A Diagrammatic Category for the Representation Theory of $U_q(\mathfrak{sl}_n)$

by

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Professor Vaughan Jones, Chair
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Professor Steve Evans

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Abstract

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This thesis provides a partial answer to a question posed by Greg Kuperberg in [24] and again by Justin Roberts as problem 12.18 in *Problems on invariants of knots and 3-manifolds* [28], essentially:

> Can we describe the category of representations of the quantum group $U_q(\mathfrak{sl}_n)$ (thinking of it as a spherical category) via generators and relations?

For each $n \geq 0$, I define a certain tensor category of trivalent graphs, modulo isotopy, and construct a functor from this category onto (a full subcategory of) the category of representations of the quantum group $U_q(\mathfrak{sl}_n)$. One would like to describe completely the kernel of this functor, by providing generators for the tensor category ideal. The resulting quotient of the diagrammatic category would then be a category equivalent to the representation category of $U_q(\mathfrak{sl}_n)$.

I make significant progress towards this, describing certain elements of the kernel, and some obstructions to further elements. It remains a conjecture that these elements really generate the kernel. The argument is essentially the following. Take some trivalent graph in the diagrammatic category for some value of n, and consider the morphism of $U_q(\mathfrak{sl}_n)$ representations it is sent too. Forgetting the full action of $U_q(\mathfrak{sl}_n)$, keeping only a $U_q(\mathfrak{sl}_{n-1})$ action, the source and target representations branch into direct sums, and the morphism becomes a matrix of maps of $U_q(\mathfrak{sl}_{n-1})$ representations. Arguing inductively now, we attempt to write each such matrix entry as a linear combination of diagrams for $n-1$. This gives a functor $d\mathcal{GT}$ between diagrammatic categories, realising the forgetful functor at the representation theory level. Now, if a certain linear combination of diagrams for n is to be in the kernel of the representation functor, each matrix entry of $d\mathcal{GT}$ applied to that linear combination must already be in the kernel of the representation functor one level down. This allows us to perform inductive calculations, both establishing families of elements of the kernel, and finding obstructions to other linear combinations being in the kernel.

This thesis is available electronically from the arXiv, at `arXiv:0704.1503`, and at `http://tqft.net/thesis`.

Contents

Acknowledgements

First I'd like to thank Vaughan for being such a great advisor. I'm glad he took a chance on me, and I hope he's not too disappointed by the complete absence of subfactors in what follows! I'm grateful for so many things; good advice, lots of mathematics, constant interest and encouragement, gentle reminders to keep climbing and work in balance, windsurfing lessons, and a great friendship.

Thanks also to Dror Bar-Natan and Kevin Walker. I've learnt a ton from each of them, thoroughly enjoyed working with them, and look forward to more! And further thanks to Greg Kuperberg, for introducing me to the subject this thesis treats, and taking interest in my work. Conversations with Joel Kamnitzer, Mikhail Khovanov, Ari Nieh, Ben Webster, Noah Snyder and Justin Roberts helped me along the way.

Thanks to my family, Pam, Graham and Adele Morrison, for endless love and support. And finally, thanks to my friends Nina White, Yossi Farjoun, Rahel Wachs, Erica Mikesh, and Carl Mautner.

Chapter 1

Introduction

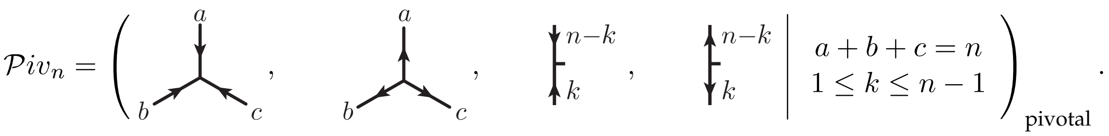
1.1 Summary

The eventual goal is to provide a diagrammatic presentation of the representation theory of $U_q(\mathfrak{sl}_n)$. The present work describes a category of diagrams, along with certain relations amongst these diagrams, and a functor from these diagrams to the representation theory. Further, I conjecture that the relations given are in fact all of them—that this functor is an equivalence of categories.

We begin by defining a 'freely generated category of diagrams', and show that there's a well-defined functor from this category to the category of representations of $U_q(\mathfrak{sl}_n)$. Essentially, this is a matter of realising that the representation category is a *pivotal category*, and, as a pivotal category, it is finitely generated. It's then a matter of trying to find the kernel of this functor; if we could do this, the quotient by the kernel would give the desired diagrammatic category equivalent to the representation category.

This work extends Kuperberg's work [24] on $\mathbf{Rep}\, U_q(\mathfrak{sl}_3)$, and agrees with a previously conjectured [21] description of $\mathbf{Rep}\, U_q(\mathfrak{sl}_4)$. Some, but not all, of the relations have been presented previously in the context of the quantum link invariants [11, 27]. There's a detailed discussion of connections with previous work in §6.

For each $n \geq 0$ we have a category of (linear combinations of) diagrams



with edges labelled by integers 1 through $n-1$, generated by two types of trivalent vertices, and orientation reversing 'tags', as shown above. We allow arbitrary planar isotopies of the diagrams. This category has no relations; it is a free pivotal category.

We can construct a functor from this category into the representation theory

$$\mathbf{Rep}_n : \mathcal{P}iv_n \to \mathbf{Rep}\, U_q(\mathfrak{sl}_n)\,.$$

This functor is well-defined, in that isotopic diagrams give the same maps between representations. Our primary goal is thus to understand this functor, and to answer two questions:

1. Is $\mathbf{Rep}_n$ full? That is, do we obtain all morphisms between representations?
2. What is the kernel of $\mathbf{Rep}_n$? When do different diagrams give the same maps of representations? Can we describe a diagrammatic quotient category which is equivalent to the representation theory?

The first question has a relatively straightforward answer. Simply because we do not hit every object, we can not get all of $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$, but if we lower our expectations to the subcategory containing only the fundamental representations, and their tensor products, then the functor is in fact full. Kuperberg gave a proof of this fact for $n = 3$, by recognising the image of the functor using a Tannaka-Krein type theorem. This argument continues to work with only slight modifications for all n. I'll also give a direct proof using quantum Schur-Weyl duality, in §3.5.

The second question has proved more difficult. Partial answers have been known for some time. I will describe a new method for discovering elements of the kernel, based on branching. This method also gives us a limited ability to find obstructions for further relations.

The core of the idea is that there is a forgetful functor

$$\mathcal{GT} : \mathbf{Rep}\, U_q(\mathfrak{sl}_n) \to \mathbf{Rep}\, U_q(\mathfrak{sl}_{n-1}),$$

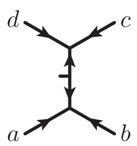
which forgets the full $U_q(\mathfrak{sl}_n)$ action but does not change the underlying linear maps, and that this should be reflected somehow in the diagrams. A diagram in $\mathcal{P}iv_n$ 'represents' some morphism in $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$; thinking of this as a morphism in $\mathbf{Rep}\, U_q(\mathfrak{sl}_{n-1})$ via $\mathcal{GT}$, we can hope to represent it by diagrams in $\mathcal{P}iv_{n-1}$. This hope is borne out—in §4 I construct a functor $d\mathcal{GT} : \mathcal{P}iv_n \to \mathrm{Mat}(\mathcal{P}iv_{n-1})$ (and explain what a 'matrix category' is), in such a way that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{P}iv_n & \xrightarrow{\mathbf{Rep}_n} & \mathbf{Rep}\, U_q(\mathfrak{sl}_n) \\
\big\downarrow{\scriptstyle d\mathcal{GT}} & & \big\downarrow{\scriptstyle \mathcal{GT}} \\
\mathrm{Mat}(\mathcal{P}iv_{n-1}) & \xrightarrow{\mathrm{Mat}(\mathbf{Rep}_{n-1})} & \mathbf{Rep}\, U_q(\mathfrak{sl}_{n-1})
\end{array}$$

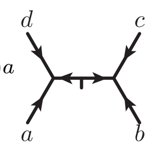
With this functor on hand, we can begin determining the kernel of $\mathbf{Rep}_n$. In particular, given a morphism in $\mathcal{P}iv_n$ (that is, some linear combination of diagrams) we can consider the image under $d\mathcal{GT}$. This is a matrix of (linear combinations of) diagrams in $\mathcal{P}iv_{n-1}$. Then the original diagrammatic morphism becomes zero in the $U_q(\mathfrak{sl}_n)$ representation theory exactly if each entry in this matrix of diagrams is zero in the $U_q(\mathfrak{sl}_{n-1})$ representation theory. Thus, if we understand the kernel of $\mathbf{Rep}_{n-1}$, we can obtain quite strong restrictions on the kernel of $\mathbf{Rep}_n$. Of course, the kernels of $\mathbf{Rep}_2$ and $\mathbf{Rep}_3$ are well known, given by the relations in the Temperley-Lieb category and Kuperberg's spider for $\mathfrak{sl}_3$. Moreover, the kernel of $\mathbf{Rep}_1$ is *really* easy to describe. The method described allows us to work up from these, to obtain relations for all $U_q(\mathfrak{sl}_n)$.

In §5 I use this approach to find three families of relations, and to show that these relations are the only ones of certain types. It remains a conjecture that the proposed relations are in fact complete.

The first family are the $I = H$ relations, essentially corresponding to $6 - j$ symbols:

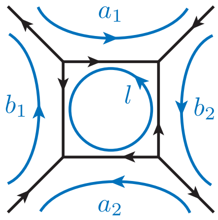


$= (-1)^{(n+1)a}$

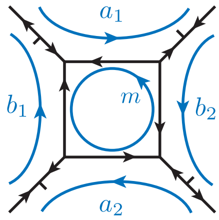


.

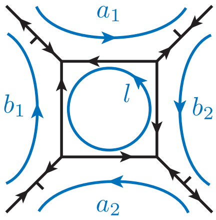
For a given boundary, there are two types of squares, and each can be written as a linear combination of the others. I call these relations the 'square-switch' relations. When $n + \Sigma a - \Sigma b \geq 0$, for $\max b \leq l \leq \min a + n$ we have



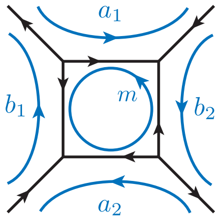
$$= \sum_{m=\max a}^{\min b} \begin{bmatrix} n+\Sigma a - \Sigma b \\ m + l - \Sigma b \end{bmatrix}_q$$



and when $n + \Sigma a - \Sigma b \leq 0$, for $\max a \leq l \leq \min b$ we have



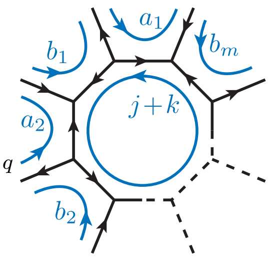
$$= \sum_{m=\max b}^{n+\min a} \begin{bmatrix} \Sigma b - n - \Sigma a \\ m + l - \Sigma a - n \end{bmatrix}_q$$



.

Finally, there are relations amongst polygons of arbitrarily large size (but with a cutoff for each n), called the 'Kekulé ' relations. For each $\Sigma b \leq j \leq \Sigma a + n - 1$,

$$\sum_{k=-\sum b}^{-\sum a+1} (-1)^{j+k} \begin{bmatrix} j+k-\max b \\ j - \Sigma b \end{bmatrix}_q \begin{bmatrix} \min a + n - j - k \\ \Sigma a + n - 1 - j \end{bmatrix}_q$$



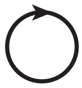
$= 0.$

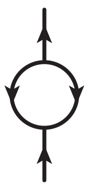
1.2 The Temperley-Lieb algebras

The $n = 2$ part of this story has, unsurprisingly, been understood for a long time. The Temperley-Lieb category gives a diagrammatic presentation of the morphisms between tensor powers of the standard representation of $U_q(\mathfrak{sl}_2)$. The objects of this category are natural numbers, and the morphisms from n to m are $\mathbb{Z}[q, q^{-1}]$-linear combinations of diagrams drawn in a horizontal strip consisting of non-intersecting arcs, with n arc endpoints on the bottom edge of the strip, and m on the top edge. (Notice, in particular, that I'm an optimist, not a pessimist; time goes up the page.) Composition of morphisms is achieved by gluing diagrams, removing each closed circle in exchange for a factor of $[2]_q$.

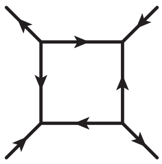
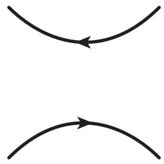
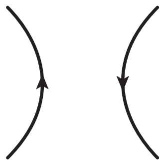
1.3 Kuperberg's spiders

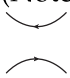
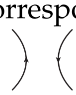
The $n = 3$ story, dates back to around Kuperberg's paper [24]. There he defines the notion of a 'spider' (in this work, we use the parallel notion of a pivotal category), and constructs the spiders for each of the rank 2 Lie algebras $A_2 = \mathfrak{su}(3)$, $B_2 = \mathfrak{sp}(4)$ and G_2 and their quantum analogues. Translated into a category, his A_2 spider has objects words in $(+, -)$, and morphisms (linear combinations of) oriented trivalent graphs drawn in a horizontal strip, with orientations of boundary points along the top and bottom edges coinciding with the target and source word objects, and each trivalent vertex either 'oriented inwards' or 'oriented outwards', subject to the relations

 $= [3]_q = q^2 + 1 + q^{-2}$ (1.3.1)

 $= -[2]_q$  (1.3.2)

and

 $=$  $+$  . (1.3.3)

(Note that there's a 'typo' in Equation (2) of [24], corresponding to Equation (1.3.3) above; the  term has been replaced by another copy of the  term.)

Kuperberg proves that this category is equivalent to a full subcategory of the category of representations of the quantum group $U_q(\mathfrak{sl}_3)$; the subcategory with objects arbitrary tensor products of the two 3-dimensional representations. It is essentially an equinumeration proof, showing that the number of diagrams (modulo the above relations) with a given boundary agrees with the dimension of the appropriate $U_q(\mathfrak{sl}_3)$ invariant space. I'm unable to give an analogous equinumeration argument in what follows.

Note that the $n = 3$ special case of my construction will not quite reproduce Kuperberg's relations above; the bigon relation will involve a $+ [2]_q$, not a $- [2]_q$. This is just a normalisation issue, resolved by multiplying each vertex by $\sqrt{-1}$.

Chapter 2

The 'diagrammatic' category $\mathcal{S}ym_n$

Just as permutations form groups, planar diagrams up to planar isotopy form pivotal categories. In what follows, we'll define a certain 'free (strict) pivotal category', $\mathcal{P}iv_n$ along with a slight modification called $\mathcal{S}ym_n$ obtained by adding some symmetries and some relations for degenerate cases. Essentially, $\mathcal{P}iv_n$ will be the category of trivalent graphs, with edges carrying both orientations and labels 1 through $n-1$, up to planar isotopy.

For lack of a better place, I'll introduce the notion of a matrix category here; given any category $\mathcal{C}$ in which the Hom spaces are guaranteed to be abelian groups, we can form a new category $\mathrm{Mat}\,(\mathcal{C})$, whose objects are formal finite direct sums of objects in $\mathcal{C}$, and whose morphisms are matrices of appropriate morphisms in $\mathcal{C}$. Composition of morphisms is just matrix multiplication (here's where we need the abelian group structure on Hom spaces). If the category already had direct sums, then there's a natural isomorphism $\mathcal{C} \cong \mathrm{Mat}\,(\mathcal{C})$.

2.1 Pivotal categories

I'll use the formalism of pivotal categories in the following. This formalism is essentially interchangeable with that of spiders, due to Kuperberg [24], or of planar algebras, due to Jones [13].[1]

A pivotal category is a monoidal category[2] $\mathcal{C}$ equipped with

1. a cofunctor $* : \mathcal{C}^{op} \to \mathcal{C}$, called the dual,
2. a natural isomorphism $\tau : 1_{\mathcal{C}} \to **$,
3. a natural isomorphism $\gamma : \otimes \circ (* \times *) \to * \circ \otimes^{op}$,
4. an isomorphism $e \to e^*$, where e is the neutral object for tensor product, and
5. for each object $c \in \mathcal{C}$, a 'pairing' morphism $p_c : c^* \otimes c \to e$.

