applied the airlock relation in calculating ϕ, and the blister relation in calculating μ. Notice here that $\mu = 0$, making the cancellation of the isomorphisms markedly simple; there's no error term. We thus obtain exactly the complex associated to , but shifted up in homological height by $+2$, and in grading by $+8$.

The simplifying map itself a composition of the simplifying isomorphisms followed by the homotopy equivalence killing off the contractible pieces. The homotopy equivalence is 0 at height 0, $\begin{pmatrix} -\gamma\psi^{-1} & 0 & 1 \end{pmatrix} = \begin{pmatrix} -z & 0 & 1 \end{pmatrix}$ at height 1, and the identity at height 2. Composing with the simplifying isomorphisms gives the map in the statement of this lemma. □

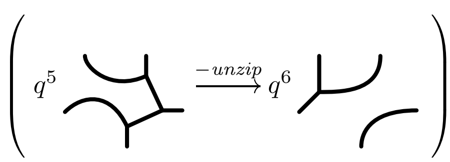
Analogously, we have the somewhat more awkward

Lemma 4.4. *The simplification algorithm provides a simple homotopy equivalence between the complex*



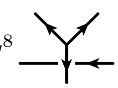
(4.3)

and the complex

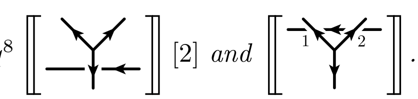


via the map

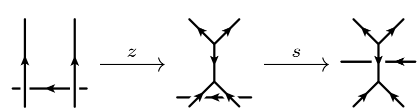
$$s'_0 = \left(0\right) \qquad s'_1 = \left(-z \circ d \;\; 1\right) \qquad s'_2 = \left(r\right).$$

This second complex isn't quite the complex associated to q^8  [2]*; the differential has been negated. Thus the map*

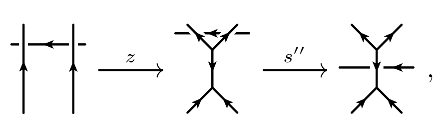
$$s''_0 = \left(0\right) \qquad s''_1 = \left(z \circ d \;\; -1\right) \qquad s''_2 = \left(r\right).$$

is a simple homotopy equivalence between q^8 .

Lemma 4.5. *The two compositions*



and



using the maps defined in the previous two lemmas, are equal.

Proof. Easy arithmetic (just in $\mathbb{Z}$, not even foam arithmetic). □

We now need a few facts about cones.

Definition 4.6. Given a chain map $f : A^\bullet \to B^\bullet$, the cone over f is $C(f)^\bullet = A^{\bullet+1} \oplus B^\bullet$, with differential

$$d_{C(f)} = \begin{pmatrix} d_A & 0 \\ f & -d_B \end{pmatrix}$$

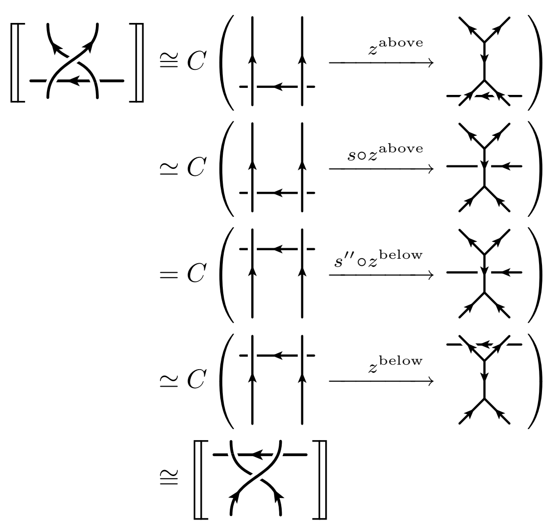
Lemma 4.7. *If $f : A^\bullet \to B^\bullet$ is a chain map, $r : B^\bullet \to C^\bullet$ is a simple homotopy equivalence throwing away contractible components (e.g. a simplification map, like those appearing above) and $\mathfrak{i} : C^\bullet \to B^\bullet$ is the inverse of r, then the cone $C(rf)$ is homotopic to the cone $C(f)$, via*

$$C(f)^\bullet = A^{\bullet+1} \oplus B^\bullet \underset{\left(\begin{smallmatrix} 1 & 0 \\ -hf & \mathfrak{i} \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} 1 & 0 \\ 0 & r \end{smallmatrix}\right)}{\rightleftarrows}} A^{\bullet+1} \oplus C^\bullet = C(rf)^\bullet$$

Remark. If instead $f : B^\bullet \to A^\bullet$, then the cone $C(f\mathfrak{i})$ is homotopic to $C(f)$ via

$$C(f)^\bullet = B^{\bullet+1} \oplus A^\bullet \underset{\left(\begin{smallmatrix} \mathfrak{i} & 0 \\ 0 & 1 \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} r & 0 \\ hf & 1 \end{smallmatrix}\right)}{\rightleftarrows}} C^{\bullet+1} \oplus A^\bullet = C(f\mathfrak{i})^\bullet$$

Together, the previous four lemmas provide a proof of invariance under one variation of the R3 move, via the categorified Kauffman trick.



The equality on the third line is simply Lemma 4.5.

The other R3 move requires similar calculations.

4.3. *Tangle cobordisms*

We've almost, but not quite, provided enough detail here to check that the $\mathfrak{su}_3$ cobordism theory is functorial on the nose, not just up to sign. The calculations for the third Reidemeister move would have to be made slightly more explicit, and then a great many movie moves (unfortunately, there are lots of different orientations to deal with!) need to be checked.

Conjecture 4.8. *The $\mathfrak{su}_3$ cobordism theory is functorial; in particular the sign problems seen in the $\mathfrak{su}_2$ case [1,8,4] don't occur.*

Remark. This conjecture has two sources of support. Firstly, the representation theoretic origin of the sign problem in $\mathfrak{su}_2$, namely that the standard representation is self-dual, but only *antisymmetrically* so, is simply irrelevant: the standard representation of $\mathfrak{su}_3$ isn't self-dual at all. Secondly, looking at §4.2.1, we see that the coefficients of the first and last terms of the 'unsimplifying' map for the first Reidemeister move are equal. This easily implies that the movie moves only involving the first Reidemeister move, MM12 and MM13 (in [1]'s numbering), come out right. These moves had already failed in the $\mathfrak{su}_2$ case.

5. Decategorification

5.1. *What is decategorification?*

As with quantization [5], while categorification is an art, decategorification is a functor; it's just a fancy name for taking the Grothendieck group[22]. Even so, our situation requires slightly unusual treatment.

Usually, given an abelian category, we would form the free $\mathbb{Z}$-module on the set of objects, and add one relation $A = B + C$ for every short exact sequence $0 \to B \to A \to C \to 0$.

In the cobordism categories we're interested in, there are no notions of kernels, images, or exactness. However, our categories still have direct sums, so we instead add relations $A = B + C$ whenever $A \cong B \oplus C$. You can think of the result as the 'split Grothendieck group', which still makes sense in this context.

It's easy to see that we can also decategorify a canopolis; starting with a planar algebra of categories, we obtain a planar algebra of $\mathbb{Z}$-modules.

When we decategorify a graded category, we remember the grading data and form a $\mathbb{Z}[q, q^{-1}]$-module instead of a $\mathbb{Z}$-module.

5.2. *A direct argument for $\mathfrak{su}_2$*

Our first result describes the decategorification of the Bar-Natan canopolis of $\mathfrak{su}_2$.

