This thesis provides a partial answer to a question posed by Greg Kuperberg in [13] and again by Justin Roberts as problem 12.18 in *Problems on invariants of knots and 3-manifolds* [16], essentially:

> Can one describe the category of representations of the quantum group $U_q(\mathfrak{sl}_n)$ (thought of as a spherical category) via generators and relations?

An obvious generalisation of the question replaces $U_q(\mathfrak{sl}_n)$ with an arbitrary quantum group.

Spherical categories (essentially tensor categories with duals) permit a diagrammatic calculus, in which we can modify diagrams by arbitrary planar isotopies. Understanding such a category in terms of generators and relations ensures that all identities are 'local'. The original motivation for wanting such a local diagrammatic calculus came from the skein theories for quantum knot invariants; it turns out that many of the examples of tensor categories with duals (and all those coming from quantum groups) are also braided categories, and so naturally provide invariants of knots and tangles. Local rules for simplifying diagrams allow 'divide and conquer' style calculations of these invariants — we can simplify small subtangles of a larger tangle before gluing the pieces together. More recently, a new motivation has been discovered, coming from physics and topological quantum computing, which I'll discuss later.

Answers to special cases of the generators and relations question are known. The Temperley-Lieb category gives a generators and relations description of the representation theory of $U_q(\mathfrak{sl}_2)$. Kuperberg's results in [13] give a diagrammatic presentation of the representations theories of the three rank 2 complex Lie algebras $\mathfrak{sl}_3, \mathfrak{so}_5$ and $\mathfrak{g}_2$, and their corresponding quantum groups. In fact, a computer implementation of the 'divide and conquer' method of calculating the quantum knot invariants for the rank 2 quantum groups is available as part of the `KnotTheory`‘ package available at [2].

For each $n \geq 0$, I define a certain tensor category of trivalent graphs, modulo isotopy, and construct a functor from this category onto the category of representations of the quantum group $U_q(\mathfrak{sl}_n)$. (Actually, the functor will only be onto the full subcategory of tensor products of fundamental representations, which is still 'big enough' in the sense that every irreducible representation appears as a subobject of such a tensor product.) One would like to describe completely the kernel of this functor, by providing generators. The resulting quotient of the diagrammatic category would then be a category equivalent to the representation category of $U_q(\mathfrak{sl}_n)$.

I make significant progress towards this, describing certain elements of the kernel, and some obstructions to further elements. It remains a conjecture that these elements really generate the kernel. The argument is essentially the following. Take some trivalent graph in the diagrammatic category for some value of n, and consider the morphism of $U_q(\mathfrak{sl}_n)$ representations it is sent to. Forgetting the full action of $U_q(\mathfrak{sl}_n)$, keeping only a $U_q(\mathfrak{sl}_{n-1})$ action, the source and target representations branch into direct sums, and the morphism becomes a matrix of maps of $U_q(\mathfrak{sl}_{n-1})$ representations. Arguing inductively now, we attempt to write each such matrix entry as a linear combination of diagrams for $n-1$. This gives a functor $d\mathcal{GT}$ between diagrammatic categories, realising the forgetful functor at the representation theory level. Now, if a certain linear combination of diagrams for n is to be in the kernel of the representation functor, each matrix entry of $d\mathcal{GT}$

applied to that linear combination must already be in the kernel of the representation functor one level down. This allows us to perform inductive calculations, both establishing families of elements of the kernel, and finding obstructions to other linear combinations being in the kernel.

The results here recover the relations proposed to $n = 4$ in [11], and provide some further evidence for a conjecture there that there are no further relations in this case. One of the two interesting families of relations I describe have appeared previously; proposed without proof in [15], and proved to be relations in [6]. My results providing obstructions actually contradict a conjecture appearing in [6].

Inspired by the 'topological states' apparently explaining the fractional quantum Hall effect (in high magnetic field, low temperature, two-dimensional electron gases), various people [3, 12, 17] have proposed building real systems based on lattice models whose physics are described by the tensor category $\mathcal{D}$ of representations of some quantum group at a root of unity. More precisely, the quantum states of the system should form a representation of the category $\mathcal{D}$; objects in $\mathcal{D}$ should specify 'boundary conditions', so there is a Hilbert space for each object, and we should be able to act on the system via morphisms in $\mathcal{D}$. The ground states with boundary conditions $\mathcal{O}$ should be identifiable with $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{O})$. In such a system, one would have an underlying microscopic medium, generally a 2-dimensional lattice, with each site having some finite dimensional space of quantum states. Then some rule allows you to interpret a basis state for the entire medium as a diagram. The simplest example might a square lattice, with the space of states for each edge being $\mathbb{C}^2$. The obvious basis is then in bijection with the set of collections of open paths and closed loops on the lattice. These diagrams generally won't look like the allowable diagrams in the tensor category $\mathcal{D}$ we're trying to engineer — for example Schur's lemma guarantees that, in the category of representations of a quantum group, we should never see diagrams with univalent vertices, because there are no maps from the trivial representation to any other irreducible. We might imagine then turning on some Hamiltonian which adds an energy penalty to basis states corresponding to disallowed diagrams. The (highly degenerate) ground states of such a system are then arbitrary complex linear combinations of particular lattice-embeddings of diagrams from $\mathcal{D}$. We then imagine turning on further terms in our Hamiltonian, which firstly enforce isotopy invariance (that is, which add an energy penalty to states which give different coefficients to isotopic diagrams) and secondly enforce any identities in our category $\mathcal{D}$.

Actually implementing such a device promises to be extremely difficult. Any physically practicable Hamiltonian must act locally; for example by nearest neighbour or next-to-nearest neighbour interactions. However those Hamiltonians which have been studied so far and are thought to result in 'topological phases', that is, systems as described above, result in uninteresting ones (such as $\mathbb{Z}/2\mathbb{Z}$ gauge theory) in which the braiding is trivial. On the other hand, there are rigourous statements to the effect that certain local (although extraordinarily difficult to engineer) Hamiltonians do result in interesting topological phases [4, 5]. This situation makes it extremely desirable to obtain local descriptions of the small examples of braided tensor categories; the locality of the description is essential, because the Hamiltonian which implements the relations must itself be local. (It's perhaps worth saying that nothing in my thesis addresses braidings or roots of unity phenomena, both of which are essential for the physical picture.) This thesis can then be thought of as the beginning of this sort of description of the braided tensor categories coming from $U_q(\mathfrak{sl}_n)$ for all n. Of course, only the small values of n are at all likely to be interesting to physicists! So far their interest has concentrated on simple cases such as $U_q(\mathfrak{sl}_2)$ at 4th and 5th roots of unity.

This thesis is certainly only an intermediate step. It remains to find an argument that the proposed relations really are all of them. Formulas for the braided structure essentially appear in the work of [15], but need translating into my conventions. Diagrammatic formulas for the inclusions of arbitrary irreducibles into tensor products of fundamental representations are known at $n = 3$, thanks to [11], but need to be worked out beyond that. These are essential to describing the quotients of the category at roots of unity. Finally, one might try to categorify everything in sight, following [7, 1] for $n = 2$, and [8] and my work with Ari Nieh [14] for $n = 3$, hoping to find a topological alternative to the matrix factorization model [9, 10] of Khovanov-Rozansky homology.

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