[1] These alternatives are perhaps more desirable, as the pivotal category view forces us to make a artificial distinction between the domain and codomain of a morphism. I'll have to keep reminding you this distinction doesn't matter, in what follows. On the other hand, the categorical setup allows us to more easily incorporate the notion of direct sum.

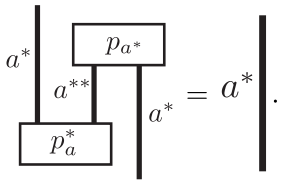
[2] For our purposes, we need only consider strict monoidal categories, where, amongst other things, the tensor product is associative on the nose, not just up to an isomorphism. The definitions given here must be modified for non-strict monoidal categories.

with the natural isomorphisms satisfying certain coherence conditions, and the pairing morphisms certain axioms, all given in [2].[3] The primary example of a pivotal category is the category of representations of an involutory or ribbon Hopf algebra [2]. A pivotal functor between two pivotal categories should intertwine the duality cofunctors, commute with the isomorphisms $e \to e^*$, and take pairing morphisms to pairing morphisms. It should also intertwine the natural isomorphisms τ; that is, for a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, we require $\tau^{\mathcal{D}}_{\mathcal{F}(a)} = \mathcal{F}(\tau^{\mathcal{C}}_a)$. There's a similar condition for γ.

In the case that τ and γ are simply the identity on each object (and $e = e^*$), we say the pivotal category is strict. Unfortunately, representations of a Hopf algebra do not generally form a strict pivotal category; τ cannot be the identity. However, every pivotal category is equivalent to a strict pivotal category [2]. We'll take advantage of this later!

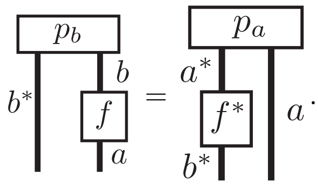
In a pivotal category with $e = e^*$ and γ the identity (but not necessarily also τ), the axioms satisfied by the natural isomorphism τ simplify to the requirements that it's a tensor natural transformation: $\tau_{a\otimes b} = \tau_a \otimes \tau_b$, and that $\tau^*_a : a^{***} \to a^*$ and $\tau_{a^*} : a^* \to a^{***}$ are inverse morphisms.[4] In this same situation, the axioms for the pairing morphisms simplify to:

1. For each object a,



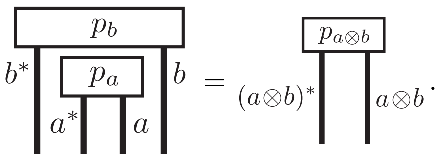
(2.1.1)

2. For each morphism $f : a \to b$,



(2.1.2)

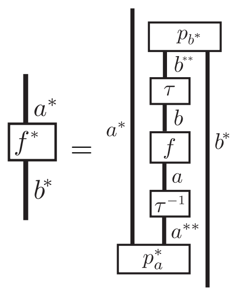
3. For all objects a and b,



(2.1.3)

From these, we can derive

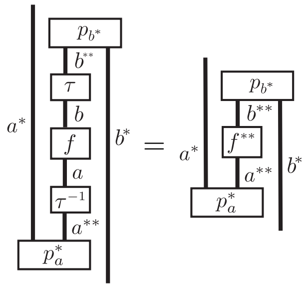
Lemma 2.1.1. *The dual of a morphism can be written in terms of the original morphism, components of* τ*, and the pairing morphisms as*



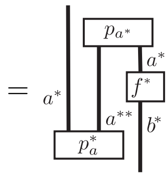
[3]Actually, [6] is an earlier reference for the strict case (see below), and they cite a preprint of [15] for the full version; however, the published version of that paper ended up introducing a slightly different notion, of 'autonomy' for duals.

[4]This condition might also be stated as τ commuting with the cofunctor $*$.

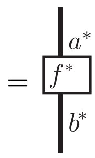
Proof.



using by the naturality of τ, which becomes



by Equation (2.1.2), and finally

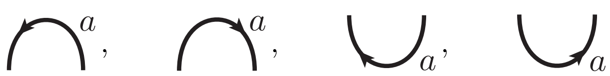


by Equation (2.1.1). $\square$

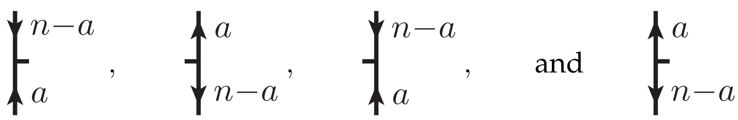
Note that the category of matrices over a pivotal category is still pivotal, in an essentially obvious way.

2.2 Quotients of a free tensor category

To begin with, let's just define a free (strict) monoidal category on some generating morphisms, which we'll call $\mathcal{T}_n$. We'll then add some relations implementing planar isotopy to obtain $\mathcal{P}iv_n$, and some more relations to obtain $\mathcal{S}ym_n$. The objects of $\mathcal{T}_n$ form a monoid under tensor product, with neutral object 0 (sometimes also called n), generated by the set $\{1, \ldots, n-1, 1^*, \ldots, (n-1)^*\}$. The 'generating morphisms' are diagrams

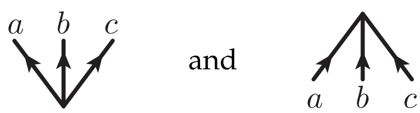


for each $a = 1, \ldots, n-1$ (but not for the 'dual integers' $1^*, \ldots, (n-1)^*$), along with

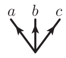


(2.2.1)

again for each $a = 1, \ldots, n-1$, and finally



(2.2.2)

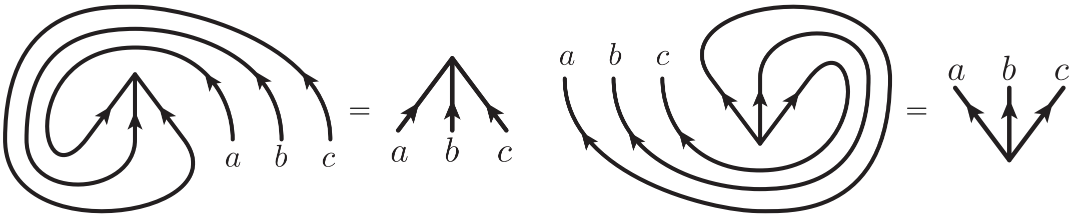
for each $a, b, c = 0, 1, \ldots, n-1$ such that $a+b+c = n$. We'll sometimes speak of the 'type' of a vertex; the first, outgoing, vertex here is of $+$-type, the second, incoming, vertex is of $-$-type. We say a vertex is 'degenerate' if one of its edges is labelled with 0. Notice there are no nondegenerate trivalent vertices for $n = 2$, and exactly one of each type for $n = 3$. To read off the source of such a morphism, you read across the lower boundary of the diagram; each endpoint of an arc labelled a gives a tensor factor of the source object, either a if the arc is oriented upwards, or a^* if the arc is oriented downwards. To read the target, simply read across the upper boundary. Thus the source of 

is 0, and the target is $a \otimes b \otimes c$. All morphisms are then generated from these, by formal tensor product and composition, subject only to the usual identities of a tensor category.

Next $\mathcal{P}iv_n$. This category has exactly the same objects. The morphisms, however, are arbitrary trivalent graphs drawn in a strip, which look locally like one of the pictures above, up to planar isotopy fixing the boundary of the strip. (Any boundary points of the graph must lie on the boundary of the strip.) Thus the graphs are oriented, with each edge carrying a label 1 through n, and edges only ever meet bivalently as in Equation (2.2.1) or trivalently as in Equation (2.2.2). Being a little more careful, we should ask that the diagrams have product structure near the boundary, that this is preserved throughout the isotopies, and that small discs around the trivalent vertices are carried around rigidly by the isotopies, so we can always see the ordering (not just the cyclic ordering) of the three edges incident at a vertex. (Note, though, that we're not excited about being able to see this ordering; we're going to quotient it out in a moment.) The source and target of such a graph can be read off from the graph exactly as described for the generators of $\mathcal{T}_n$ above.

Next, we make this category into a strict pivotal category. For this, we need to define a duality functor, specify the evaluation morphisms, and then check the axioms of §2.1. The duality functor on objects is defined by $0^* = 0$ and otherwise $(k)^* = k^*, (k^*)^* = k$. On morphisms, it's a π rotation of the strip the graph is drawn in. Clearly the double dual functor $**$ is the identity on the nose. The evaluation morphisms for $a = 1, \ldots, n-1$ are 'leftwards-oriented' cap diagrams; the evaluation morphisms for $a = 1^*, \ldots, n-1^*$ are 'rightwards-oriented'. The axioms in Equation (2.1.1) and (2.1.2) are then satisfied automatically, because we allow isotopy of diagrams. The evaluation morphisms for iterated tensor products are just the nested cap diagrams, with the unique orientations and labels matching the required source object. This definition ensures the axiom of Equation (2.1.3) is satisfied.

There's a (tensor) functor from $\mathcal{T}_n$ to $\mathcal{P}iv_n$, which I'll call Draw. Simply take a morphism in $\mathcal{T}_n$, which can be written as a composition of tensor products of generating morphisms, and *draw* the corresponding diagram, using the usual rules of stacking boxes to represent composition, and juxtaposing boxes side by side to represent tensor product. The resulting diagram can then be interpreted as a morphism in $\mathcal{P}iv_n$. This is well-defined by the usual nonsense of [14], that the identities relating tensor product and composition in a tensor category correspond to 'rigid' isotopies (that is, isotopies which do not rotate boxes). The functor is obviously full; or at least, obviously modulo some Morse theory. The kernel of this functor is generated by the extra isotopies we allow in $\mathcal{P}iv_n$. Thus, as a tensor ideal of $\mathcal{T}_n$, $\ker \mathrm{Draw}$ is generated by



(2.2.3)

(2.2.4)

(2.2.5)

and

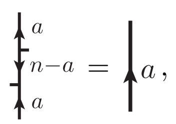
(2.2.6)

There are other obvious variations of Equations (2.2.3) (rotating the vertex other way) and (2.2.5) (tags pointing the other way), but these follow easily from the ones given here. Whenever we want to define a functor on $\mathcal{P}iv_n$ by defining it on generators, we need to check these morphisms are in the kernel.

Finally, we can define the category we're really interested in, which I'll call $\mathcal{S}ym_n$. We'll add just a few more relations to $\mathcal{P}iv_n$; these will be motivated shortly when we define a functor from $\mathcal{T}_n$ to the representations category of $U_q(\mathfrak{sl}_n)$. This functor will descend to the quotient $\mathcal{P}iv_n$, and then to the quotient $\mathcal{S}ym_n$, and the relations we add from $\mathcal{P}iv_n$ to $\mathcal{S}ym_n$ will be precisely the parts of the kernel of this functor which only involve a single generator. The real work of this thesis is, of course, understanding the rest of that kernel! We add relations insisting that the trivalent vertices are rotationally symmetric

(2.2.7)

that opposite tags cancel



(2.2.8)

that dual of a tag is a ± 1 multiple of a tag

(2.2.9)

and that trivalent vertices 'degenerate' to tags

(2.2.10)

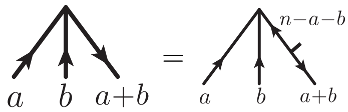
Notice here we're implicitly using the canonical identifications between the objects $0 \otimes a$, a, and $a \otimes 0$, available because our tensor categories are strict.

Clearly the element of $\ker \mathrm{Draw}$ in Equation (2.2.3) can be constructed by tensor product and composition out of the briefer rotations in Equation (2.2.7), and so in checking the well-definedness of a functor on $\mathcal{S}ym_n$, we only need to worry about the latter.

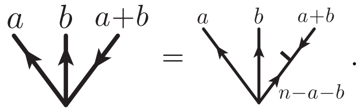
2.3 Flow vertices

We'll now introduce two new types of vertices. You could add them as diagrammatic generators, then impose as relations the formulas below, but it's less cumbersome to just think of them as a convenient notation. In each of these vertices, there will be some 'incoming' and some 'outgoing' edges, and the sum of the incoming edges will be the same as the sum of the outgoing edges.

Definition 2.3.1. The 'flow vertices' are



and



The convention here is that the 'hidden tag' lies on the 'thick' edge, and points counterclockwise. These extra vertices will be convenient in what follows, hiding a profusion of tags. 'Splitting' vertices are of $+$-type, 'merging' vertices are of $-$-type.

2.4 Polygonal webs

To specify the kernel of the representation functor, in §5, we'll need to introduce some notations for 'polygonal webs'. These webs will come in two families, the '$\mathcal{P}$' family and the '$\mathcal{Q}$' family. In each family, the vertices around the polygon will alternate in type. A boundary edge which is connected to a $+$-vertex in a $\mathcal{P}$-polygon will be connected to a $-$-vertex in a $\mathcal{Q}$-polygon.

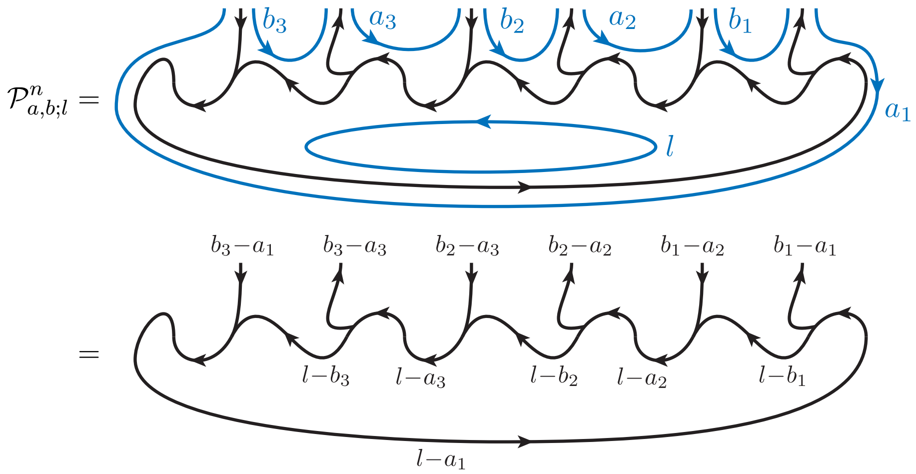
For $a, b \in \mathbb{Z}^k$ define[5] the boundary label pattern

$$\mathcal{L}(a,b) = (b_k - a_1, -) \otimes (b_k - a_k, +) \otimes \cdots \otimes (b_2 - a_2, +) \otimes (b_1 - a_2, -) \otimes (b_1 - a_1, +).$$

We'll now define some elements of $\mathrm{Hom}_{\mathcal{S}ym^n}(\emptyset, \mathcal{L}(a,b))$, $\mathcal{P}^n_{a,b;l}$ for $\max b \leq l \leq \min a + n$ and

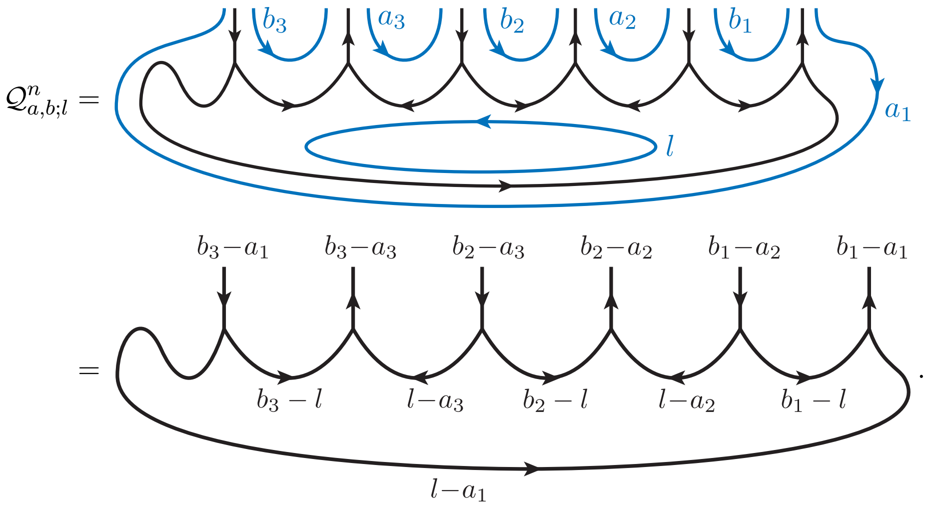
[5] I realise this definition is 'backwards', or at least easier to read from right to left than from left to right. Sorry—I only realised too late.