Definition 5.1. The Temperley-Lieb planar algebra $\mathcal{TL}$ is the free planar algebra of $\mathbb{Z}[q, q^{-1}]$-modules with no generators, modulo the relation $\bigcirc = q + q^{-1}$. (Its objects are $\mathbb{Z}[q, q^{-1}]$-linear combinations of planar tangle diagrams modulo that relation.) The planar algebra $\mathcal{TL}$ is isomorphic to the representation theory of $U_q(\mathfrak{sl}_2)$, or, more precisely, to the full subcategory with objects restricted to the standard representation, and tensor powers.

Theorem 5.2. *The (graded!) decategorification of the Bar-Natan canopolis* $\mathbf{Mat}\left(\mathcal{C}ob\left(\mathfrak{su}_2\right)\right)$ *is the Temperley-Lieb planar algebra.*

Proof. The argument splits into two parts.

The first half is easy. We must show that the relation $\bigcirc = q + q^{-1}$ holds in the decategorification of $\mathbf{Mat}\left(\mathcal{C}ob\left(\mathfrak{su}_2\right)\right)$; that is, $\bigcirc \cong q\,\emptyset \oplus q^{-1}\,\emptyset$ in $\mathbf{Mat}\left(\mathcal{C}ob\left(\mathfrak{su}_2\right)\right)$. This has already been done for us by [2].

Now for the other half. We need to show that there are no more relations in the decategorification than we one we've just seen.

Suppose we have some isomorphism $\phi : \oplus_D n_D D \cong \oplus_D n'_D D$, where each D is a non-elliptic diagram. We need to show that the multiplicities n_D and n'_D appearing on either side agree for each diagram D. Fix any particular diagram Δ, let

$$J = \bigoplus_{D \neq \Delta} n_D D$$

(J stands for 'junk'),

$$J' = \bigoplus_{D \neq \Delta} n'_D D,$$

and write both $\phi : n_\Delta \Delta \oplus J \to n'_\Delta \Delta \oplus J'$ and its inverse $\phi^{-1} : n'_\Delta \Delta \oplus J' \to n_\Delta \Delta \oplus J$ as 2×2 matrices:

$$\phi = \begin{pmatrix} \phi_{00} : n_\Delta \Delta \to n'_\Delta \Delta & \phi_{01} : J \to n'_\Delta \Delta \\ \phi_{10} : n_\Delta \Delta \to J' & \phi_{11} : J \to J' \end{pmatrix}$$

$$\phi^{-1} = \begin{pmatrix} \phi^{-1}_{00} : n'_\Delta \Delta \to n_\Delta \Delta & \phi^{-1}_{01} : J' \to n_\Delta \Delta \\ \phi^{-1}_{10} : n'_\Delta \Delta \to J & \phi^{-1}_{11} : J' \to J \end{pmatrix}$$

Looking at the top-left entry of the composition $\phi\phi^{-1}$, we see that $\phi_{00}\phi^{-1}_{00} + \phi_{01}\phi^{-1}_{10}$ must be the identity on $n_\Delta \Delta$. Notice that $\phi_{01}\phi^{-1}_{10}$ is a linear combination of endomorphisms of Δ, each of which factors through some non-elliptic object other than Δ. Therefore, by Corollary 2.3, their gradings are all strictly negative, so $\phi_{01}\phi^{-1}_{10}$ lives entirely in negative grading. Consequently, $\phi_{00}\phi^{-1}_{00}$ is equal to the identity, plus terms with strictly negative grading. By the same argument, $\phi^{-1}_{00}\phi_{00}$ has the same form. Furthermore, because all the entries of ϕ_{00} and ϕ^{-1}_{00} are in non-positive grading, we must have $\left(\phi_{00}\phi^{-1}_{00}\right)_0 = \left(\phi_{00}\right)_0 \left(\phi^{-1}_{00}\right)_0$ and $\left(\phi^{-1}_{00}\phi_{00}\right)_0 = \left(\phi^{-1}_{00}\right)_0 \left(\phi_{00}\right)_0$. (Here the final subscript 0 indicates the grading 0 piece.) Therefore, both $\left(\phi_{00}\right)_0 \left(\phi^{-1}_{00}\right)_0$ and $\left(\phi^{-1}_{00}\right)_0 \left(\phi_{00}\right)_0$ are identity matrices. By Corollary 2.3, the entries of $\left(\phi_{00}\right)_0$ and $\left(\phi^{-1}_{00}\right)_0$ are simply multiples of the identity on Δ. So these two matrices are, essentially, invertible matrices over R, and therefore square [23]! This gets us the desired result: $n_\Delta = n'_\Delta$. □

See also §10 of [1], on 'trace groups', for another way to recover the Temperley-Lieb planar algebra from this canopolis. (In fact, the construction there doesn't really start in the same place; it uses the pure cobordism category, whereas our decategorification only makes sense on the category of matrices over the cobordism category, where direct sum is defined.)

5.3. *... and why it doesn't work for* $\mathfrak{su}_3$

We wish to prove that the decategorification of $\mathbf{Mat}\left(\mathcal{C}ob\left(\mathfrak{su}_3\right)\right)$ is the $\mathfrak{su}_3$ spider: the planar algebra of webs modulo the relations in Equation 1.1.

A proof along the lines of the previous section won't work for the $\mathfrak{su}_3$ canopolis, simply because we have no guarantee that non-identity morphisms between non-elliptic diagrams are in negative degree. In fact, Theorem 5.3 below shows that this is false. Without this, we can't argue that (in the notation of the proof of Theorem 5.2) $\left(\phi_{00}\phi_{00}^{-1}\right)_0 = \left(\phi_{00}\right)_0\left(\phi_{00}^{-1}\right)_0$.

While we think it would be nice to have a proof of a $\mathfrak{su}_3$ decategorification statement purely in terms of the $\mathfrak{su}_3$ cobordism category, we'll fail at this for now, and instead describe in §5.4 a proof that relies on some $\mathfrak{su}_3$ representation theory.

We'll now show that both Corollary 2.2 and Corollary 2.3 describing the morphisms in the $\mathfrak{su}_2$ category fail in the $\mathfrak{su}_3$ category.

Theorem 5.3. *There are morphisms between non-elliptic objects in zero grading, and in arbitrarily large positive gradings.*

Proof. See Figure 5.1 for the first example of a grading zero cobordism between non-elliptic objects. We can easily count the total grading; going from the first frame to the second, we create 6 circles, for a grading of +12, and going from the third frame to the fourth we do 12 'zips', for a grading of −12.

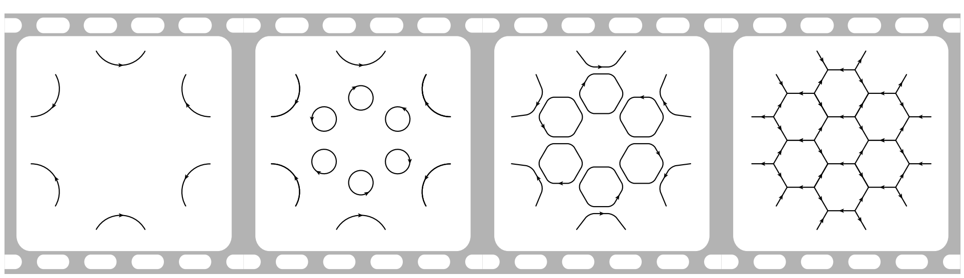


Fig. 5.1. The simplest example of a grading zero cobordism between non-elliptic objects which is not an identity cobordism.

Calling this cobordism x and the time-reversed version x^*, observe that x^*x is a (nonzero!) multiple of the identity on the initial frame of Figure 5.1 (and in particular, $x \neq 0$). This is an exercise in the repeated application of the 'bamboo' relation, and a few closed foam evaluations.

We leave the construction of positive grading morphisms as an exercise to the reader. (Hint: if you perform a sequence of zips which produce a non-elliptic diagram with some extra circles, then kill the resulting circles, the total grading is minus the Euler characteristic of the graph dual to the unzipped edges.) □

We'll return to the consequences of this phenomenon in §5.5.