$\mathcal{Q}^n_{a,b;l}$ for $\max a \leq l \leq \min b$ by



(2.4.1)

and



(2.4.2)

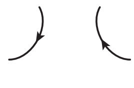
(The diagrams are for $k = 3$, but you should understand the obvious generalisation for any $k \in \mathbb{N}$.) You should consider the first of each pair of diagrams simply as notation for the second. Each edge label is a signed sum of the 'flows labels' on either side, determining signs by relative orientations. It's trivial[6] to see that for web diagrams with only '2 in, 1 out' and '1 in, 2 out' vertices, it's always possible to pick a set of flow labels corresponding to an allowable set of edge labels. Not every set of flow labels, however, gives admissible edge labels, because the edge labels must be between 0 and n. The allowable flow labels for the $\mathcal{P}$- and $\mathcal{Q}$-polygons are exactly those for which $a_i, a_{i+1} \leq b_i \leq n + a_i, n + a_{i+1}$. Further, there's a $\mathbb{Z}$ redundancy in flow labels; adding a constant to every flow label in a diagram doesn't actually change anything. Taking this into account, there's a finite set of pairs a, b for each n and k. The inequalities on the 'internal flow label' l for both $\mathcal{P}^n_{a,b;l}$ and $\mathcal{Q}^n_{a,b;l}$ simply demand that all the internal edges have labels between 0 and n, inclusive.

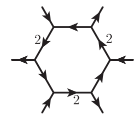
We denote the subspace of $\mathrm{Hom}_{\mathcal{S}ym^n}(\emptyset, \mathcal{L}(a,b))$ spanned by all the $\mathcal{P}$-type polygons by $\mathcal{AP}^n_{a,b}$, and the subspace spanned by the $\mathcal{Q}$-type polygons by $\mathcal{AQ}^n_{a,b}$. The space $\mathcal{AP}^n_{a,b}$ is

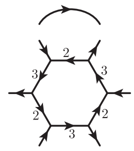
[6] Actually, perhaps only trivial after acknowledging that the disk in which the diagrams are drawn is simply connected.

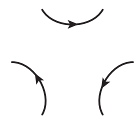
$\min a - \max b + n - 1$ dimensional (or 0 dimensional when this quantity is negative), and the space $\mathcal{AQ}^n_{a,b}$ is $\max a - \min b + 1$ dimensional.

A word of warning; a and b each having k elements does not necessarily mean that $\mathcal{P}^n_{a,b;l}$ or $\mathcal{Q}^n_{a,b;l}$ are honest $2k$-gons. This can fail in two ways. First of all, if some $a_i = b_i$, $a_i + n = b_i$, $a_{i+1} = b_i$ or $a_{i+1} + n = b_i$, then one of the external edges carries a trivial label. Further, when l takes on one of its extremal allowed values, at least one of the internal edges of the polygon becomes trivial, and the web becomes a tree, or a disjoint union of trees and arcs. For example (ignoring the distinction between source and target of morphisms; strictly speaking these should all be drawn with all boundary points at the top of the diagram),

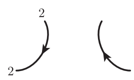
$\mathcal{P}^4_{(0,0,0),(1,1,1);1} =$ 

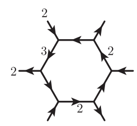
$\mathcal{P}^4_{(0,0,0),(1,1,1);2} =$ 

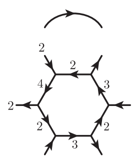
$\mathcal{P}^4_{(0,0,0),(1,1,1);3} =$ 

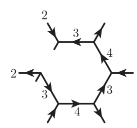
$\mathcal{P}^4_{(0,0,0),(1,1,1);4} =$ 

and

$\mathcal{P}^5_{(0,1,1),(2,2,2);2} =$ 

$\mathcal{P}^5_{(0,1,1),(2,2,2);3} =$ 

$\mathcal{P}^5_{(0,1,1),(2,2,2);4} =$ 

$\mathcal{P}^5_{(0,1,1),(2,2,2);5} =$ 

Finally, it's actually possible for a $\mathcal{P}$-type polygon and a $\mathcal{Q}$-type polygon to be equal in $\mathcal{S}ym_n$. This can only happen in the case that a and b each have length 2, or at any length, when either a or b is constant. This only involves polygons with extreme values of the internal flow label l. Specifically

Lemma 2.4.1. *If a and b are each of length 2,*

$$\mathcal{P}^n_{a,b;\max b} = \mathcal{Q}^n_{a,b;\min b}$$

and

$$\mathcal{P}^n_{a,b;\min a+n} = \mathcal{Q}^n_{a,b;\max a}.$$

Further, even if a and b have length greater than 2, when a is a constant vector $a = \overrightarrow{a}$

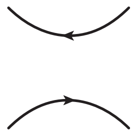
$$\mathcal{P}^n_{\overrightarrow{a},b;a+n} = \mathcal{Q}^n_{\overrightarrow{a},b;a}$$

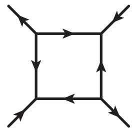
and when b is a constant vector $b = \overrightarrow{b}$

$$\mathcal{P}^n_{a,\overrightarrow{b};b} = \mathcal{Q}^n_{a,\overrightarrow{b};b}$$

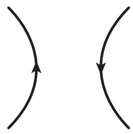
in $\mathcal{S}ym$. Otherwise, the $\mathcal{P}$- and $\mathcal{Q}$-polygons are linearly independent in $\mathcal{S}ym_n$. In particular, $\dim(\mathcal{AP}^n_{a,b} \cap \mathcal{AQ}^n_{a,b})$ is 0, 1 or 2 dimensional, depending on whether neither a nor b are constant, one is, or either both are or a and b have length 2.

For example

$\mathcal{P}^3_{(0,0),(1,1);1} =$  $= \mathcal{Q}^3_{(0,0),(1,1);1},$

$\mathcal{P}^3_{(0,0),(1,1);2} =$  ,

and

$\mathcal{P}^3_{(0,0),(1,1);3} =$  $= \mathcal{Q}^3_{(0,0),(1,1);0}.$

2.4.1 Rotations

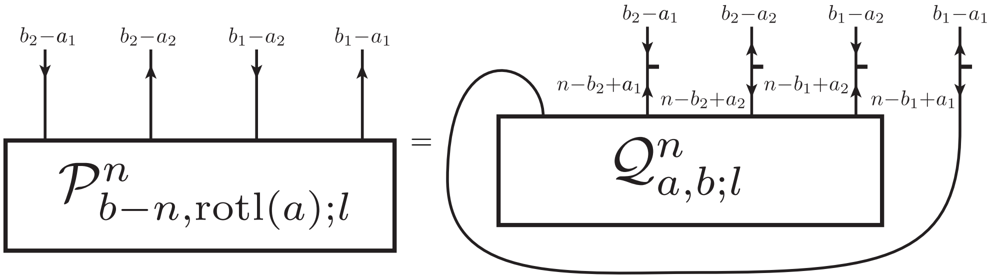
It's important to point out that the two types of polygons described above are actually closely related. In fact, by a 'rotation', and adding some tags, we can get from one to the other. This will be important in our later descriptions of the kernel of the representation functor. Unsurprisingly, this symmetry between the types of polygons is reflected in the kernel, and we'll save half the effort in each proof, essentially only having to deal with one of the two types.

First, by $\mathrm{rotl} : \mathbb{Z}^k \to \mathbb{Z}^k$ we just mean 'rotate left':

$$\mathrm{rotl}(a_1, a_2, \ldots, a_k) = (a_2, \ldots, a_k, a_1),$$

and by $\mathrm{rotr} : \mathbb{Z}^k \to \mathbb{Z}^k$ 'rotate right'.

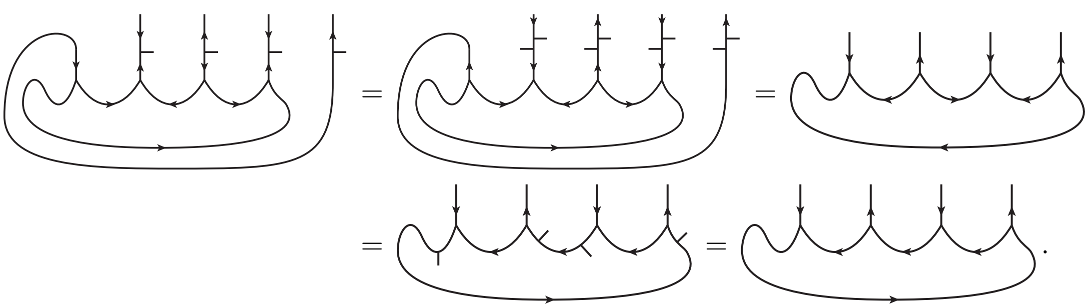
Lemma 2.4.2. *Writing the identity out by explicit tensor products and compositions would be tedious; much easier is to write it diagrammatically first off:*



, (2.4.3)

although keep in mind that we intend the obvious generalisation to arbitrary size polygons, not just squares.

Proof. It's just a matter of using isotopy, the definition of the 'flow vertices' in §2.3 and cancelling tags by Equation (2.2.8).



. □

Notice that it's actually not important which way the tags point in Equation (2.4.3); we could reverse them, since by Equation (2.2.9) we'd just pick up an overall sign of

$$(-1)^{(n+1)(2\Sigma b-2\Sigma a)} = 1.$$

We'll sometimes write Equation (2.4.3) as

$$\mathcal{P}^n_{b-n,\mathrm{rotl}(a);l} = \mathrm{drotl}\left(\mathcal{Q}^n_{a,b;l}\right)$$

or equivalently, re-indexing

$$\mathcal{P}^n_{a,b;l} = \mathrm{drotl}\left(\mathcal{Q}^n_{\mathrm{rotr}(b),a+n;l}\right)$$

where the operation drotl is implicitly defined by Equation (2.4.3). (You should read drotl as 'diagrammatic rotation', and perhaps have the 'd' prefix remind you that we're not just rotating, but also adding a tag, called d in the representation theory, to each external edge.) The corresponding identity writing a $\mathcal{Q}$-polygon in terms of a $\mathcal{P}$-polygon is

$$\mathcal{Q}^n_{\mathrm{rotr}(b),a+n;l} = \mathrm{drotl}\left(\mathcal{P}^n_{a,b;l}\right)$$

or

$$\mathcal{Q}^n_{a,b;l} = \mathrm{drotl}\left(\mathcal{P}^n_{b-n,\mathrm{rotl}(a);l}\right).$$

Chapter 3

Just enough representation theory

In this chapter, we'll describe quite a bit of representation theory, but hopefully only what's necessary for later! There are no new results in this section, although possibly Proposition 3.5.8, explaining roughly that 'the fundamental representation theory of $U_q(\mathfrak{sl}_n)$ is generated by triple invariants' has never really been written down.

3.1 The Lie algebra $\mathfrak{sl}_n$

The Lie algebra $\mathfrak{sl}_n$ of traceless, n-by-n complex matrices contains the commutative Cartan subalgebra $\mathfrak{h}$ of diagonal matrices, spanned by $H_i = E_{i,i} - E_{i+1,i+1}$ for $i = 1, \ldots, n-1$. (Here $E_{i,j}$ is simply the matrix with a single nonzero entry, a 1 in the (i,j) position.) When $\mathfrak{sl}_n$ acts on a vector space V, the action of $\mathfrak{h}$ splits V into eigenspaces called weight spaces, with eigenvalues in $\mathfrak{h}^*$. The points

$$\Lambda = \{\lambda \in \mathfrak{h}^* \mid \lambda(H_i) \in \mathbb{Z}\} \cong \mathbb{Z}^{n-1}$$

form the weight lattice. We call $\mathfrak{h}^*_+ = \{\lambda \in \mathfrak{h}^* \mid \lambda(H_i) \geq 0\}$ the positive Weyl chamber, and the lattice points in it, $\Lambda_+ = \Lambda \cap \mathfrak{h}^*_+$, the dominant weights. The positive Weyl chamber looks like $\mathbb{R}^{n-1}_{\geq 0}$, and the dominant weights like $\mathbb{N}^{n-1}$, generated by certain fundamental weights, $\lambda_1, \ldots, \lambda_{n-1}$, the dual basis to $\{H_i\} \subset \mathfrak{h}$.

Under the adjoint action of $\mathfrak{sl}_n$ on itself, $\mathfrak{sl}_n$ splits into $\mathfrak{h}$, as the 0-weight space, and one dimensional root spaces, each spanned by a root vector $E_{i \neq j}$ with weight $[H_k, E_{i,j}] = (\delta_{i,k} + \delta_{j,k+1} - \delta_{i,k+1} - \delta_{j,k}) E_{i,j}$. In particular, the simple roots are defined $E_i^+ = E_{i,i+1}$, for $i = 1, \ldots, n-1$. Similarly define $E_i^- = E_{i+1,i}$. Together, E_i^+, E_i^- and H_i for $i = 1, \ldots, n-1$ generate $\mathfrak{sl}_n$ as a Lie algebra.

The universal enveloping algebra $U(\mathfrak{sl}_n)$ is a Hopf algebra generated as an associative algebra by symbols E_i^+, E_i^- and H_i for $i = 1, \ldots, n-1$, subject only to the relations that the commutator of two symbols agrees with the Lie bracket in $\mathfrak{sl}_n$. (We won't bother specify the other Hopf algebra structure, the comultiplication, counit or antipode. See the next section for all the details for quantum $\mathfrak{sl}_n$.) Trivially, $\mathfrak{sl}_n$ and $U(\mathfrak{sl}_n)$ have the same representation theory. We denote the subalgebra generated by H_i and $E_i^{\pm}$ for $i = 1, \ldots, n-1$ by $U^{\pm}\ (\mathfrak{sl}_n)$.

3.2 The quantum groups $U_q(\mathfrak{sl}_n)$

We now recall the q-deformation of the Hopf algebra $U(\mathfrak{sl}_n)$. Unfortunately we can't straightforwardly recover $U(\mathfrak{sl}_n)$ by setting $q = 1$, but we will see in §3.3 that this is the case at the level of representation theory.

Let $\mathcal{A} = \mathbb{Q}(q)$ be the field of rational functions in an indeterminate q, with coefficients in $\mathbb{Q}$. The quantum group $U_q(\mathfrak{sl}_n)$ is a Hopf algebra over $\mathcal{A}$, generated as an associative algebra by symbols X_i^+, X_i^-, K_i and K_i^{-1}, for $i = 1, \ldots, n-1$, subject to the relations

$$\begin{aligned}
K_i K_i^{-1} &= 1 = K_i^{-1} K_i \\
K_i K_j &= K_j K_i \\
K_i X_j^{\pm} &= q^{\pm(i,j)} X_j^{\pm} K_i \qquad \text{where } (i,j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \text{ or} \\ 0 & \text{if } |i-j| \geq 2 \end{cases} \\
X_i^+ X_j^- - X_j^- X_i^+ &= \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \\
X_i^{\pm} X_j^{\pm} &= X_j^{\pm} X_i^{\pm} \qquad \text{when } |i-j| \geq 2
\end{aligned}$$

and the quantum Serre relations

$$X_i^{\pm 2} X_j^{\pm} - [2]_q X_i^{\pm} X_j^{\pm} X_i^{\pm} + X_j^{\pm} X_i^{\pm 2} = 0 \qquad \text{when } |i-j| = 1.$$

The Hopf algebra structure is given by the following formulas for the comultiplication Δ, counit ε and antipode S.

$$\begin{aligned}
\Delta(K_i) &= K_i \otimes K_i, \\
\Delta(X_i^+) &= X_i^+ \otimes K_i + 1 \otimes X_i^+, \\
\Delta(X_i^-) &= X_i^- \otimes 1 + K_i^{-1} \otimes X_i^-,
\end{aligned}$$

$$\varepsilon(K_i) = 1 \qquad \varepsilon(X_i^{\pm}) = 0,$$

$$S(K_i) = K_i^{-1} \qquad S(X_i^+) = -X_i^+ K_i^{-1} \qquad S(X_i^i) = -K_i X_i^-.$$

(See, for example, [4, §9.1] for verification that these do indeed fit together to form a Hopf algebra.) There are Hopf subalgebras $U_q(\mathfrak{sl}_n)^{\pm}$, leaving out the generators $X_i^{\mp}$.

3.3 Representations

The finite-dimensional irreducible representations of both $U(\mathfrak{sl}_n)$ and $U_q(\mathfrak{sl}_n)$ can be succinctly catalogued.

A representation V of $U(\mathfrak{sl}_n)$ is said to be a highest weight representation if there is a weight vector $v \in V$ such that $V = U(\mathfrak{sl}_n)^-(v)$. The representation is said to have weight λ if v has weight λ. As with $\mathfrak{sl}_n$, representations of $U_q(\mathfrak{sl}_n)$ split into the eigenspaces of the action of the commutative subalgebra generated by the K_i and K_i^{-1}. These eigenspaces are again called weight spaces. A representation V of $U_q(\mathfrak{sl}_n)$ is called a high weight representation if it contains some weight vector v so $V = U_q(\mathfrak{sl}_n)^-(v)$. The finite-dimensional irreducible representations (henceforth called irreps) are then classified by:

Proposition 3.3.1. *Every irrep of $U(\mathfrak{sl}_n)$ or of $U_q(\mathfrak{sl}_n)$ is a highest weight representation, and there is precisely one irrep with weight λ for each $\lambda \in \Lambda_+$, the set of dominant weights.*

Proof. See [7, §23] for the classical case, [4, §10.1] for the quantum case. □

Further, the finite dimensional representation theories are tensor categories and the tensor product of two irreps decomposes uniquely as a direct sum of other irreps. The combinatorics of the the tensor product structures in $\mathbf{Rep}\, U(\mathfrak{sl}_n)$ and $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ agree. A nice way to say this is that the Grothendieck rings of $\mathbf{Rep}\, U(\mathfrak{sl}_n)$ and $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ are isomorphic, identifying irreps with the same highest weight [4, §10.1].

3.3.1 Fundamental representations

Amongst the finite dimensional irreducible representations, there are some particularly simple ones, whose highest weights are the fundamental weights. These are called the fundamental representations, and there are $n-1$ of them for $U(\mathfrak{sl}_n)$ or $U_q(\mathfrak{sl}_n)$.

We'll write, in either case, $V_{a<n}$ for the fundamental representation with weight λ_a. For $U(\mathfrak{sl}_n)$ this is just the representation $\wedge^a\mathbb{C}^n$. It will be quite convenient to agree that $V_{0<n}$ and $V_{n<n}$ (poor notation, I admit!) both denote the trivial representation.

We can now define $\mathbf{FundRep}U_q(\mathfrak{sl}_n)$; it's the full subcategory of $\mathbf{Rep}\,U_q(\mathfrak{sl}_n)$, whose objects are generated by tensor product and duality from the trivial representation $\mathcal{A}$, and the fundamental representations $V_{a<n}$. Note that there are no direct sums in $\mathbf{FundRep}U_q(\mathfrak{sl}_n)$.

All dominant weights are additively generated by fundamental weights, and this is reflected in the representation theory; the irrep with high weight $\lambda = \sum_a m_a\lambda_a$ is contained, with multiplicity one, as a direct summand in the tensor product $\otimes_a V_{a<n}^{\otimes m_a}$.

This observation explains why it is satisfactory to study just $\mathbf{FundRep}U_q(\mathfrak{sl}_n)$, instead of the full representation theory. Every irrep, while not necessarily allowed as an object of $\mathbf{FundRep}U_q(\mathfrak{sl}_n)$, reappears in the Karoubi envelope [17, 37], since any irrep appears as a subrepresentation of some tensor product of fundamental representations. In fact, there's a canonical equivalence of categories

$$\mathrm{Kar}\left(\mathbf{FundRep}U_q(\mathfrak{sl}_n)\right) \cong \mathbf{Rep}\,U_q(\mathfrak{sl}_n)\,.$$

3.3.2 The Gel'fand-Tsetlin basis

We'll now define the Gel'fand-Tsetlin basis [9, 10], a canonical basis which arises from the nice multiplicity free splitting rules for $U_q(\mathfrak{sl}_n) \hookrightarrow U_q(\mathfrak{sl}_{n+1})$. (Here, and hereafter, this inclusion is just $K_i \mapsto K_i, X_i^\pm \mapsto X_i^\pm$.)

Lemma 3.3.2. *If V is an irrep of $U_q(\mathfrak{sl}_{n+1})$, then as a representation of $U_q(\mathfrak{sl}_n)$ each irrep appearing in V appears exactly once.*

More specifically, if V is the $U_q(\mathfrak{sl}_{n+1})$ irrep with highest weight $(\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$, then an $U_q(\mathfrak{sl}_n)$ irrep W of weight $(\mu_1, \mu_2, \ldots, \mu_n)$ appears (with multiplicity one) if and only if 'μ fits inside λ', that is

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_n \leq \mu_n \leq \lambda_{n+1}.$$

In particular, the fundamental irreps $V_{a<n}$ break up as

$$V_{a<n} \cong V_{a-1<n-1} \oplus V_{a<n-1}.$$

Proof. See [4, §14.1.A], [23]. □

This allows us to inductively define ordered bases for $U_q(\mathfrak{sl}_n)$ irreps, at least projectively. This was first done in the quantum case in [12]. Choose, without much effort, a basis for the only irreducible representation of $U_q(\mathfrak{sl}_1)$, the trivial representation $\mathbb{C}(q)$. Now for any representation V of $U_q(\mathfrak{sl}_{n+1})$, decompose V over $U_q(\mathfrak{sl}_n)$, as $V \cong \bigoplus_\alpha W_\alpha$, ordered lexicographically by highest weight, and define the Gel'fand-Tsetlin basis of V to be the concatenation of the inclusions of the bases for each W_α into V. These inclusions are unique up to complex multiples, and we thus obtain a canonical projective basis.

We'll call this forgetful functor the 'Gel'fand-Tsetlin' functor, $\mathcal{GT} : \mathbf{Rep}\,U_q(\mathfrak{sl}_n) \to \mathbf{Rep}\,U_q(\mathfrak{sl}_{n-1})$. Restricted to the fundamental part of the representation theory (see §3.3.1), it becomes a functor $\mathcal{GT} : \mathbf{FundRep}U_q(\mathfrak{sl}_n) \to \mathrm{Mat}\left(\mathbf{FundRep}U_q(\mathfrak{sl}_n)\right)$.

Next, I'll describe in gory detail the action of $U_q(\mathfrak{sl}_n)$ on each of its fundamental representations $V_{a<n}$, using the Gel'fand-Tsetlin decomposition. We introduce maps p_{-1} and p_0, the

$U_q(\mathfrak{sl}_{n-1})$-linear projections of $V_{a<n} \twoheadrightarrow V_{a-1<n-1}$ and $V_{a<n} \twoheadrightarrow V_{a<n-1}$. We also introduce the inclusions $\mathfrak{i}_{-1} : V_{a-1<n-1} \hookrightarrow V_{a<n}$ and $\mathfrak{i}_0 : V_{a<n-1} \hookrightarrow V_{a<n}$.

Proposition 3.3.3. *We can describe the action of $U_q(\mathfrak{sl}_n)$ on $V_{a<n}$ recursively as follows. On $V_{0<n}$ and $V_{n<n}$, $U_q(\mathfrak{sl}_n)$ acts trivially: $X_i^{\pm}$ by 0, and K_i by 1. On the non-trivial representations, we have*

$$X^+_{n-1|V_{a<n}} = \mathfrak{i}_{-1}\mathfrak{i}_0 p_{-1} p_0, \tag{3.3.1}$$

$$X^-_{n-1|V_{a<n}} = \mathfrak{i}_0\mathfrak{i}_{-1} p_0 p_{-1},$$

$$K_{n-1|V_{a<n}} = \mathfrak{i}_{-1}(\mathfrak{i}_{-1}p_{-1} + q\mathfrak{i}_0 p_0)p_{-1} + \mathfrak{i}_0(q^{-1}\mathfrak{i}_{-1}p_{-1} + \mathfrak{i}_0 p_0)p_0, \tag{3.3.2}$$

and for $Z \in U_q(\mathfrak{sl}_{n-1})$

$$Z_{|V_{a<n}} = \mathfrak{i}_{-1} Z_{|V_{a-1<n-1}} p_{-1} + \mathfrak{i}_0 Z_{|V_{a<n-1}} p_0. \tag{3.3.3}$$

Remark. Relative to the direct sum decomposition

$$V_{a<n} \cong (V_{a-2<n-2} \oplus V_{a-1<n-2}) \oplus (V_{a-1<n-2} \oplus V_{a<n-2})$$

under $U_q(\mathfrak{sl}_{n-2})$, we can write these as matrices, as

$$X^+_{n-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X^-_{n-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_{n-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad Z = \begin{pmatrix} Z & 0 & 0 & 0 \\ 0 & Z & 0 & 0 \\ 0 & 0 & Z & 0 \\ 0 & 0 & 0 & Z \end{pmatrix}.$$

Proof. We need to check the relations in $U_q(\mathfrak{sl}_n)$ involving $X^{\pm}_{n-1}$ and K_{n-1}. We're inductively assuming the others hold. To begin, the K_i's automatically commute, as they're diagonal with respect to the direct sum decomposition. Further, it's clear from the definitions that $X^{\pm}_{n-1}$ and K_{n-1} commute with the subalgebra $U_q(\mathfrak{sl}_{n-2})$. We then need to check the relation $K_i X_j^{\pm} = q^{\pm(i,j)} X_j^{\pm} K_i$, for $|i-j| \leq 1$. Now

$$\begin{aligned} K_{n-1}X^+_{n-1} &= (\mathfrak{i}_{-1}(\mathfrak{i}_{-1}p_{-1} + q\mathfrak{i}_0 p_0)p_{-1} + \mathfrak{i}_0(q^{-1}\mathfrak{i}_{-1}p_{-1} + \mathfrak{i}_0 p_0)p_0)\mathfrak{i}_{-1}\mathfrak{i}_0 p_{-1}p_0 \\ &= q\mathfrak{i}_{-1}\mathfrak{i}_0 p_{-1}p_0, \end{aligned}$$

while

$$\begin{aligned} X^+_{n-1}K_{n-1} &= \mathfrak{i}_{-1}\mathfrak{i}_0 p_{-1}p_0(\mathfrak{i}_{-1}(\mathfrak{i}_{-1}p_{-1} + q\mathfrak{i}_0 p_0)p_{-1} + \mathfrak{i}_0(q^{-1}\mathfrak{i}_{-1}p_{-1} + \mathfrak{i}_0 p_0)p_0) \\ &= q^{-1}\mathfrak{i}_{-1}\mathfrak{i}_0 p_{-1}p_0, \end{aligned}$$

so $K_{n-1}X^+_{n-1} = q^2 X^+_{n-1}K_{n-1}$ and similarly $K_{n-1}X^-_{n-1} = q^{-2}X^-_{n-1}K_{n-1}$. Further

$$\begin{aligned} K_{n_2}X^+_{n-1} &= (\mathfrak{i}_{-1}(\mathfrak{i}_{-1}(\mathfrak{i}_{-1}p_{-1} + q\mathfrak{i}_0 p_0)p_{-1} + \mathfrak{i}_0(q^{-1}\mathfrak{i}_{-1}p_{-1} + \mathfrak{i}_0 p_0)p_0)p_{-1} + \\ &\quad + \mathfrak{i}_0(\mathfrak{i}_{-1}(\mathfrak{i}_{-1}p_{-1} + q\mathfrak{i}_0 p_0)p_{-1} + \mathfrak{i}_0(q^{-1}\mathfrak{i}_{-1}p_{-1} + \mathfrak{i}_0 p_0)p_0)p_0)\mathfrak{i}_{-1}\mathfrak{i}_0 p_{-1}p_0 \\ &= \mathfrak{i}_{-1}\mathfrak{i}_0(q^{-1}\mathfrak{i}_{-1}p_{-1} + \mathfrak{i}_0 p_0)p_{-1}p_0, \end{aligned}$$

while

$$
\begin{aligned}
X^+_{n-1}K_{n-2} &= \mathfrak{i}_{-1}\mathfrak{i}_0 p_{-1}p_0(\mathfrak{i}_{-1}(\mathfrak{i}_{-1}(\mathfrak{i}_{-1}p_{-1} + q\mathfrak{i}_0 p_0)p_{-1} + \mathfrak{i}_0(q^{-1}\mathfrak{i}_{-1}p_{-1} + \mathfrak{i}_0 p_0)p_0)p_{-1} + \\
&\quad + \mathfrak{i}_0(\mathfrak{i}_{-1}(\mathfrak{i}_{-1}p_{-1} + q\mathfrak{i}_0 p_0)p_{-1} + \mathfrak{i}_0(q^{-1}\mathfrak{i}_{-1}p_{-1} + \mathfrak{i}_0 p_0)p_0)p_0) \\
&= \mathfrak{i}_{-1}\mathfrak{i}_0(\mathfrak{i}_{-1}p_{-1} + q\mathfrak{i}_0 p_0)p_{-1}p_0,
\end{aligned}
$$

so $K_{n-2}X^+_{n-1} = q^{-1}X^+_{n-1}K_{n-2}$. By similar calculations $K_{n-2}X^-_{n-1} = qX^-_{n-1}K_{n-2}$, $K_{n-1}X^+_{n-2} = q^{-1}X^+_{n-2}K_{n-1}$ and $K_{n-1}X^-_{n-2} = qX^-_{n-2}K_{n-1}$.

Next, we check $X^+_i X^-_j - X^-_j X^+_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q-q^{-1}}$, for $i = j = n-1$, and for $i = n-1, j = n-2$.

$$
\begin{aligned}
X^+_{n-1}X^-_{n-1} &= \mathfrak{i}_{-1}\mathfrak{i}_0 p_{-1}p_0\mathfrak{i}_0\mathfrak{i}_{-1}p_0 p_{-1} \\
&= \mathfrak{i}_{-1}\mathfrak{i}_0 p_0 p_{-1} \\
X^-_{n-1}X^+_{n-1} &= \mathfrak{i}_0\mathfrak{i}_{-1}p_0 p_{-1}\mathfrak{i}_{-1}\mathfrak{i}_0 p_{-1}p_0, \\
&= \mathfrak{i}_0\mathfrak{i}_{-1}p_{-1}p_0
\end{aligned}
$$

while

$$
\begin{aligned}
K_{n-1} - K^{-1}_{n-1} &= \mathfrak{i}_{-1}(\mathfrak{i}_{-1}p_{-1} + q\mathfrak{i}_0 p_0)p_{-1} + \mathfrak{i}_0(q^{-1}\mathfrak{i}_{-1}p_{-1} + \mathfrak{i}_0 p_0)p_0 - \\
&\quad - \mathfrak{i}_{-1}(\mathfrak{i}_{-1}p_{-1} + q^{-1}\mathfrak{i}_0 p_0)p_{-1} - \mathfrak{i}_0(q\mathfrak{i}_{-1}p_{-1} + \mathfrak{i}_0 p_0)p_0 \\
&= (q - q^{-1})\mathfrak{i}_{-1}\mathfrak{i}_0 p_0 p_{-1} + (q^{-1} - q)\mathfrak{i}_0\mathfrak{i}_{-1}p_{-1}p_0,
\end{aligned}
$$

so $X^+_{n-1}X^-_{n-1} - X^-_{n-1}X^+_{n-1} = \frac{K_{n-1} - K^{-1}_{n-1}}{q-q^{-1}}$, as desired, and

$$
\begin{aligned}
X^+_{n-1}X^-_{n-2} &= \mathfrak{i}_{-1}\mathfrak{i}_0 p_{-1}p_0(\mathfrak{i}_{-1}\mathfrak{i}_0\mathfrak{i}_{-1}p_0 p_{-1}p_{-1} + \mathfrak{i}_0\mathfrak{i}_0\mathfrak{i}_{-1}p_0 p_{-1}p_0) \\
&= \mathfrak{i}_{-1}\mathfrak{i}_0 p_{-1}\mathfrak{i}_0\mathfrak{i}_{-1}p_0 p_{-1}p_0 = 0 \\
X^-_{n-2}X^+_{n-1} &= (\mathfrak{i}_{-1}\mathfrak{i}_0\mathfrak{i}_{-1}p_0 p_{-1}p_{-1} + \mathfrak{i}_0\mathfrak{i}_0\mathfrak{i}_{-1}p_0 p_{-1}p_0)\mathfrak{i}_{-1}\mathfrak{i}_0 p_{-1}p_0 \\
&= \mathfrak{i}_{-1}\mathfrak{i}_0\mathfrak{i}_{-1}p_0 p_{-1}\mathfrak{i}_0 p_{-1}p_0 = 0
\end{aligned}
$$

so $X^+_{n-1}X^-_{n-2} - X^+_{n-1}X^-_{n-2} = 0$.