5.4. *Nondegeneracy*

5.4.1. *Nondegeneracy for $\mathfrak{su}_2$*

Let $\mathcal{TL}_k$ denote the space of Temperley-Lieb diagrams with k endpoints, modulo the usual relation $\bigcirc = q + q^{-1}$. We define a symmetric $\mathbb{Z}[q, q^{-1}]$-bilinear pairing $\langle\, ,\, \rangle_{\mathfrak{su}_2} : \mathcal{TL}_k \times \mathcal{TL}_k \to \mathbb{Z}[q, q^{-1}]$ by gluing the k endpoints together, and evaluating the resulting closed diagram.

Proposition 5.4. *The pairing $\langle\, ,\, \rangle_{\mathfrak{su}_2}$ is non-degenerate on non-elliptic diagrams.*

The following argument first appeared in [14].

Proof. [Proof. 'Diagonal dominance' [21]] Fix k. We'll show that the determinant of the matrix for the pairing (with respect to the diagrammatic basis) is nonzero. This will follow easily from the fact that the term in the determinant corresponding to the product of the diagonal entries has strictly higher q-degree than any other term.

Each entry of the matrix is of the form $(q+q^{-1})^k$, where k is the number of loops formed when two basis diagrams are glued together. Pairing a diagram with itself produces strictly more loops than pairing it with any other diagram, and hence the highest value of k appearing in any row appears only on the diagonal. □

The main result of this section is that this pairing actually tells us the graded dimension of the space of morphisms between two particular (unshifted) diagrams in $\mathcal{C}ob\,(\mathfrak{su}_2)$.

Proposition 5.5. *For A and B in $\mathcal{TL}_k$, $\langle A, B\rangle_{\mathfrak{su}_2} = q^{\frac{k}{2}} \dim_q \operatorname{Hom}(A, B)$*

Proof. [The easy proof specific to $\mathfrak{su}_2$.]

First, note that $\langle A, B\rangle_{\mathfrak{su}_2} = (q + q^{-1})^l$, where l is the number of boundary components of $A \cup B \cup \partial \times [0, 1]$. By Proposition 2.1, the morphism space $\operatorname{Hom}(A, B)$ is generated by 2^l cobordisms consisting of l connected surfaces, each of which has Euler characteristic ± 1. The degree of such a cobordism is equal to $\chi(C) - k/2$, so $\dim_q \operatorname{Hom}(A, B) = (q + q^{-1})^l q^{-\frac{k}{2}}$, and the result follows. □

However, because we have no simple classification of morphisms in $\mathcal{C}ob\,(\mathfrak{su}_3)$, this argument does not apply to that case. We therefore give a second proof of Proposition 5.5, this one using geometric techniques that work equally well on foams.

Proof. [A proof that will generalize.]

Lemma 5.6 ($\mathfrak{su}_2$ Reduction lemma). *Suppose B contains a circle, and let $B^\bullet$ denote B with that circle removed. Then* $\dim_q \operatorname{Hom}(A, B) = (q + q^{-1}) \dim_q \operatorname{Hom}(A, B^\bullet)$, *and* $\langle A, B\rangle_{\mathfrak{su}_2} = (q + q^{-1}) \langle A, B^\bullet\rangle_{\mathfrak{su}_2}$. *The same result applies to removing a circle from A.*

Proof.
The first equality follows from the delooping isomorphism in [2], and the second from the definition of the Temperley-Lieb algebra. □

Lemma 5.7 ($\mathfrak{su}_2$ Shellback lemma). *Suppose B is non-elliptic and contains an arc α between two adjacent boundary points. Let B' denote B with α removed, and let A' denote A with the corresponding boundary points joined by an arc α'. (Note that $\partial A' = \partial B'$ has two fewer points than ∂A.) Then* $\dim_q \operatorname{Hom}(A, B) = q^{-1} \dim_q \operatorname{Hom}(A', B')$.

Proof. Although a direct argument using canopolis operations is possible, it is far easier to think of this operation as pulling α 'down the wall' of $A \cup B \cup \partial \times [0, 1]$. Because $A \cup B \cup \partial \times [0, 1]$ and $A' \cup B' \cup \partial \times [0, 1]$ are isotopic on the surface of the cylinder, there is an obvious induced isomorphism between $\operatorname{Hom}(A, B)$ and $\operatorname{Hom}(A', B')$. The only difference is in the gradings, which are shifted because of the change in number of boundary points. □

To prove Proposition 5.5, first observe that it holds when A and B are empty diagrams.

Assume that B is empty. Since ∂A is empty, A is a disjoint union of loops, and we can apply Lemma 5.6 repeatedly to reduce to the previous case.

Assume B is non-empty. Then either B contains a circle, or B contains an arc connecting adjacent boundary points. If it contains a circle, we apply Lemma 5.6. Otherwise, we apply Lemma 5.7. The result follows by induction on the number of edges in B.

We can extend this pairing to sums of diagrams:

$$\langle A, B + C\rangle_{\mathfrak{su}_2} = q^{\frac{k}{2}} \dim_q \operatorname{Hom}(A, B \oplus C).$$

(This is just observing that Hom respects direct sums.)

Together, Proposition 5.4 and Proposition 5.5 combine to yield a simple proof of Theorem 5.2. Essentially, knowing that the Hom pairing is nondegenerate on non-elliptic diagrams guarantees that there are no isomorphisms amongst non-elliptic diagrams:

Proof. [Alternate proof of Theorem 5.2] Suppose that $\oplus n_i D_i$ and $\oplus n'_i D_i$ are isomorphic objects in $\mathbf{Mat}(\mathcal{Cob}(\mathfrak{su}_2))$, with each D_i being a non-elliptic object. Then for any object C, $\dim_q \operatorname{Hom}(\oplus n_i D_i, C) = \dim_q \operatorname{Hom}(\oplus n_i' D_i, C)$. Therefore,

$$\left\langle \sum n_i D_i - \sum n_i' D_i, C \right\rangle_{\mathfrak{su}_2} = 0$$

and $\sum n_i D_i = \sum n_i' D_i$ in the Temperley-Lieb algebra. There are no relations amongst non-elliptic objects in the Temperley-Lieb planar algebra, and so $n_i = n_i'$ for each i. □

5.4.2. *Nondegeneracy for $\mathfrak{su}_3$*

We now have a new plan for a decategorification statement for $\mathfrak{su}_3$; prove an analogue of Proposition 5.5, prove an analogue of Proposition 5.4, and then follow the alternate proof of the $\mathfrak{su}_2$ decategorification statement given at the end of the previous section.

To this end, we define a pairing $\langle\, , \rangle_{\mathfrak{su}_3}$ on spider diagrams with identical boundary. Let $\langle A, B\rangle_{\mathfrak{su}_3}$ be the evaluation of the closed web resulting from reversing the orientations of A, then gluing A and B along their boundary. (This is $\langle A^*, B\rangle$ in the notation of §3.3.3.)

Proposition 5.8. *For spider diagrams A and B with boundary ∂, $\langle A, B\rangle_{\mathfrak{su}_3} = q^k \dim_q \operatorname{Hom}(A, B)$, where $k = |\partial|$.*

We'll need two lemmas first. (It might be helpful to recall the isomorphisms from Theorem 3.11 at this point.)

Lemma 5.9 ($\mathfrak{su}_3$ Reduction lemma).

Suppose B contains a circle, and let $B^\bullet$ denote B with that circle removed. Then $\dim_q \operatorname{Hom}(A, B) = (q^2 + 1 + q^{-2}) \dim_q \operatorname{Hom}(A, B^\bullet)$, and $\langle A, B\rangle_{\mathfrak{su}_3} = (q^2 + 1 + q^{-2}) \langle A, B^\bullet\rangle_{\mathfrak{su}_3}$.

Similarly, assume B contains a bigon, and let $B^!$ denote B with that bigon deleted and replaced by an edge. Then $\dim_q \operatorname{Hom}(A, B) = (q + q^{-1}) \dim_q \operatorname{Hom}\left(A, B^!\right)$, and $\langle A, B\rangle_{\mathfrak{su}_3} = (q + q^{-1}) \left\langle A, B^!\right\rangle_{\mathfrak{su}_3}$.

Lastly, suppose B contains a square, and let $B^\sharp$ and $B^\flat$ denote B with the two possible smoothings where opposite sides of the square are erased. Then $\dim_q \operatorname{Hom}(A, B) = \dim_q \operatorname{Hom}\left(A, B^\sharp \oplus B^\flat\right)$, and $\langle A, B\rangle_{\mathfrak{su}_3} = \left\langle A, B^\sharp + B^\flat\right\rangle_{\mathfrak{su}_3}$.

Analogous statements hold for A.

Proof. The equalities of morphism dimensions come directly from the isomorphisms in Theorem 3.11. The equalities of pairings are exactly Kuperberg's spider relations. □

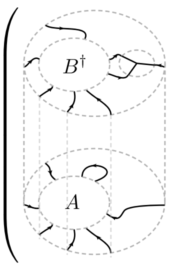
Lemma 5.10 ($\mathfrak{su}_3$ Shellback lemma).

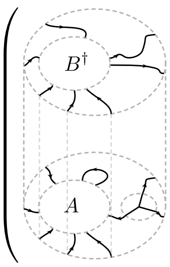
Suppose B is non-elliptic and contains an arc α between two adjacent boundary points. Let B' denote B with α removed, and let A' denote A with the *corresponding boundary points joined by an arc* α'*. Then* $\dim_q \mathrm{Hom}\,(A, B) = q^{-2} \dim_q \mathrm{Hom}\,(A', B')$*.*

Suppose B *has a trivalent vertex* v *with an edge* β *touching* ∂*. Let* $B^\dagger$ *denote* B *with* v *removed and the other edges of* v *now terminating at* ∂*. Let* $A^\dagger$ *denote* A *with an extra vertex* v' *added at the appropriate boundary point, and two edges connecting it to the boundary.*

Then $\dim_q \mathrm{Hom}\,(A, B) = q \dim_q \mathrm{Hom}\left(A^\dagger, B^\dagger\right)$*.*

A picture is worth far, far more than the words in the preceding paragraph:

$\dim_q \mathrm{Hom}$ 

$= q \dim_q \mathrm{Hom}$ 

Proof. The first statement is simply Lemma 5.7, modified to fit the grading on $\mathfrak{su}_3$ foams.

The second looks more frightening, but it is proved by exactly the same argument: $A \cup B \cup \partial \times [0, 1]$ and $A^\dagger \cup B^\dagger \cup \partial \times [0, 1]$ are isotopic on the surface of the cylinder, so 'dragging v down the wall' changes $\mathrm{Hom}\,(A, B)$ only by a grading shift. The power of q reflects that $\partial A^\dagger$ has one more point than ∂A. □

Thus armed, we have a

Proof. [Proof of Proposition 5.8] The proposition clearly holds when both A and B are empty diagrams.

Assume that B is empty. Then A is a closed web, and we can apply Lemma 5.9 repeatedly to reduce to the previous case.

Assume B is non-empty. If B contains a circle, bigon, or square, we apply Lemma 5.9. Otherwise, B has no closed components, and ∂B is non-empty. In this case, either we can find a trivalent vertex v adjacent to the boundary, or B is a disjoint union of arcs, and we can find an arc α connecting two adjacent boundary points. Either one will allow us to use Lemma 5.10. The result follows by induction on the number of edges in B. □

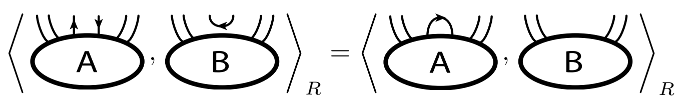
*Remark.*The geometrically-inclined reader may take the above nonsense with grading shifts as evidence that a canopolis is not the most natural setting for our seamed cobordisms. Indeed, we claim that their native habitat is a 'spatial algebra', a higher-dimensional variant of a planar algebra.

Proposition 5.11. *The pairing* $\langle\, , \rangle_{\mathfrak{su}_3}$ *is non-degenerate.*

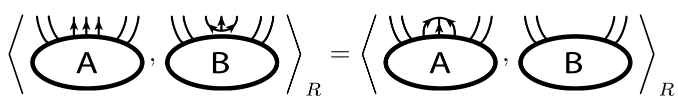
It suffices to prove nondegeneracy at $q = 1$, because this implies that it holds for generic q. The proof of this statement will require an equivalent algebraic definition of $\langle\, ,\, \rangle_{\mathfrak{su}_3}$. We can interpret any spider diagram with boundary ∂ as the set of invariant tensors in $V^{\otimes\partial}$, where V is the fundamental representation of $\mathfrak{su}_3$. There is a standard Hermitian inner product on V. If A and B are spider diagrams with identical boundary, let $\langle A, B\rangle_R$ denote the extension of this inner product to tensor products of V and V^*. Clearly $\langle\, ,\, \rangle_R$ is nondegenerate. It remains to show that $\langle\, ,\, \rangle_{\mathfrak{su}_3} = \langle\, ,\, \rangle_R$. We will proceed, as above, by induction on A and B.

First, if $\partial = \emptyset$, then the two pairings coincide by [15]. For dealing with nonempty boundaries, we prove the following lemma, which is most easily stated in pictures:

Lemma 5.12.



and



The corresponding statements with other orientations also hold, but we omit those calculations.

Here, only the middle parts of the diagrams are meant literally; the number of side strands is irrelevant. In a nutshell, this says that pieces of spider diagrams can be dragged between 'floor' and 'ceiling' without changing the value of $\langle\, ,\, \rangle_R$. Since we know this to be the case for $\langle\, ,\, \rangle_{\mathfrak{su}_3}$ by Lemma 5.10, the equality between $\langle\, ,\, \rangle_{\mathfrak{su}_3}$ at $q = 1$ and $\langle\, ,\, \rangle_R$ follows from this lemma by induction on the size of B.

Proof.

Translating pictures to symbols, the first statement says:

$\big\langle A, (\mathrm{id} \otimes$  $\otimes\, \mathrm{id}) \circ \mathrm{B}\big\rangle_R = \big\langle(\mathrm{id} \otimes$  $\otimes\, \mathrm{id}) \circ \mathrm{A}, \mathrm{B}\big\rangle_R$

and the second that

$\big\langle A, (\mathrm{id} \otimes$  $\otimes\, \mathrm{id}) \circ \mathrm{B}\big\rangle_R = \big\langle(\mathrm{id} \otimes$  $\otimes\, \mathrm{id}) \circ \mathrm{A}, \mathrm{B}\big\rangle_R$

Let $\{e_i\}$ be a basis for V and $\{f^i\}$ the dual basis. We write out these pictures explicitly:

 $= e_1 \otimes f_1 + e_2 \otimes f_2 + e_3 \otimes f_3$

 $= f_1 \otimes e_1 + f_2 \otimes e_2 + f_3 \otimes e_3$

 $= \sum_{\sigma \in S_3} (-1)^{\operatorname{sgn}(\sigma)} e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}$

 $= \sum_{\sigma \in S_3} (-1)^{\operatorname{sgn}(\sigma)} f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes f_{\sigma(3)}$

Then the lemma follows from the definition of the inner product: $\langle e_i, e_j \rangle_R = \delta_{ij} = \langle f_i, f_j \rangle_R$. □

Theorem 5.13. *The graded decategorification of the canopolis* $\mathbf{Mat}(\mathcal{C}ob(\mathfrak{su}_3))$ *is Kuperberg's* $\mathfrak{su}_3$ *spider.*

Remark. See the next section, however, for a conjecture which goes further.