Finally, we don't need to check the Serre relations. By a result of [16], if you defined a quantum group without the Serre relations, you'd see them reappear in the ideal of elements acting by zero on all finite dimensional representations. □

Corollary 3.3.4. *We'll collect here some formulas for the generators acting on the dual representations. Abusing notation somewhat, we write* $\mathfrak{i}_0 : V^*_{a<n-1} \hookrightarrow V^*_{a<n}$, $\mathfrak{i}_{-1} : V^*_{a-1<n-1} \hookrightarrow V^*_{a<n}$, $p_0 : V^*_{a<n} \twoheadrightarrow V^*_{a<n-1}$ *and* $p_{-1} : V^*_{a<n} \twoheadrightarrow V^*_{a-1<n-1}$. *With these conventions,* $(\mathfrak{i}_{0|V_{a<n-1}})^* = p_{0|V^*_{a<n}}$, *and so on. Then*

$$
\begin{aligned}
X^+_{n-1|V^*_{a<n}} &= -q\mathfrak{i}_0\mathfrak{i}_{-1}p_0 p_{-1}, \\
X^-_{n-1|V^*_{a<n}} &= -q^{-1}\mathfrak{i}_{-1}\mathfrak{i}_0 p_{-1}p_0, \\
K_{n-1|V^*_{a<n}} &= \mathfrak{i}_{-1}(\mathfrak{i}_{-1}p_{-1} + q^{-1}\mathfrak{i}_0 p_0)p_{-1} + \mathfrak{i}_0(q^{\mathfrak{i}}_{-1}p_{-1} + \mathfrak{i}_0 p_0)p_0,
\end{aligned}
$$

and for $Z \in U_q(\mathfrak{sl}_{n-1})$

$$
Z_{|V^*_{a<n}} = \mathfrak{i}_{-1}Z_{|V^*_{a-1<n-1}}p_{-1} + \mathfrak{i}_0 Z_{|V^*_{a<n-1}}p_0.
$$

3.4 Strictifying $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$

We'll now make something of a digression into abstract nonsense. (But it's good abstract nonsense!) It's worth understanding at this point that while $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ isn't a strict pivotal category, it can be 'strictified'. That is, the natural isomorphism $\tau : \mathbf{1} \to **$ isn't the identity, but $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ is equivalent to another pivotal category in which that 'pivotal isomorphism' τ *is* the identity. This strictification will be a full subcategory of $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ (as a tensor category), but with a new, modified duality functor.

While this discussion may seem a little esoteric, it actually pays off! Our eventual goal is a diagrammatic category equivalent to $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$, with diagrams which are only defined up to planar isotopy. Such a pivotal category will automatically be strict. The 'strictification' we perform in this section bridges part of the inevitable gap between such a diagrammatic category, and the conventionally defined representation category.

Of course, it would be possible to have defined $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ in the first place in a way which made it strict as a pivotal category, but this would have required a strange and unmotivated definition of the dual of a morphism. Hopefully in the strictification we describe here, you'll see the exact origin of this unfortunate dual.

First of all, let's explicitly describe the pivotal category structure on $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$, in order to justify the statement that it is not strict[1]. We need to describe the duality functor $*$, pairing maps, and the 'pivotal' natural isomorphism $\tau : \mathbf{1} \to **$.

The contravariant[2] duality functor $*$ on $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ is given by the usual duality functor for linear maps. We need to dress up duals of representations of $U_q(\mathfrak{sl}_n)$ as representations again, which we do via the the antipode S, just as we use the comultiplication to turn tensor products of representations into representations. Thus for V some representation of $U_q(\mathfrak{sl}_n)$, $Z \in U_q(\mathfrak{sl}_n)$, $f \in V^*$ and $v \in V$, we say $(Zf)(v) = f(S(Z)v)$.

The pairing maps $p_V : V^* \otimes V \to \mathcal{A}$ are just the usual duality pairing maps for duals of vector spaces. That these are maps of representations follows easily from the Hopf algebra axioms. Be careful, however, to remember that the other vector space pairing map $V \otimes V^* \to \mathcal{A}$ is not generally a map of representations of the Hopf algebra. Similarly, the copairing $c_V : \mathcal{A} \to V \otimes V^*$ is equivariant, while $\mathcal{A} \to V^* \otimes V$ is not.

The pivotal isomorphism is where things get interesting, and we make use of the particular structure of the quantum group $U_q(\mathfrak{sl}_n)$.

Definition 3.4.1. We'll define the element $\tau_n = K^\rho = \prod_{j=1}^{n-1} K_j^{j(n-j)}$ of $U_q(\mathfrak{sl}_n)$.

Lemma 3.4.2. *Abusing notation, this gives an isomorphism of $U_q(\mathfrak{sl}_n)$ representations, $\tau_n : V \xrightarrow{\cong} V^{**}$, providing the components of the pivotal natural isomorphism.*

Proof. This relies on the formulas for the antipode acting on $U_q(\mathfrak{sl}_n)$; we need to check

$$\tau_n X_i^\pm = S^2(X_i^\pm)\tau_n.$$

Since $S^2(X_i^\pm) = q^{\pm 2} X_i^\pm$, this becomes the condition

$$\tau_n X_i^\pm \tau_n^{-1} = q^{\pm 2} X_i^\pm,$$

which follows immediately from the definition of τ_n and the commutation relations in the quantum group.

[1] Not strict as a pivotal category, that is. We've defined it in such a way that it's strict as a tensor category, meaning we don't bother explicitly reassociating tensor products.

[2] In fact, it's doubly contravariant; both with respect to composition, and tensor product.

We also need to check that τ satisfies the axioms for a pivotal natural isomorphism. Since the K_i are group-like in $U_q(\mathfrak{sl}_n)$ (i.e. $\Delta(K_i) = K_i \otimes K_i$), τ is a tensor natural transformation. Further

$$\begin{aligned} \tau^*_{V|V^{***}} \circ \tau_{V^*|V^*} &= (K^\rho)^*{}_{|V^{***}} \circ K^\rho{}_{|V^*} \\ &= (K^\rho) S(K^\rho) \\ &= 1, \end{aligned}$$

so τ^*_V and τ_{V^*} are inverses, as required. □

To construct the strictification of $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$, written as $\mathcal{S}\,\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$, we begin by defining it as a tensor category, taking the full tensor subcategory of $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ whose objects are generated by tensor product (but not duality) from the trivial representation $\mathcal{A}$, $V_{a<n}$ and $V^*_{a<n}$, for each $a = 1, \ldots, n-1$. Before describing the pivotal structure on $\mathcal{S}\,\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$, we might as well specify the equivalence; already there's only one sensible choice. We can include $\mathcal{S}\,\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ into $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ in one direction. For the other, we simply send $V^{*(n)}_{a<n}$ (that is, the n-th iterated dual) to $V_{a<n}$ or $V^*_{a<n}$, depending on whether n is even or odd, and modify each morphism by pre- and post-composing with the unique[3] appropriate tensor product of compositions of τ and τ^{-1}. Clearly the composition $\mathbf{Rep}\, U_q(\mathfrak{sl}_n) \to \mathcal{S}\,\mathbf{Rep}\, U_q(\mathfrak{sl}_n) \to \mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ is not the identity functor, but equally easily it's naturally isomorphic to the identity, again via appropriate tensor products of compositions of τ. Here, we're doing nothing more than identifying objects in a category related by a certain family of isomorphisms, and pointing out that the result is equivalent to what we started with!

The interesting aspect comes in the pivotal structure on $\mathcal{S}\,\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$. Of course, we define the duality cofunctor on objects of $\mathcal{S}\,\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ so it exchanges $V_{a<n}$ and $V^*_{a<n}$. It turns out (inevitably, according to the strictification result of [2], but explicitly here) that if we define pairing maps on $\mathcal{S}\,\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ by taking the image of the pairing maps for $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$, we make $\mathcal{S}\,\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ a strict pivotal category (and of course, we thus make the equivalence of categories from the previous paragraph an equivalence of pivotal categories). Explicitly, then, the pairing morphisms on $\mathcal{S}\,\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ are $p^{\mathcal{S}}_a : V^*_{a<n} \otimes V_{a<n} \to \mathcal{A} = p_a$ and $p^{\mathcal{S}}_{a^*} : V_{a<n} \otimes V^*_{a<n} \to \mathcal{A} = p_{a^*} \circ (\tau_a \otimes \mathbf{1}_{a^*})$. We've yet to define the duality cofunctor at the level of morphisms in $\mathcal{S}\,\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$; there's in fact a unique definition forced on us by Lemma 2.1.1. Thus

$$f^* = (\mathbf{1}_{a^*} \otimes p_{b^*}) \circ (\mathbf{1}_{a^*} \otimes (\tau_b \circ f \circ \tau_a^{-1}) \otimes \mathbf{1}_{b^*})) \circ (p^*_a \otimes \mathbf{1}_{b^*}).$$

The processes of strictifying, and of limiting our attention to the fundamental part of the representation theory, are independent; the discussion above applies exactly to $\mathbf{FundRep} U_q(\mathfrak{sl}_n)$. In that case, we obtain the full subcategory $\mathcal{S}\mathbf{FundRep} U_q(\mathfrak{sl}_n)$.

3.5 Generators for FundRep$U_q(\mathfrak{sl}_n)$

In this section we'll define certain morphisms in $\mathbf{FundRep} U_q(\mathfrak{sl}_n)$, which we'll term 'elementary' morphisms. Later, in Proposition 3.5.8 we'll show that they generate all of $\mathbf{FundRep} U_q(\mathfrak{sl}_n)$ as a tensor category.

We already have the duality pairing and copairing maps, discussed above, $p_V : V^* \otimes V \to \mathcal{A}$, and $c_V : \mathcal{A} \to V \otimes V^*$, for each fundamental representation $V = V_{a<n}$, and for arbitrary iterated duals $V = V^{*(k)}_{a<n}$. We also have the non-identity isomorphisms τ identifying objects with their double duals. Beyond those, we'll introduce some more interesting maps:

[3] It's unique, given the axioms for τ described in §2.1, and established for our particular τ in Lemma 3.4.2.

- a map which identifies a fundamental representations with the dual of another fundamental representation, $d_{a,n} : V_{a<n} \xrightarrow{\cong} V^*_{n-a<n}$, along with the inverses of these maps,
- a 'triple invariant', living in the tensor product of three fundamental representations $v^n_{a,b,c} : \mathcal{A} \to V_{a<n} \otimes V_{b<n} \otimes V_{c<n}$ with $a+b+c=n$, and
- a 'triple coinvariant', $w^n_{a,b,c} : V_{a<n} \otimes V_{b<n} \otimes V_{c<n} \to \mathcal{A}$, again with $a+b+c=n$.

Definition 3.5.1. The map $d_{a,n} : V_{a<n} \to V^*_{n-a<n}$ is specified recursively by

$$\begin{aligned}
&d_{0,n} : 1 \mapsto 1^*, \\
&d_{n,n} : 1 \mapsto 1^*, \\
&d_{a,n} = \mathfrak{i}_0 d_{a-1,n-1} p_{-1} + (-q)^{-a} \mathfrak{i}_{-1} d_{a,n-1} p_0.
\end{aligned}$$

Lemma 3.5.2. *The map $d_{a,n}$ is a map of $U_q(\mathfrak{sl}_n)$ representations.*

Proof. Since $d_{a,n}$ is defined in terms of $U_q(\mathfrak{sl}_{n-1})$ equivariant maps, we need only check $d_{a,n}$ commutes with $X^{\pm}_{n-1}$ and K_{n-1}. We'll do one calculation explicitly:

$$\begin{aligned}
d_{a,n} X^+_{n-1} &= \left(\mathfrak{i}_0 d_{a-1,n-1} p_{-1} + (-q)^{-a} \mathfrak{i}_{-1} d_{a,n-1} p_0\right) \mathfrak{i}_{-1} \mathfrak{i}_0 p_{-1} p_0 \\
&= \mathfrak{i}_0 d_{a-1,n-1} \mathfrak{i}_0 p_{-1} p_0 \\
&= \mathfrak{i}_0 \left(\mathfrak{i}_0 d_{a-2,n-2} p_{-1} + (-q)^{1-a} \mathfrak{i}_{-1} d_{a-1,n-2} p_0\right) \mathfrak{i}_0 p_{-1} p_0 \\
&= (-q)^{1-a} \mathfrak{i}_0 \mathfrak{i}_{-1} d_{a-1,n-2} p_{-1} p_0,
\end{aligned}$$

while

$$\begin{aligned}
X^+_{n-1} d_{a,n} &= -q \mathfrak{i}_0 \mathfrak{i}_{-1} p_0 p_{-1} \left(\mathfrak{i}_0 d_{a-1,n-1} p_{-1} + (-q)^{-a} \mathfrak{i}_{-1} d_{a,n-1} p_0\right) \\
&= (-q)^{1-a} \mathfrak{i}_0 \mathfrak{i}_{-1} p_0 d_{a,n-1} p_0) \\
&= (-q)^{1-a} \mathfrak{i}_0 \mathfrak{i}_{-1} p_0 \left(\mathfrak{i}_0 d_{a-1,n-2} p_{-1} + (-q)^{-a} \mathfrak{i}_{-1} d_{a,n-2} p_0\right) p_0) \\
&= (-q)^{1-a} \mathfrak{i}_0 \mathfrak{i}_{-1} d_{a-1,n-2} p_{-1} p_0).
\end{aligned}$$

The other two cases (commuting with X^-_{n-1} and K_{n-1}) are pretty much the same. □

Proposition 3.5.3. *The duals of the maps $d_{a,n}$ satisfy:*

$$d^*_{a,n} \tau_n = (-1)^{(n+1)a} d_{n-a,n}. \tag{3.5.1}$$

We need two lemmas before proving this.