Proof. Given Proposition 5.8, the alternate proof of Theorem 5.2 works *mutatis mutandis.* □

5.5. *The Karoubi envelope*

We now return to the example of a degree zero non-identity morphism from Theorem 5.3. Recall we had named the cobordism shown there in Figure 5.1 x, and x^* denoted its time reversal. We proved $x \neq 0$ by showing x^*x was a (nonzero!) multiple of the identity on the first frame.

Composing the other way round, xx^* is a (multiple of a) projection on the final frame of Figure 5.1. Normalizing correctly, let's call the projection p. This projection p certainly has an image in the foam category; just the initial frame. However, $1 - p$, while necessarily also being a projection, does not have an image. (For a projection $p^2 = p : \mathcal{O} \to \mathcal{O}$ in an arbitrary linear category, an image is pair of morphisms $r : \mathcal{O} \to \mathcal{O}'$ and $i : \mathcal{O}' \to \mathcal{O}$, such that $p = i \circ r$, and $i \circ r = \mathbf{1}_{\mathcal{O}'}$.) A clumsy way to see this is to compute the pairing matrix for all non-elliptic diagrams with the prescribed boundary; there's just a single pair of off diagonal entries with maximal q degree, corresponding via Proposition 5.8 to the maps r and i for the projection p, leaving no room for maps r and i for the projection $1 - p$.

We might suggest fixing this 'problem' by passing to the Karoubi envelope (see [3] and references therein) of the foam category, which artificially creates images for every projection. There, we can make a conjecture relating the minimal projections appearing in the foam category to the dual canonical basis.

Conjecture 5.14. *The Grothendieck group of* $\mathbf{Kar}(\mathcal{C}ob(\mathfrak{su}_3))$ *is the same as that of* $\mathcal{C}ob(\mathfrak{su}_3)$*, namely the* $\mathfrak{su}_3$ *spider.*

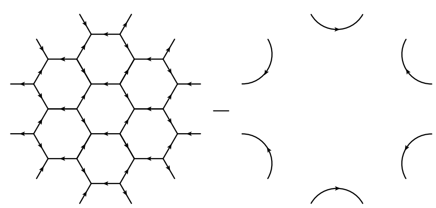
In particular, there is an ordering $\prec$ *of the objects of in* $\mathcal{C}ob(\mathfrak{su}_3)$ *(the ordering generated by 'cap' and 'unzip' will probably do), and a bijection between non-elliptic* *diagrams in $\mathcal{C}ob(\mathfrak{su}_3)$ and minimal idempotents in $\mathbf{Kar}(\mathcal{C}ob(\mathfrak{su}_3))$, $D \leftrightarrow p_D$ such that*

$$\mathbf{1}_D \cong p_D \oplus \bigoplus_i q^{n_i} p_{D_i}$$

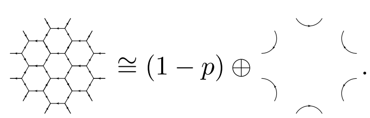
for some collection of diagrams $D_i \prec D$, and grading shifts n_i. Equivalently, when we write $\mathbf{1}_D$ as a sum of minimal projections, there is one 'new' projection, which we might think of as the 'leading term', plus 'old' projections, each equivalent to the new projection associated to some simpler diagram.

Conjecture 5.15. *Further, the basis for $\mathbf{Kar}(\mathcal{C}ob(\mathfrak{su}_3))$ coming from the minimal idempotents is the dual canonical basis of the $\mathfrak{su}_3$ spider.*

The immediate evidence for these conjectures is provided by the work of Khovanov and Kuperberg in [11]. There, they show that the first non-elliptic diagram which is not a dual canonical basis element is the final frame of the movie in Figure 5.1. Instead, in the space $\mathrm{Inv}\left(\left(V^{\otimes 2} \otimes V^{*\otimes 2}\right)^{\otimes 3}\right)$, they find that while 511 of the dual canonical basis vectors are given by non-elliptic diagrams, the 512-th is given by



This is exactly the behavior described by the conjectures above. Up until this point, every identity map on a non-elliptic diagram has been a minimal idempotent. However, in the Karoubi envelope, we have (identifying diagrams with their identity maps)



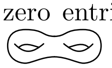
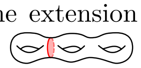
6. Calculations

Over $\mathbb{Q}$, at least, the $\mathfrak{su}_3$ invariant is completely computable for links.

Lemma 6.1. *For any link, there is a homotopy representative (in fact, a simple homotopy representative) of the associated complex in the category with objects being direct sums of graded empty diagrams and only the zero morphism.*

Proof. By applying isomorphisms, we can reduce the complex for a link to one in which the objects are all direct sums of graded empty diagrams. The morphisms are then matrices over $\mathbb{Q}$; any non-zero entry is invertible, and so there is an associated contractible direct summand, which we can remove using Lemma 4.1. □

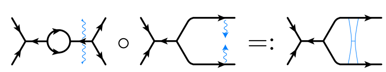
This essentially says that over $\mathbb{Q}$, the homotopy type of the invariant is characterized by its Poincaré polynomial, and that we lose nothing by having a topological rather than algebraic construction.

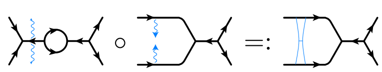
Over $\mathbb{Z}[\frac{1}{2}, \frac{1}{3}]$, it's more complicated; we can still reduce all objects to the empty diagram, but there may be 'integral torsion'; the differentials may still have non-zero entries. In the extension described in §A.2, in which we relax the relations  $= 0$ and  $= 0$, there may be further torsion associated to the polynomial ring generated by these two foams.

6.1. *The* $(2, n)$ *torus knots*

We now calculate the complex associated to the two strand braid σ^n, and from that the knot homology of its closure, the $(2, n)$ torus knot.

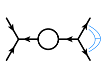
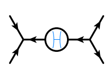
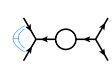
To begin, we introduce some notation for cobordisms,

$$\psi_R = \mathrm{unzip}_R \circ \mathrm{zip}_R =$$


$$\psi_L = \mathrm{unzip}_L \circ \mathrm{zip}_L =$$


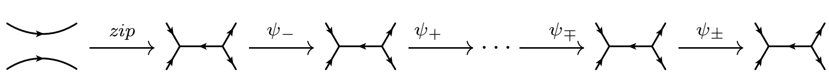
along with $\psi_\pm = \frac{1}{2}(\psi_R \pm \psi_L)$. These cobordisms satisfy some simple relations, namely that $\psi_R^2 = \psi_L^2 = 0$, by the double bagel relation from Equation (3.13), and $\psi_R\psi_L = \psi_L\psi_R$. As a consequence, $\psi_\pm\psi_\mp = 0$.

We'll further define, (harmlessly reusing names)

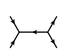
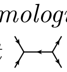
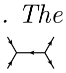
$\psi_R =$  $\psi_C =$  $\psi_L =$ .

We now calculate the complex associated to a 2-twist.