Lemma 3.5.4.

$$\prod_{j=1}^{n-1} K^j_{j\,|V_{a<n}} = q^{n-a} \mathfrak{i}_{-1} p_{-1} + q^{-a} \mathfrak{i}_0 p_0.$$

Proof. Arguing inductively, and using the formula for $K_{n-1|V_{a<n}}$ from Equation (3.3.2), we find

$$\begin{aligned}
\prod_{j=1}^{n-1} K^j_{j\,|V_{a<n}} &= \left(\mathfrak{i}_{-1} \left(\prod_{j=1}^{n-2} K^j_{j\,|V_{a-1<n-1}}\right) p_{-1} + \mathfrak{i}_0 \left(\prod_{j=1}^{n-2} K^j_{j\,|V_{a<n-1}}\right) p_0\right) K^{n-1}_{n-1\,|V_{a<n}} \\
&= \left(\mathfrak{i}_{-1}(q^{n-a} \mathfrak{i}_{-1} p_{-1} + q^{-a+1} \mathfrak{i}_0 p_0) p_{-1} + \mathfrak{i}_0 (q^{n-a-1} \mathfrak{i}_{-1} p_{-1} + q^{-a} \mathfrak{i}_0 p_0) p_0\right) \times \\
&\quad \times \left(\mathfrak{i}_{-1}(\mathfrak{i}_{-1} p_{-1} + q^{n-1} \mathfrak{i}_0 p_0) p_{-1} + \mathfrak{i}_0 (q^{-n+1} \mathfrak{i}_{-1} p_{-1} + \mathfrak{i}_0 p_0) p_0\right) \\
&= q^{n-a} \mathfrak{i}_{-1} \mathfrak{i}_{-1} p_{-1} p_{-1} + q^{n-a} \mathfrak{i}_{-1} \mathfrak{i}_0 p_0 p_{-1} + q^{-a} \mathfrak{i}_0 \mathfrak{i}_{-1} p_{-1} p_0 + q^{-a} \mathfrak{i}_0 \mathfrak{i}_0 p_0 p_0 \\
&= q^{n-a} \mathfrak{i}_{-1} p_{-1} + q^{-a} \mathfrak{i}_0 p_0.
\end{aligned}$$

□

Lemma 3.5.5.
$$\tau_{n|V_{a<n}} = q^{n-a}\mathfrak{i}_{-1}\tau_{n-1|V_{a-1<n-1}}p_{-1} + q^{-a}\mathfrak{i}_0\tau_{n-1|V_{a<n-1}}p_0$$

Proof. Making use of Equation (3.3.3) and Lemma 3.5.4:

$$\begin{aligned}
\tau_{n|V_{a<n}} &= \tau_{n-1|V_{a<n}} \times \prod_{j=1}^{n-1} K^j_{j|V_{a<n}} \\
&= \left(\mathfrak{i}_{-1}\tau_{n-1|V_{a-1<n-1}}p_{-1} + \mathfrak{i}_0\tau_{n-1|V_{a<n-1}}p_0\right) \times \left(q^{n-a}\mathfrak{i}_{-1}p_{-1} + q^{-a}\mathfrak{i}_0p_0\right) \\
&= q^{n-a}\mathfrak{i}_{-1}\tau_{n-1|V_{a-1<n-1}}p_{-1} + q^{-a}\mathfrak{i}_0\tau_{n-1|V_{a<n-1}}p_0.
\end{aligned}$$
□

Proof of Proposition 3.5.3. The proposition certainly holds for $a = 0$ or $a = n$, where all three of the maps $d^*_{a,n}, \tau_n$ and $d_{n-a,n}$ are just the identity, and the sign is $+1$. Otherwise, we proceed inductively. First, write

$$d^*_{a,n} = \mathfrak{i}_{-1}d^*_{a-1,n-1}p_0 + (-q)^{-a}\mathfrak{i}_0d^*_{a,n-1}p_{-1}.$$

Using Lemma 3.5.5 we then have

$$\begin{aligned}
d^*_{a,n}\tau_n &= \left(\mathfrak{i}_{-1}d^*_{a-1,n-1}p_0 + (-q)^{-a}\mathfrak{i}_0d^*_{a,n-1}p_{-1}\right)\left(q^a\mathfrak{i}_{-1}\tau_{n-1}p_{-1} + q^{a-n}\mathfrak{i}_0\tau_{n-1}p_0\right) \\
&= (-q)^{-a}q^a\mathfrak{i}_0d^*_{a,n-1}\tau_{n-1}p_{-1} + q^{a-n}\mathfrak{i}_{-1}d^*_{a-1,n-1}\tau_{n-1}p_0 \\
&= (-1)^a\mathfrak{i}_0(-1)^{na}d_{n-1-a,n-1}p_{-1} + q^{a-n}\mathfrak{i}_{-1}(-1)^{n(a-1)}d_{n-a,n-1}p_0 \\
&= (-1)^{(n+1)a}\mathfrak{i}_0d_{n-1-a,n-1}p_{-1} + q^{a-n}(-1)^{n(a-1)}\mathfrak{i}_{-1}d_{n-a,n-1}p_0,
\end{aligned}$$

while

$$\begin{aligned}
(-1)^{(n+1)a}d_{n-a,n} &= (-1)^{(n+1)a}(\mathfrak{i}_0d_{n-a-1,n-1}p_{-1} + (-q)^{a-n}\mathfrak{i}_{-1}d_{n-a,n-1}p_0) \\
&= (-1)^{(n+1)a}\mathfrak{i}_0d_{n-a-1,n-1}p_{-1} + q^{a-n}(-1)^{n(a-1)}\mathfrak{i}_{-1}d_{n-a,n-1}p_0).
\end{aligned}$$
□

Finally, we need to define the triple invariants $v^n_{a,b,c}$ and $w^n_{a,b,c}$.

Definition 3.5.6. When $a + b + c = m$, we define $v^n_{a,b,c} \in V_{a<n} \otimes V_{b<n} \otimes V_{c<n}$ and $w^n_{a,b,c} \in V^*_{a<n} \otimes V^*_{b<n} \otimes V^*_{c<n}$ by the formulas

$$\begin{aligned}
v^0_{0,0,0} &= 1 \otimes 1 \otimes 1, \\
w^0_{0,0,0} &= 1^* \otimes 1^* \otimes 1^*, \\
v^n_{a,b,c} &= (-1)^c q^{b+c}(\mathfrak{i}_{-1} \otimes \mathfrak{i}_0 \otimes \mathfrak{i}_0)(v^{n-1}_{a-1,b,c}) + \\
&\quad (-1)^a q^c \quad (\mathfrak{i}_0 \otimes \mathfrak{i}_{-1} \otimes \mathfrak{i}_0)(v^{n-1}_{a,b-1,c}) + \\
&\quad (-1)^b \qquad (\mathfrak{i}_0 \otimes \mathfrak{i}_0 \otimes \mathfrak{i}_{-1})(v^{n-1}_{a,b,c-1}),
\end{aligned} \tag{3.5.2}$$

and

$$\begin{aligned}
w^n_{a,b,c} &= (-1)^c \qquad (w^{n-1}_{a-1,b,c})(p_{-1} \otimes p_0 \otimes p_0) + \\
&\quad (-1)^a q^{-a} \quad (w^{n-1}_{a,b-1,c})(p_0 \otimes p_{-1} \otimes p_0) + \\
&\quad (-1)^b q^{-a-b}(w^{n-1}_{a,b,c-1})(p_0 \otimes p_0 \otimes p_{-1}).
\end{aligned} \tag{3.5.3}$$

Lemma 3.5.7. *The maps $v^n_{a,b,c}$ and $w^n_{a,b,c}$ are maps of $U_q(\mathfrak{sl}_n)$ representations.*

Proof. As in Lemma 3.5.2, we just need to check that these maps commute with $X^{\pm}_{n-1}$ and K_{n-1}. We'll do the explicit calculation for X^{+}_{n-1}; here we need to check that

$$X^{+}_{n-1|V_{a<b}\otimes V_{b<n}\otimes V_{c<b}} v^n_{a,b,c} = 0$$

and

$$w^n_{a,b,c} X^{+}_{n-1|V_{a<b}\otimes V_{b<n}\otimes V_{c<b}} = 0.$$

First, we need to know how X^{+}_{n-1} acts on the tensor product of three representations, via

$$\Delta^{(2)}(X^{+}_{n-1}) = X^{+}_{n-1} \otimes K_{n-1} \otimes K_{n-1} + 1 \otimes X^{+}_{n-1} \otimes K_{n-1} + 1 \otimes 1 \otimes X^{+}_{n-1}.$$

Next, let's use two steps of the inductive definition of $v^n_{a,b,c}$ to write

$$\begin{aligned}
v^n_{a,b,c} = q^{2b+2c}&(\mathfrak{i}_{-1}\mathfrak{i}_{-1} \otimes \mathfrak{i}_0\mathfrak{i}_0 \otimes \mathfrak{i}_0\mathfrak{i}_0)(v^{n-2}_{a-2,b,c}) + \\
q^{2c} \quad &(\mathfrak{i}_0\mathfrak{i}_0 \otimes \mathfrak{i}_{-1}\mathfrak{i}_{-1} \otimes \mathfrak{i}_0\mathfrak{i}_0)(v^{n-2}_{a,b-2,c}) + \\
&(\mathfrak{i}_0\mathfrak{i}_0 \otimes \mathfrak{i}_0\mathfrak{i}_0 \otimes \mathfrak{i}_{-1}\mathfrak{i}_{-1})(v^{n-2}_{a,b,c-2}) + \\
(-1)^{a+b}q^c&(-\mathfrak{i}_0\mathfrak{i}_0 \otimes \mathfrak{i}_{-1}\mathfrak{i}_0 \otimes \mathfrak{i}_0\mathfrak{i}_{-1} + q^{-1}\mathfrak{i}_0\mathfrak{i}_0 \otimes \mathfrak{i}_0\mathfrak{i}_{-1} \otimes \mathfrak{i}_{-1}\mathfrak{i}_0)(v^{n-2}_{a,b-1,c-1}) + \\
(-1)^{b+c}q^{b+c}&(\mathfrak{i}_{-1}\mathfrak{i}_0 \otimes \mathfrak{i}_0\mathfrak{i}_0 \otimes \mathfrak{i}_0\mathfrak{i}_{-1} - q^{-1}\mathfrak{i}_0\mathfrak{i}_{-1} \otimes \mathfrak{i}_0\mathfrak{i}_0 \otimes \mathfrak{i}_{-1}\mathfrak{i}_0)(v^{n-2}_{a-1,b,c-1}) + \\
(-1)^{a+c}q^{b+2c}&(-\mathfrak{i}_{-1}\mathfrak{i}_0 \otimes \mathfrak{i}_0\mathfrak{i}_{-1} \otimes \mathfrak{i}_0\mathfrak{i}_0 + q^{-1}\mathfrak{i}_0\mathfrak{i}_{-1} \otimes \mathfrak{i}_{-1}\mathfrak{i}_0 \otimes \mathfrak{i}_0\mathfrak{i}_0)(v^{n-2}_{a-1,b-1,c}).
\end{aligned}$$

We'll now show X^{+}_{n-1} kills each term (meaning each line, as displayed above) separately. In each case, it follows immediately from the formulas for X^{+}_{n-1} and K_{n-1} in Equations (3.3.1) and (3.3.2). In the first three terms, we use $X^{+}_{n-1}\mathfrak{i}_{-1}\mathfrak{i}_{-1} = X^{+}_{n_1}\mathfrak{i}_0\mathfrak{i}_0 = 0$. In the fourth term, we see

$$\begin{aligned}
\Delta^{(2)}(X^{+}_{n-1})&(-\mathfrak{i}_0\mathfrak{i}_0 \otimes \mathfrak{i}_{-1}\mathfrak{i}_0 \otimes \mathfrak{i}_0\mathfrak{i}_{-1} + q^{-1}\mathfrak{i}_0\mathfrak{i}_0 \otimes \mathfrak{i}_0\mathfrak{i}_{-1} \otimes \mathfrak{i}_{-1}\mathfrak{i}_0) \\
&= -\mathfrak{i}_0\mathfrak{i}_0 \otimes \mathfrak{i}_{-1}\mathfrak{i}_0 \otimes \mathfrak{i}_{-1}\mathfrak{i}_0 + q^{-1}q\mathfrak{i}_0\mathfrak{i}_0 \otimes \mathfrak{i}_{-1}\mathfrak{i}_0 \otimes \mathfrak{i}_{-1}\mathfrak{i}_0 \\
&= 0.
\end{aligned}$$

Here only one of the three terms of $\Delta^{(2)}(X^{+}_{n-1})$ acts nontrivially on each of the two terms. The fifth and sixth terms are exactly analogous.

For $w^n_{a,b,c}$ we use the same trick; write it in terms of $w^{n-2}_{a-2,b,c}$, $w^{n-2}_{a,b-2,c}$, $w^{n-2}_{a,b,c-2}$, $w^{n-2}_{a,b-1,c-1}$, $w^{n-2}_{a-1,b,c-1}$ and $w^{n-2}_{a-1,b-1,c}$, and show that each of these terms multiplied by X^{+}_{n-1} gives zero separately. □

Remark. Going through this proof carefully, you'll see that it would still work with some variation allowed in constants the definitions of $v^n_{a,b,c}$ and of $w^n_{a,b,c}$ in Equations (3.5.2) and (3.5.3). However, given the normalisation for $d_{a,n}$ that we've chosen in Definition 3.5.1, these normalisation constants are pinned down by Lemmas 4.2.4 and 4.2.6 below.

Later, we'll discuss relationships between these elementary morphisms, but for now we want to justify our interest in them.

Proposition 3.5.8. *The elementary morphisms generate, via tensor product, composition and linear combination, all the morphisms in* $\mathbf{FundRep}U_q(\mathfrak{sl}_n)$.

Remark. Certainly, they can't generate all of $\mathbf{Rep}\,U_q(\mathfrak{sl}_n)$, simply because the sources and targets of elementary morphisms are all in $\mathbf{FundRep}U_q(\mathfrak{sl}_n)$!

Remark. A fairly abstract proof of this fact for $n = 3$ has been given by Kuperberg [24].[4] Briefly, he specialises to $q = 1$, and considers the subcategory of $\mathbf{FundRep}U(\mathfrak{sl}_3) \cong \mathbf{FundRep}SU(3)$

[4] His other results additionally give a direct proof for $n = 3$, although quite different from the one here.

generated by the elementary morphisms. He extends this by formally adding kernels and cokernels of morphisms (that is, by taking the Karoubi envelope). This extension being (equivalent to) all of $\mathbf{Rep}\, SU(3)$ is enough to obtain the result. To see this, he notes that the extension is the sort of category to which an appropriate Tannaka-Krein theorem applies, allowing him to say that it is the representation category of some compact Lie group. Some arguments about the symmetries of the 'triple invariant' morphisms allow him to conclude that this group must in fact be $SU(3)$. While I presume this proof can be extended to cover all n, I prefer to give a more direct proof, based on the quantum version of Frobenius-Schur duality.

Proof. It's a sort of bootstrap argument. First, we notice that the action of the braid group $\mathfrak{B}_m$ on tensor powers of the standard representation $V_{1<n}$ can be written in terms of elementary morphisms. Next, we recall that the braid group action generates all the endomorphisms of $V_{1<n}^{\otimes m}$, and finally, we show how to map arbitrary tensor products of fundamental representations into a tensor power of the standard representation, using only elementary morphisms.

These ideas are encapsulated in the following four lemmas.

Lemma 3.5.9. *The action of the braid group $\mathfrak{B}_m$ on $V_{1<n}^{\otimes m}$, given by R-matrices, can be written in terms of elementary morphisms.*

Proof. We only need to prove the result for $V_{1<n}^{\otimes 2}$. There, we can be particularly lazy, taking advantage of the fact that $\dim \mathrm{Hom}_{U_q(\mathfrak{sl}_n)}\left(V_{1<n}^{\otimes 2}, V_{1<n}^{\otimes 2}\right) = 2$. (There are many paths to seeing this, the path of least effort perhaps being that $\mathrm{Hom}_{U_q(\mathfrak{sl}_n)}\left(V_{1<n}^{\otimes 2}, V_{1<n}^{\otimes 2}\right)$ is canonically isomorphic to $\mathrm{Hom}_{U_q(\mathfrak{sl}_n)}\left(V_{1<n} \otimes V_{1<n}^{*}, V_{1<n}^{*} \otimes V_{1<n}\right)$, and that both the source and target representations there decompose into the direct sum of the adjoint representation and the trivial representation.) The map

$$p = (w_{1,1,n-2}^{n} \otimes \mathbf{1}_{V_{1<n}} \otimes \mathbf{1}_{V_{1<n}}) \circ (\mathbf{1}_{V_{1<n}} \otimes \mathbf{1}_{V_{1<n}} \otimes v_{n-2,1,1}^{n})$$

is not a multiple of the identity, so every endomorphism of $V_{1<n}^{\otimes 2}$ is a linear combination of compositions of elementary morphisms, in particular the braiding. □

Remark. In fact, the map associated to a positive crossing is $q^{n-1}\mathbf{1}_{V_{1<n}^{\otimes 2}} - q^{n}p$. It's easy to prove that this, along with the negative crossing (obtained by replacing q with q^{-1}), is the only linear combination of $\mathbf{1}$ and p which satisfies the braid relation. One could presumably also check that this agrees with the explicit formulas given in [4, §8.3 and §10.1][5].

Lemma 3.5.10. *The image of $\mathfrak{B}_m$ linearly spans $\mathrm{Hom}_{U_q(\mathfrak{sl}_n)}\left(V_{1<n}^{\otimes m}, V_{1<n}^{\otimes m}\right)$.*

Proof. This is the quantum version of Schur-Weyl duality. See [4, §10.2B] and [12]. □

Lemma 3.5.11. *For any tensor product of fundamental representations $\bigotimes_i V_{\alpha_i}$, there's some natural number m and a pair of morphisms constructed out of elementary ones $\iota : \bigotimes_i V_{\alpha_i} \to V_1^{\otimes m}$ and $\pi : V_1^{\otimes m} \to \bigotimes_i V_{\alpha_i}$ such that $\pi \circ \iota = \mathbf{1}_{\bigotimes_i V_{\alpha_i}}$.*

Proof. We just need to do this for a single fundamental representation, then tensor together those morphisms. For a single fundamental representation, first define $\iota' : V_a \to V_1 \otimes V_{a-1} = (p_{V_{n-a}} \otimes \mathbf{1}_{V_1} \otimes \mathbf{1}_{V_{a-1}}) \circ (d_a \otimes v_{n-a,1,a-1}^{n})$ and $\pi' : V_1 \otimes V_{a-1} \to V_a = x(w_{1,a-1,n-a}^{n} \otimes d_a) \circ (\mathbf{1}_{V_1} \otimes \mathbf{1}_{V_{a-1}} \otimes c_{V_{n-a}})$. The composition $\pi' \circ \iota'$ is an endomorphism of the irreducible V_a, which I claim is nonzero, and so for some choice of the coefficient x is the identity. Now build the maps $\iota : V_a \to V_1^{\otimes a}$ and $\pi : V_{<1}^{\otimes a} \to V_a$ as iterated compositions of these maps. □

[5]Although be careful there — the formula for the universal R-matrix in §8.3.C is incorrect, although it shouldn't matter for such a small representation. The order of the product is backwards [5].