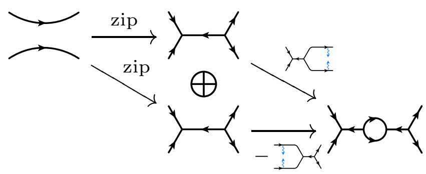
Theorem 6.2. *Assuming 2 is invertible, the invariant of* σ^n *is*

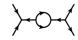


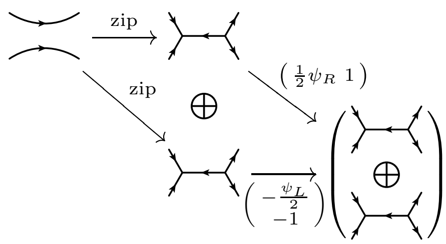
(6.1)

with *in homological height* 0*, and the final*  *in homological height* n*, so the final map is* $\psi_{(-1)^{n+1}}$*. The*  *is in grading* $2n$*, the first*  *in grading* $2n+1$*, and each subsequent*  *in grading 2 higher than the previous, so the last is in grading* $4n-1$*.*

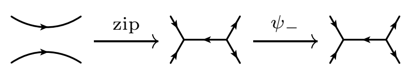
Proof. The proof is by induction on n. For $n = 1$, this complex is just the usual invariant of a positive crossing. For $n = 2$, we begin with the complex



(The sign appearing on the differential here is just the usual sign introduced by taking tensor products of complexes [6].) Reducing the object  using the debubbling isomorphism, we obtain

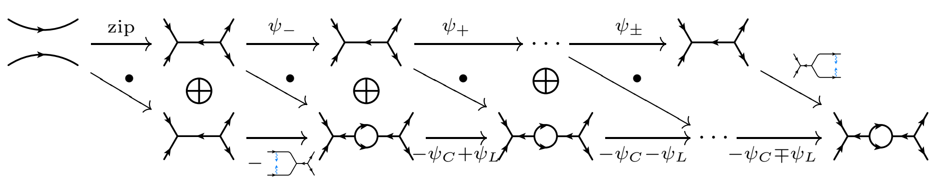


Cancelling off the matrix entry isomorphism -1 in the bottom row, using Lemma 4.1, we reach the desired complex

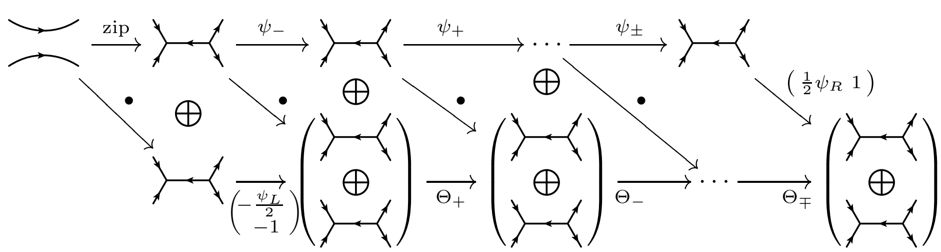


The second differential here, ψ_-, is calculated as $\frac{\psi_R}{2} - (-\frac{\psi_L}{2} \cdot (-1)^{-1} \cdot 1)$.

Now, suppose equation (6.1) holds for some $n \geq 2$. The argument is no more difficult than the $n = 2$ calculation we just did, but there's more to keep track of. To calculate **Foam** $\left(\sigma^{n+1}\right)$, we simply tensor the complex in Equation (6.1) with the two step complex for a positive crossing, producing



We now reduce every diagram in the complex with the debubbling isomorphism, obtaining

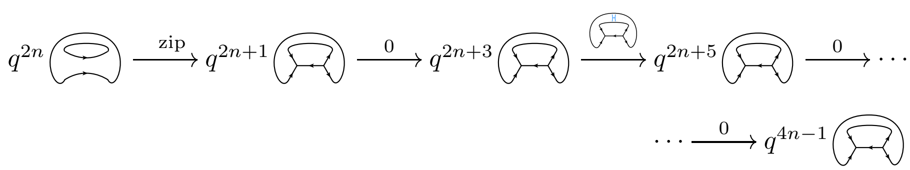


where $\Theta_\pm = \begin{pmatrix} \pm\frac{\psi_L}{2} & 0 \\ -1 & \pm\frac{\psi_L}{2} \end{pmatrix}$.

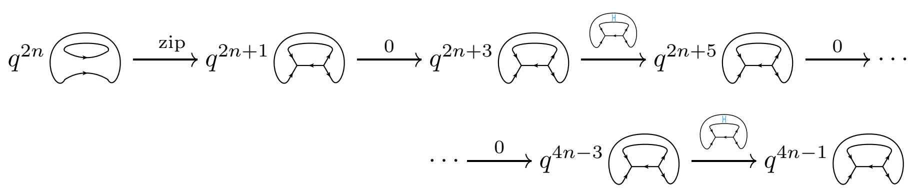
This complex contains many isomorphisms; we'll cancel off all the isomorphisms appearing as matrix entries on the horizontal arrows in the second row. This doesn't affect any of the original differentials in the first row because there are no differentials from the second row to the first. The only object in the second row that survives is the first summand at the highest homological level. The last differential is then $\frac{1}{2}\psi_R - (\mp\frac{\psi_L}{2} \cdot (-1)^{-1} \cdot 1) = \psi_\mp$, as claimed.

We leave it to the reader to check the gradings come out as claimed. □

It's now quite easy to compute the $\mathfrak{su}_3$ homology invariant for a $(2, n)$ torus knot; when we close up the braid σ^n, all the differentials ψ_- become zero, and we end up with

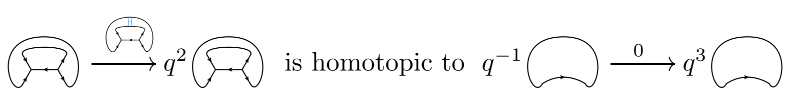


when n is even, or

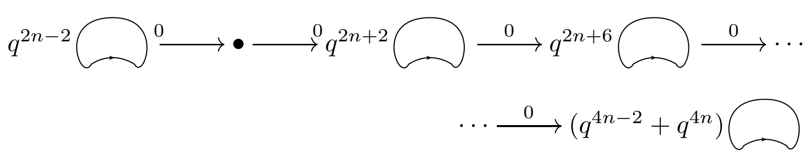


when n is odd.

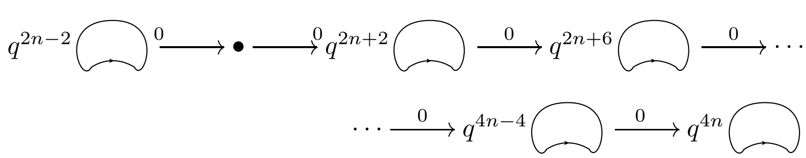
The complex 

, while the complex 

.
Making these replacements, we obtain the complexes



when n is even, or



when n is odd. (If you're paying careful attention to gradings, be extra careful here; notice that the grading on the first loop omitted by the ellipsis in the n even case is actually $2n+6$ again, not $2n+10$.)

The Poincaré polynomials are thus

$$(q^{-2}+1+q^2)q^{2n}\left(q^{-2}+(1+q^4t)(q^2t^2+q^6t^4+\cdots+q^{2n-6}t^{n-2})+(q^{2n-2}+q^{2n})t^n\right)$$

when n is even, and

$$(q^{-2}+1+q^2)q^{2n}\left(q^{-2}+(1+q^4t)(q^2t^2+q^6t^4+\cdots+q^{2n-4}t^{n-1})\right)$$

when n is odd.

The only other knot we've done calculations for is the 4_1 knot, whose $\mathfrak{su}_3$ Khovanov homology has Poincaré polynomial $(q^{-2}+1+q^2)(q^{-6}t^{-2}+q^{-2}t^{-1}+1+q^2t+q^6t^2)$.

Appendix A. This isn't quite the same as Khovanov or Mackaay-Vaz

There are three significant differences between the $\mathfrak{su}_3$ cobordism theory defined here, and the one defined by Khovanov in [10] and deformed by Mackaay and Vaz in [16]. (We assume familiarity with both of these papers throughout this section.)

The first is 'locality'. Our category is described by 'pictures modulo relations', rather than by a partition function. The knot invariant is explicitly local, defined as a map of planar algebras.

The second is that it's purely topological, in the sense that our cobordisms don't require any dots. As in the $\mathfrak{su}_2$ case, they aren't needed, and the 'sheet algebra' can be realized by topological objects.

The third is that its deformations, in the sense of Mackaay and Vaz, are also purely topological; instead of introducing three complex deformation parameters, we simply remove two relations setting certain closed foams to zero. There's a fair bit to explain here; why, by introducing only two closed foams we see everything they see with three deformation parameters, and the possibility of retaining a grading in the various degenerations of the $\mathfrak{su}_3$ theory.

A.1. *Locality*

Our local description of the foam category, using the canopolis formalism, has two principal advantages over the descriptions given in [10] and [16]. Firstly, as discussed previously in §4.1, we now have access to Bar-Natan's simplification algorithm, which allows for automatic proofs of Reidemeister invariance (§4.2), and explicit calculations (§6.1).

Secondly, we can give a clearer analysis of the different types of relations appearing the the theory.

Mackaay and Vaz begin by imposing certain relations on closed foams, sufficient for evaluation; in their notation, 3D, CN, S and Θ. In our language, their Definition 2.2 says that the category they are really interested in is the quotient by the local kernel of the category with closed webs. (Recall the appropriate definitions from §3.3.3.)

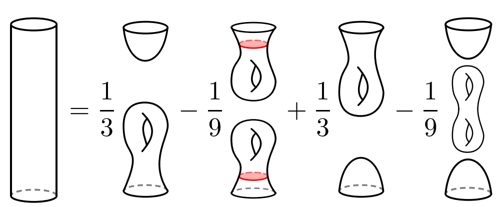
Following this definition, they derive certain relations, in Lemma 2.3. We'd like to emphasize that these relations are actually of two quite different natures. The first two, 4C (which we don't use) and RD (our 'bamboo' relation), are actually in the canopolis ideal generated by the 'evaluation' relations. On the other hand, the last two, DR and SqR (our tube and rocket relations), cannot be derived from the evaluation relations by canopolis operations, but only appear in the local kernel. Moreover, while pointing out some relations coming from the local kernel, they have no analogue of our Lemma 3.5, providing generators of the local kernel. Indeed, without a local setup, in which we can describe the local kernel as a 'canopolis ideal', it seems impossible to do this.

A.2. *Relaxing our relations*

In this section, we describe a slight generalization of our canopolis, in which we no longer impose the relations

but instead absorb these closed foams into the ground ring, calling them α and β respectively. These foams have grading -4 and -6 respectively. This change requires modifications to several subsequent parts of the paper.

The neck cutting relation gains an extra term[r]

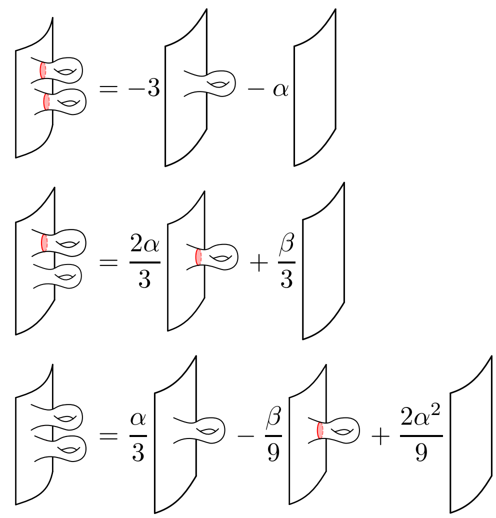


(A.1)

Consequently, there are extra terms in the sheet algebra relations, (compare Equa-

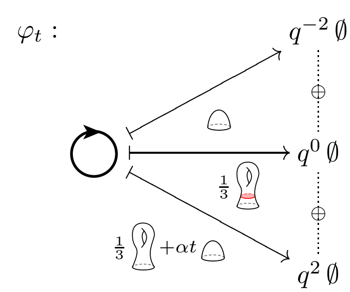
[r]The new neck cutting relation may be derived just as in §3.3.1, being slightly more careful about the dimensions of the various morphism spaces, taking into account the fact that the coefficient ring is no longer all in grading zero.

tions (3.7) and (3.8))

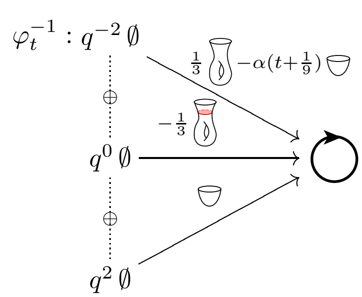


although pleasantly there are no other changes to the local relations! These relations give the $\mathfrak{su}_3$ analogue of Corollary 2.4.

The isomorphisms of Theorem 3.11 mostly survive unchanged, except the delooping isomorphism.[s] Now, somewhat strangely, we have a family of isomorphisms, indexed by a parameter $t \in S$ defined by



and

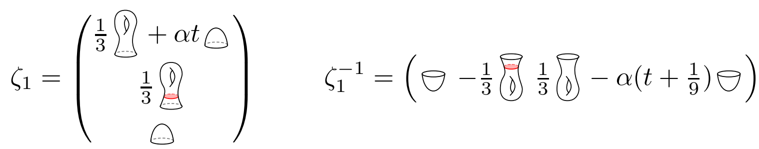


[s]The authors of [16] don't describe a delooping isomorphism.

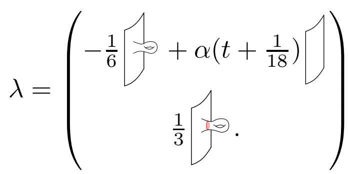
It's just as easy as it was before to check that this is an isomorphism.

Next, we turn to the isotopy invariance proofs, and check for any use of the delooping isomorphism, or the affected relations. Both Reidemeister 1 and Reidemeister 2b made use of the delooping isomorphism to simplify the complexes; it turns out that the calculation of Reidemeister 2b remains independent of which φ_t delooping isomorphism we use, and the chain homotopy we produce at the end is unchanged.

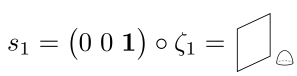
The Reidemeister 1 calculation is slightly more interesting. Using φ_t as the delooping isomorphism, we need to modify that calculation as follows. The isomorphisms become



and so in the differential in the simplified complex we see

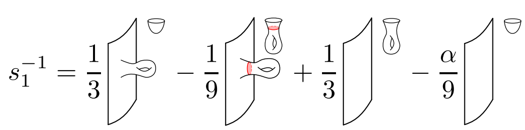


Finally then, the simplifying maps acquire an extra term,

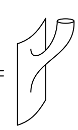


$$s_2 = 0$$

but the inverse chain homotopy acquires an extra term



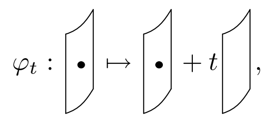
$$s_2^{-1} = 0.$$

Notice, however, that it is still the case that $s_1^{-1} =$  .

A.3. *Dots and deformation parameters*

First, recall the definition of Mackaay and Vaz of $\mathbf{Foam}_{/l}(a, b, c)$ (we've added the explicit notational dependence on a, b and c here).

There is an action of the ground ring on the collection of cobordism categories $\mathbf{Foam}_{/l}(a,b,c)$ by category equivalences:

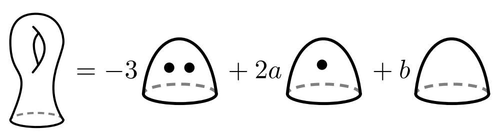


taking $\mathbf{Foam}_{/l}(a,b,c)$ to $\mathbf{Foam}_{/l}(a-3t, b+2at-3t^2, c+bt+at^2-t^3)$. It's easy to see that $\varphi_{-t}\circ\varphi_t = \mathbf{1}$, and that these maps preserve the associated filtration on the categories, but not the grading.