Lemma 3.5.12. *In fact, given two such tensor products of fundamental representations $\bigotimes_i V_{\alpha_i}$ and $\bigotimes_j V_{\beta_j}$, such that there is some nonzero $U_q(\mathfrak{sl}_n)$ map between them, it's possible to choose $\iota_\alpha, \pi_\alpha, \iota_\beta$ and π_β as in the previous lemma, with the same value of m.*

Proof. (This argument is due to Ben Webster. Thanks!) First just pick $\iota_\alpha, \pi_\alpha, \iota'_\beta$ and π'_β as in the previous lemma, with possibly different values of m, say m_α and m_β. Now, if there's some $U_q(\mathfrak{sl}_n)$ map between $\bigotimes_i V_{\alpha_i}$ and $\bigotimes_j V_{\beta_j}$, then there's also a $U_q(\mathfrak{sl}_n)$ map between $V_1^{\otimes m_\alpha}$ and $V_1^{\otimes m_\beta}$. This can happen if and only if there's some $\mathfrak{sl}_n$ map, and in fact some $SL(n)$ map, between the corresponding classical representations, $\mathbb{C}^{n \otimes m_\alpha}$ and $\mathbb{C}^{n \otimes m_\beta}$. This $SL(n)$ intertwiner guarantees that the center of $SL(n)$ acts in the same way on each representation; the element $\sqrt[n]{1} I$ acts by $1^{m_\alpha / n}$ and $1^{m_\beta / n}$ on the two representations, so m_α and m_β must be congruent mod n. Assuming, without loss of generality, that $m_\beta \leq m_\alpha$ we now define $\iota_\beta = \iota'_\beta \otimes \iota^{\otimes (m_\alpha - m_\beta)/n}$, and $\pi_\beta = \pi'_\beta \otimes \pi^{\otimes (m_\alpha - m_\beta)/n}$, where $\iota : \mathcal{A} \to V_1^{\otimes n}$ and $\pi : V_1^{\otimes n} \to \mathcal{A}$ are diagrammatic morphisms satisfying $\pi \circ \iota = \mathbf{1}_{\mathcal{A}}$. (These certainly exist!) ☐

Putting all this together, we take an arbitrary morphism $\phi : \bigotimes_i V_{\alpha_i} \to \bigotimes_j V_{\beta_j}$ in $\mathbf{FundRep} U_q(\mathfrak{sl}_n)$, and pick $\iota_\alpha, \pi_\alpha, \iota_\beta$ and π_β as in the last lemma. Then

$$\phi = \pi_\beta \circ \iota_\beta \circ \phi \circ \pi_\alpha \circ \iota_\alpha.$$

The middle of this composition, $\iota_\beta \circ \phi \circ \pi_\alpha$, is an endomorphism of $V_1^{\otimes m}$, so can be written as a linear combination of braids, and hence as a linear combination of morphisms built from elementary morphisms. ☐

3.6 The representation functor

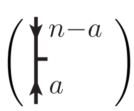
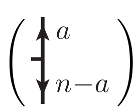
Finally, everything is in place for a definition of the representation functor $\mathbf{Rep} : \mathcal{S}ym_n \to \mathcal{S}\mathbf{FundRep} U_q(\mathfrak{sl}_n)$. (We're defining the functor into the strictification $\mathcal{S}\mathbf{FundRep} U_q(\mathfrak{sl}_n)$, as discussed in §3.4. To obtain the functor to the more usual representation theory $\mathbf{FundRep} U_q(\mathfrak{sl}_n)$, you need to compose with the equivalences of categories described in that section; of course, this is just the inclusion of a full subcategory!) First define $\mathbf{Rep}'$ on the free tensor category $\mathcal{T}_n$, by

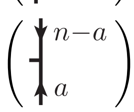
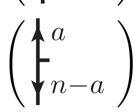
$$\mathbf{Rep}'(a \in \{1, \ldots, n-1\}) = V_{a<n}, \qquad \mathbf{Rep}'(a^*) = V^*_{a<n},$$

on objects, and

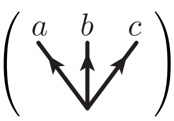
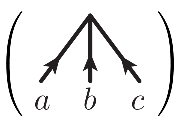
$\mathbf{Rep}'$  $= p_a$, $\mathbf{Rep}'$  $= p_{a^*} \circ (\tau_n \otimes \mathbf{1})$,

$\mathbf{Rep}'$  $= c_a$, $\mathbf{Rep}'$  $= (Id \otimes \tau_n^{-1}) \circ c_{a^*}$,

$\mathbf{Rep}'$  $= d_{a,n}$, $\mathbf{Rep}'$  $= d^{-1}_{a,n}$,

$\mathbf{Rep}'$  $= (-1)^{(n+1)a} d_{a,n}$, $\mathbf{Rep}'$  $= (-1)^{(n+1)a} d^{-1}_{a,n}$,

and

$\mathbf{Rep}'$  $= v^n_{a,b,c}$ $\mathbf{Rep}'$  $= w^n_{a,b,c}$

on morphisms.

Proposition 3.6.1. *The functor* $\mathbf{Rep}'$ *descends to a functor defined on the quotient*

$$\mathcal{S}ym_n \to \mathcal{S}\mathbf{FundRep}U_q(\mathfrak{sl}_n)\,.$$

This is proved in the next chapter; we could do it now, but the proof will read more nicely once we have better diagrams available.

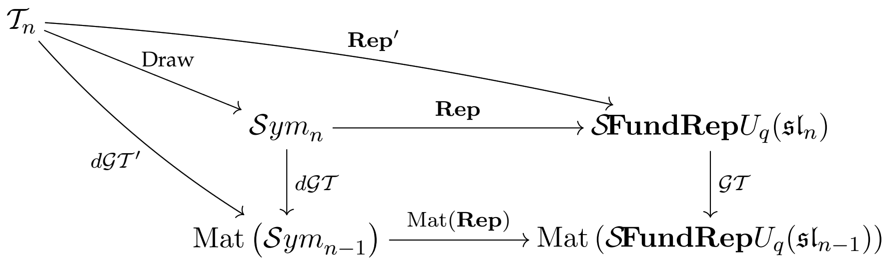
Chapter 4

The diagrammatic Gel'fand-Tsetlin functor

It's now time to define the diagrammatic Gel'fand-Tsetlin functor,

$$d\mathcal{GT} : \mathcal{S}ym_n \to \mathrm{Mat}\left(\mathcal{S}ym_{n-1}\right).$$

The first incarnation of the diagrammatic functor will be a functor defined (in §4.1) on the free version of the diagrammatic category, $d\mathcal{GT}' : \mathcal{T}_n \to \mathrm{Mat}\left(\mathcal{S}ym_{n-1}\right)$. We'll need to show that it descends to the quotient $\mathcal{S}ym_n$ (in §4.2). This definition is made so that the perimeter of the diagram



(4.0.1)

commutes.

Happily, as an easy consequence of this, we'll see that the representation functor $\mathbf{Rep} : \mathcal{T}_n \to \mathcal{S}\mathbf{FundRep}U_q(\mathfrak{sl}_n)$ also descends to the quotient $\mathcal{S}ym_n$.

At the end of the chapter in §4.4 we will describe how to compute the diagrammatic Gel'fand-Tsetlin functor, and perform a few small calculations which we'll need later.

4.1 Definition on generators

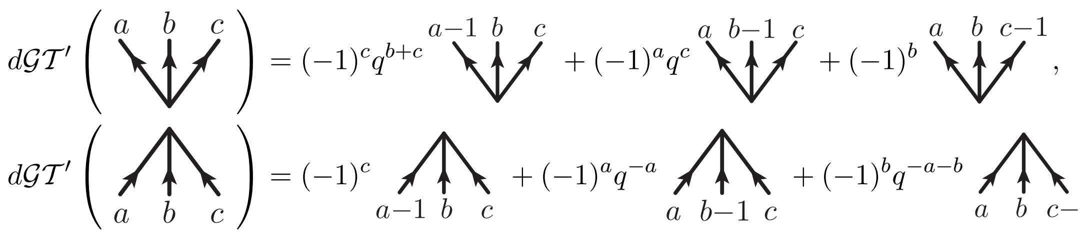
The target category for the diagrammatic Gel'fand-Tsetlin functor is the matrix category over $\mathcal{S}ym_{n-1}$. We thus need to send each object of $\mathcal{T}_n$ to a direct sum of objects of $\mathcal{S}ym_{n-1}$, and for each generating morphism of $\mathcal{T}_n$, we need to pick an appropriate matrix of morphisms in $\mathcal{S}ym_{n-1}$. On objects, we use the obvious

$$\begin{aligned} d\mathcal{GT}'(a) &= (a-1) \oplus a \\ d\mathcal{GT}'(a^*) &= (a-1)^* \oplus a^* \end{aligned} \tag{4.1.1}$$

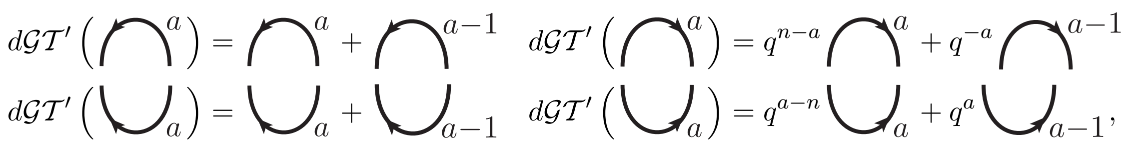
(omitting the nonsensical direct summand in the case that $a = 0, 0^*, n$ or n^*), extending to tensor products by distributing over direct sum. For morphisms we'll take advantage of the lack of 'multiplicities' in Equation 4.1.1 to save on some notation. An arbitrary morphism in

Mat $(\mathcal{S}ym_{n-1})$ has rows and columns indexed by tensor products of the fundamental (and trivial) objects and their duals, $0, 1, \ldots, n-1, 0^*, 1^*, \ldots, (n-1)^*$. Notice, however, that morphisms in the image of $d\mathcal{GT}'$ have distinct labels on each row (and also on each column). Moreover, a (diagrammatic) morphism in $\mathcal{S}ym_{n-1}$ explicitly encodes its own source and target (reading across the top and bottom boundary points). We'll thus abuse notation, and write a sum of matrix entries, instead of the actual matrix, safe in the knowledge that we can unambiguously reconstruct the matrix, working out which term should sit in each matrix entry. For example, if $f : a \to b$ is a morphism in $\mathcal{T}_n$, then $d\mathcal{GT}'(f)$ is a matrix $(a-1) \oplus a \to (b-1) \oplus b$, which we ought to write as $\left(\begin{smallmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{smallmatrix}\right)$, but will simply write as $f_{11} + f_{12} + f_{21} + f_{22}$. This notational abuse is simply for the sake of brevity when writing down matrices with many zero entries. When composing matrices written this way, you simply distribute the composition over summation, ignoring any non-composable terms.

That said, we now define $d\mathcal{GT}'$ on trivalent vertices by

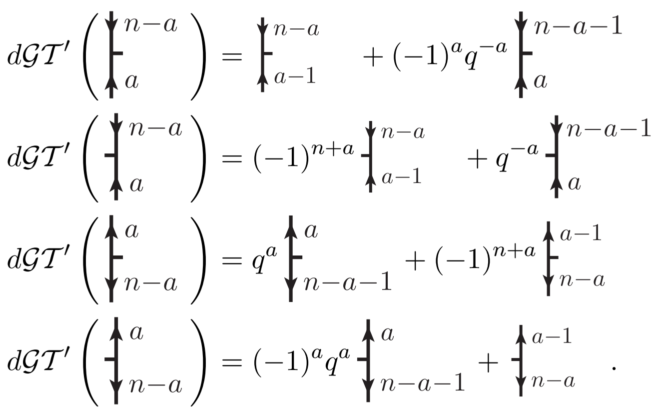


(the right hand sides of these equations are secretly 8×1 and 1×8 matrices, respectively), on the cups and caps by



(4.1.2)

and on the tags by



(4.1.3)

The functor $d\mathcal{GT}'$ then extends to all of $\mathcal{T}_n$ as a tensor functor.

Proposition 4.1.1. *The outer square of Equation* (4.0.1) *commutes.*

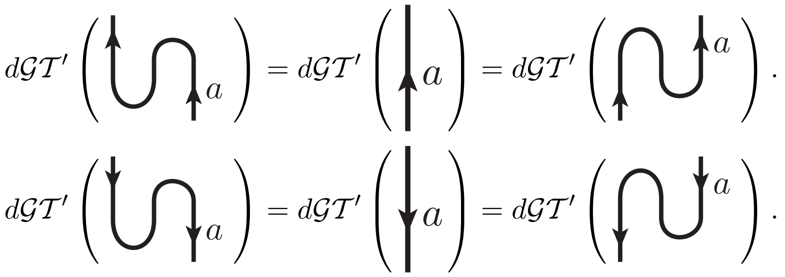
Proof. Straight from the definitions; compare the definition above with the definition of **Rep**′ from §3.6, and use the inductive Definitions 3.5.1 and 3.5.6 of the generating morphisms over in the representation theory. ☐

4.2 Descent to the quotient

Proposition 4.2.1. *The functor $d\mathcal{GT}'$ descends from $\mathcal{T}_n$ to the quotient $\mathcal{S}ym_n$, where we call it simply $d\mathcal{GT}$.*

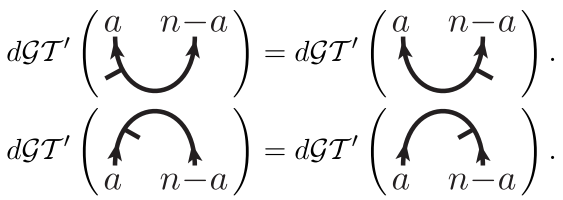
Proof. Read the following Lemmas 4.2.2, 4.2.3, 4.2.4, 4.2.5 and 4.2.6. ☐

Lemma 4.2.2. *We first check the relation in Equation* (2.2.4).

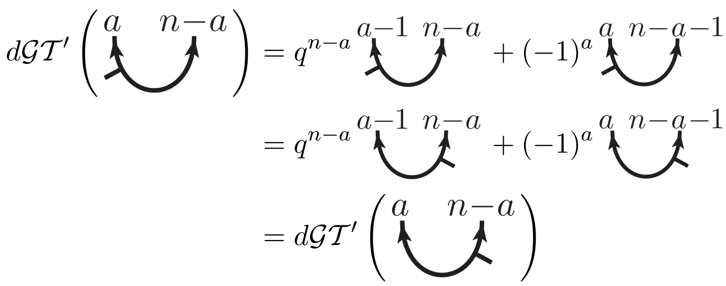


Proof. This is direct from the definitions in Equation (4.1.2), and using the fact that we can straighten strands in the target category $\mathcal{S}ym_{n-1}$. ☐

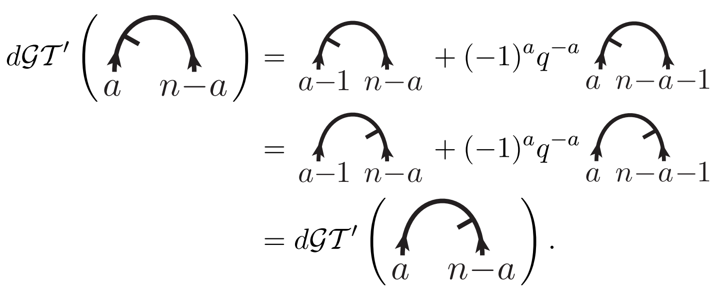
Lemma 4.2.3. *Next, the relations in Equation* (2.2.5).