In the case that a, b and c are complex numbers (so the grading is already lost), it's then easy to see that $\mathbf{Foam}_{/l}(a,b,c)$ is isomorphic to $\mathbf{Foam}_{/l}(0, b+\frac{a^2}{3}, c+\frac{ab}{3}+\frac{2a^3}{27})$, and hence we need only consider the $a=0$ case.

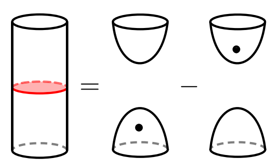
We now turn to showing that the dots appearing in the foams described by Khovanov, and by Mackaay and Vaz, have 'topological representatives'. Moreover, two out of the three 'deformation parameters' in Mackaay and Vaz's paper, b and c, also have topological representatives.

We begin by evaluating a punctured torus in the Mackaay-Vaz theory by neck cutting.

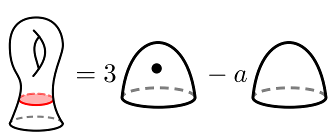


(A.2)

Next, we use the Mackaay-Vaz 'bamboo' relation,[t]

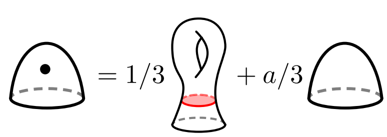


to evaluate the choking torus



(A.3)

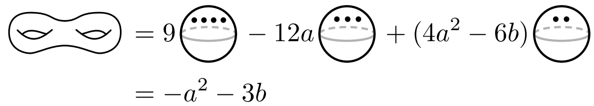
and thus



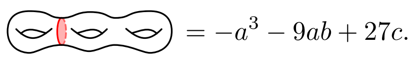
Using this, we can write any cobordism involving dots as a $\mathbb{Z}[\frac{1}{3}][a]$-linear combination of cobordisms without dots.

[t]The cyclic orientation here is lower cylinder/upper cylinder/disc.

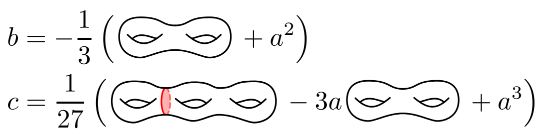
What about the parameters a,b,c? Using Equation (A.2), we obtain



and along with Equation (A.3),

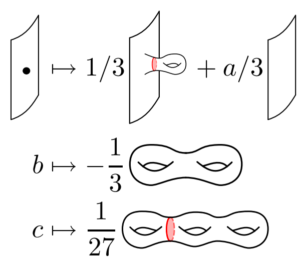


Rearranging these, we can express the deformation parameters b and c in terms of a and some closed foams.



In particular, in the special case $a = 0$, we can entirely replace the deformation parameters with closed foams.

We can now explicitly describe the correspondence between our theory and that of Mackaay and Vaz. At the level of closed spider diagrams, [u] our cobordism category is equivalent to theirs at $a = 0$, via the map



The inverse map is just inclusion; checking they're inverses involves a little cobordism arithmetic in each setup.

References

[1] Dror Bar-Natan, *Khovanov's homology for tangles and cobordisms*, Geom. Topol. **9** (2005), 1443–1499 (electronic), `arXiv:math.GT/0410495`. MR 2174270

[2] ———, *Fast Khovanov homology computations*, J. Knot Theory Ramifications **16** (2007), no. 3, 243–255, `arXiv:math.GT/0606318 MR2320156`. MR MR2320156

[3] Dror Bar-Natan and Scott Morrison, *The Karoubi envelope and Lee's degeneration of Khovanov homology*, Algebr. Geom. Topol. **6** (2006), 1459–1469 (electronic), `arXiv:math.GT/0606542 MR2253455`. MR 2253455

[4] David Clark, Scott Morrison, and Kevin Walker, *Fixing the functoriality of Khovanov homology*, `arXiv:math.GT/0701339`.

[u] We only say this because Mackaay and Vaz don't explicitly describe a formalism for open diagrams.

[5] Pavel Etingof and David Kazhdan, *Quantization of Lie bialgebras. II*, Selecta Math. (N.S.) **4** (1998), no. 2, 213–231, 233–269, `arXiv:q-alg/9701038`. MR 1669953 (2000i:17033)

[6] Sergei I. Gelfand and Yuri I. Manin, *Methods of homological algebra*, Springer-Verlag, Berlin, 1996, Translated from the 1988 Russian original. MR 1438306 (97j:18001)

[7] Jeremey Green, *JavaKh*, `http://katlas.math.toronto.edu/wiki/Khovanov_Homology`.

[8] Magnus Jacobsson, *An invariant of link cobordisms from Khovanov homology*, Algebr. Geom. Topol. **4** (2004), 1211–1251, `arXiv:math.GT/0206303`.

[9] Vaughan F. R. Jones, *Planar algebras, I*, `arXiv:math.QA/9909027`.

[10] Mikhail Khovanov, *sl(3) link homology*, Algebr. Geom. Topol. **4** (2004), 1045–1081 (electronic), `arXiv:math.QA/0304375`. MR 2100691 (2005g:57032)

[11] Mikhail Khovanov and Greg Kuperberg, *Web bases for* sl(3) *are not dual canonical*, Pacific J. Math. **188** (1999), no. 1, 129–153, `arXiv:q-alg/9712046`. MR 1680395 (2000j:17023a)

[12] Mikhail Khovanov and Lev Rozansky, *Matrix factorizations and link homology*, `arXiv:math.QA/0401268`.

[13] ———, *Matrix factorizations and link homology II*, `arXiv:math.QA/0505056`.

[14] Ki Hyoung Ko and Lawrence Smolinsky, *A combinatorial matrix in* 3*-manifold theory*, Pacific J. Math. **149** (1991), no. 2, 319–336. MR 1105701 (92d:57008)

[15] Greg Kuperberg, *Spiders for rank* 2 *Lie algebras*, Comm. Math. Phys. **180** (1996), no. 1, 109–151, `arXiv:q-alg/9712003`. MR 1403861 (97f:17005)

[16] Marco Mackaay and Pedro Vaz, *The universal sl3-link homology*, `arXiv:math.GT/0603307`.

[17] Gad Naot, *The universal Khovanov link homology theory*, Algebr. Geom. Topol. **6** (2006), 1863–1892 (electronic), `arXiv:math.GT/0603347 MR2263052`. MR MR2263052 (2007k:57022)

[18] Juan Ariel Ortiz-Navarro and Chris Truman, *Khovanov Homology and Reidemeister Torsion*, a talk at the 2006 Toronto CMS meeting, slides at `http://www.math.uiowa.edu/~jortizna/Present-CMS-06.pdf`.

[19] Dylan P. Thurston, *The algebra of knotted trivalent graphs and Turaev's shadow world*, Invariants of knots and 3-manifolds (Kyoto, 2001), Geom. Topol. Monogr., vol. 4, Geom. Topol. Publ., Coventry, 2002, `arXiv:math.GT/0311458`, pp. 337–362 (electronic). MR 2048108 (2005c:57010)

[20] Ben Webster, *Khovanov-Rozansky homology via a canopolis formalism*, Algebr. Geom. Topol. **7** (2007), 673–699, `arXiv:math.GT/0610650 MR2308960`. MR MR2308960

[21] Wikipedia, *Diagonally dominant matrix — wikipedia, the free encyclopedia*, 2006, [Online; accessed 26-December-2006].

[22] ———, *Grothendieck group — wikipedia, the free encyclopedia*, 2006, [Online; accessed 30-June-2006].

[23] ———, *Invariant basis number — wikipedia, the free encyclopedia*, 2006, [Online; accessed 24-December-2006].

This paper is available online at arXiv:`arXiv:math.GT/0612754`, and at `http://tqft.net/su3`.