Proof.

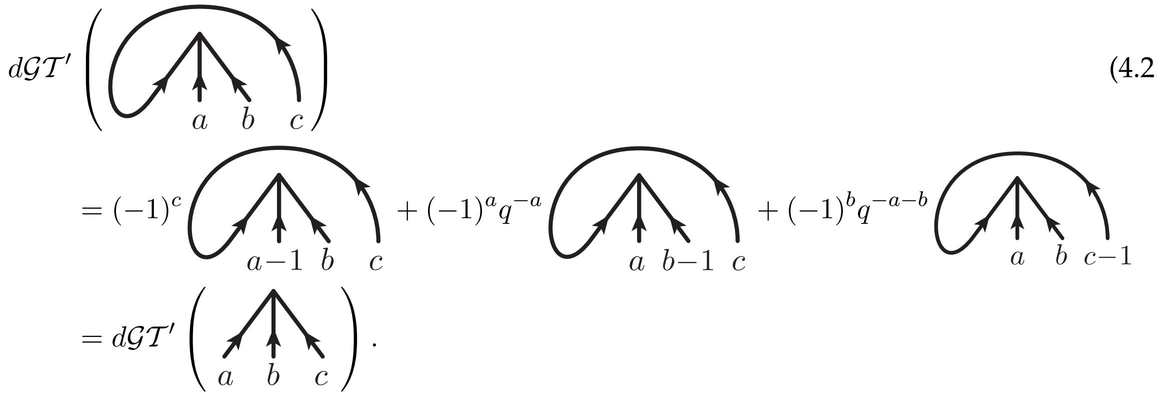


making use of the definitions in Equations (4.1.2) and (4.1.3), and the identity appearing in Equation (2.2.5) for $\mathcal{S}ym_{n-1}$, and

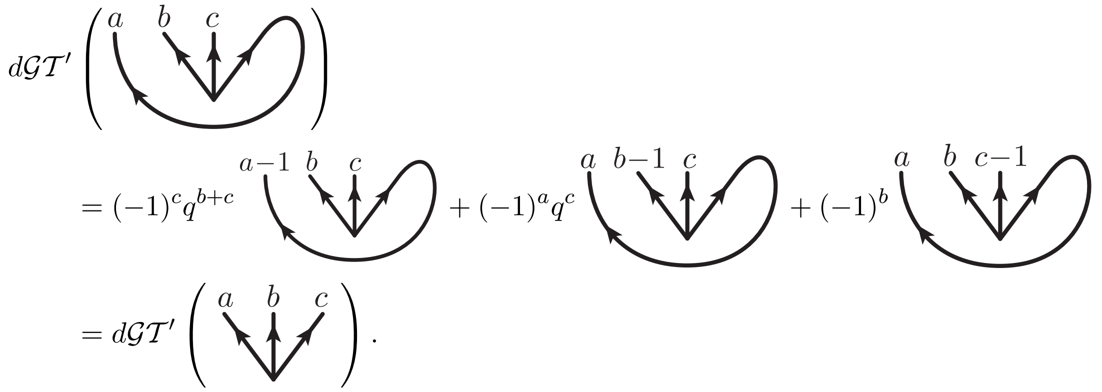


☐

Lemma 4.2.4. *We check the 'rotation relation' from Equation* (2.2.7) *is in the kernel of $d\mathcal{GT}'$, which of course implies the '2π rotation relation from Equations* (2.2.3) *is also in the kernel.*

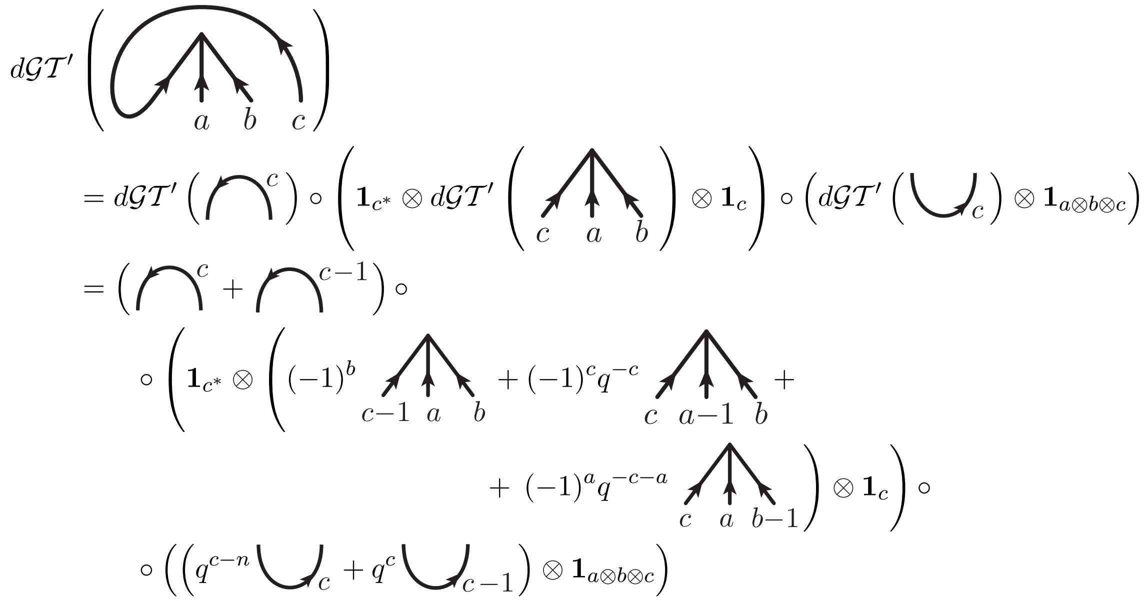


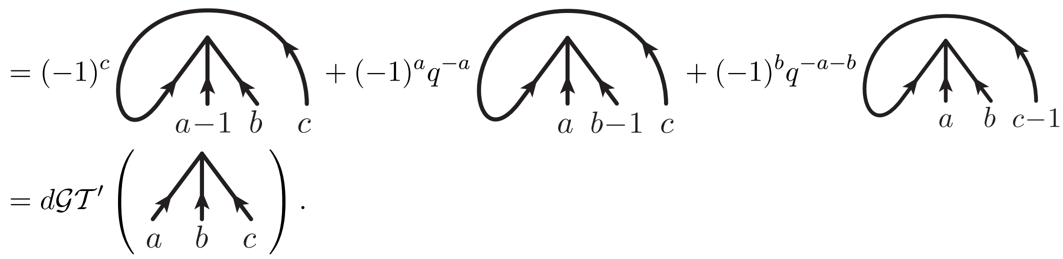
(4.2.1)



(4.2.2)

Proof. The statement of the lemma essentially contains the proof; we'll spell out Equation (4.2.1) in gory detail:

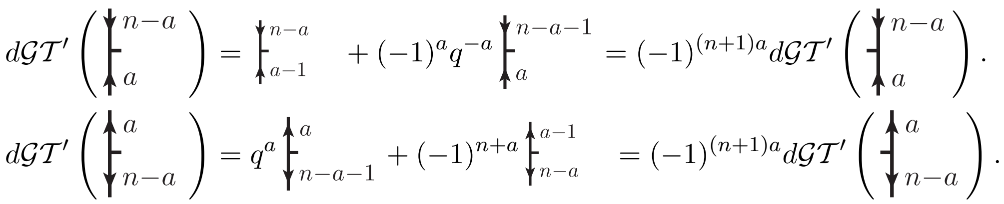




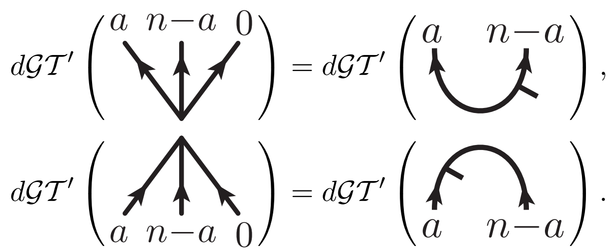
Going from the third last to the second last line, we simply throw out all non-composable cross terms. ☐

The next two lemmas are similarly direct.

Lemma 4.2.5.



Lemma 4.2.6.

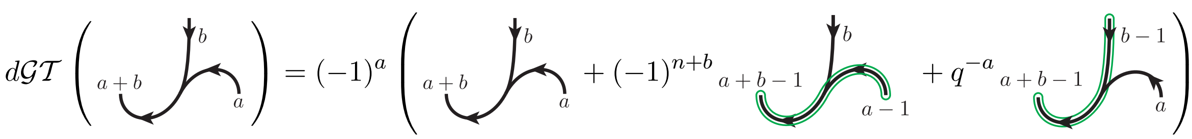


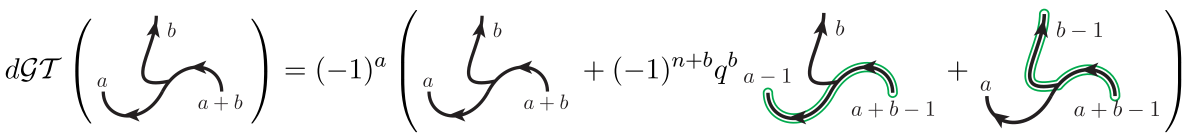
We can now give the

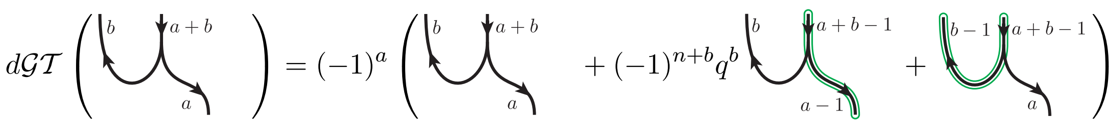
Proof of Proposition 3.6.1. Somewhat surprisingly, the fact that $d\mathcal{GT}'$ descends from $\mathcal{T}_n$ to $\mathcal{S}ym_n$ implies that $\mathbf{Rep}' : \mathcal{T}_n \to \mathcal{S}\mathbf{FundRep}U_q(\mathfrak{sl}_n)$ also descends to $\mathcal{S}ym_n$, simply because the outer square of Equation (4.0.1) commutes, as pointed out in Proposition 4.1.1. ☐

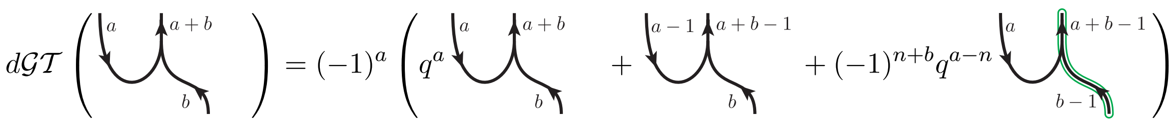
4.3 Calculations on small webs

We'll begin with some calculations of $d\mathcal{GT}$ on the flow vertices introduced in §2.3.

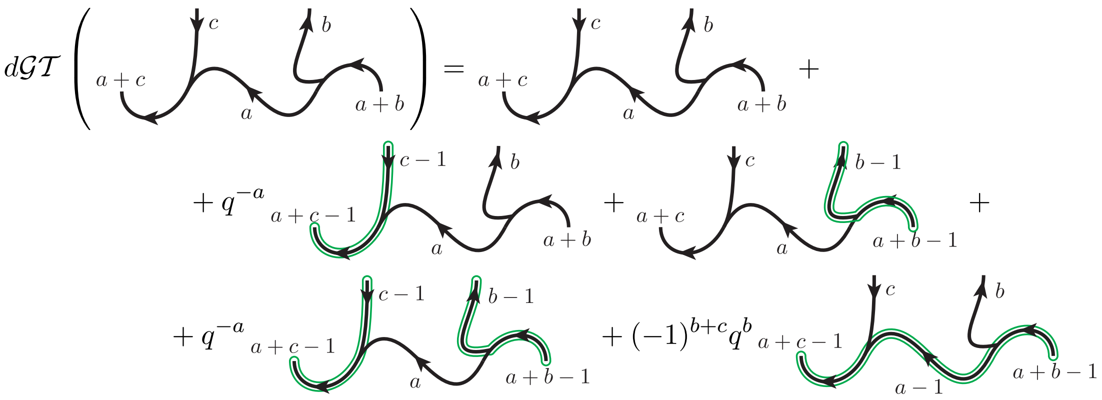




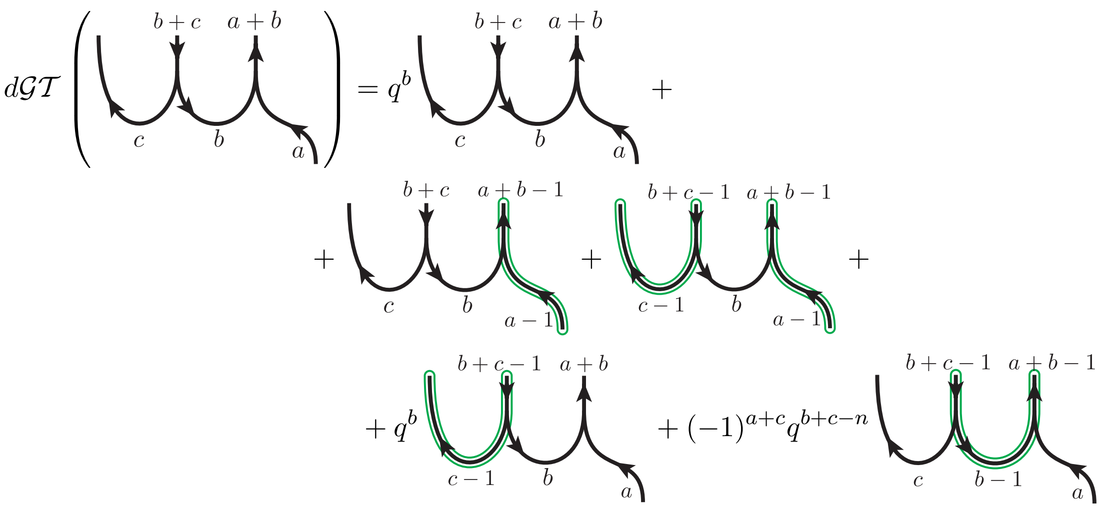




We'll also need the following formulas for two vertex webs:



(4.3.1)



(4.3.2)

4.4 A path model, and polygons.

Call a diagram in $\mathcal{S}ym_n$ in which all vertices are flow vertices and there are no tags (except those implicitly hidden inside the vertices) a *flow diagram*. In particular, the flow vertices themselves, and the $\mathcal{P}$- and $\mathcal{Q}$-polygons described in §2.4 are flow diagrams. A *reduction path* on a flow diagram is some disjoint union of cycles and paths beginning and ending at the boundary of the diagram, always matching the orientations on the edges of the diagram. We then notice that if v is a flow vertex, the three terms of $d\mathcal{GT}(v)$ correspond to the three reduction paths on the vertex. Specifically, for each reduction path, in the corresponding term of $d\mathcal{GT}(v)$ the label on each traversed edge has been reduced by one. Each term additionally has a coefficient $\pm q^k$ for some integer k.

It's easy to see that this condition must also hold for larger flow diagrams. For a reduction path π on a diagram $D \in \mathcal{S}ym_n$, write $\pi(D)$ for the diagram in $\mathcal{S}ym_{n-1}$ obtained from D by reducing the label on each edge traversed by π. Further, write $t(\pi)$ for the subset of the tensor factors in the target of D which π reaches, and $s(\pi)$ for the subset of the tensor factors in the source.

Proposition 4.4.1. *The result of applying $d\mathcal{GT}$ to a flow diagram D is a sum over the reduction paths on D:*

$$d\mathcal{GT}(D) = \sum_{\substack{\text{reduction} \\ \text{paths } \pi}} \pm q^{\bullet} \pi(D). \tag{4.4.1}$$

Further, two terms are in same matrix entry (remember, $d\mathcal{GT}$ is a functor $\mathcal{S}ym_n \to \mathrm{Mat}\left(\mathcal{S}ym_{n-1}\right)$) iff the boundaries of the two reduction paths agree.

Proof. Certainly this is true for the generators which can appear in a flow diagram; caps, cups, and flow vertices. The condition is also preserved under tensor product and composition.

To see this, suppose we have two flow diagrams D_1 and D_2 in $\mathcal{S}ym_n$, with families of reduction paths P_1 and P_2, so that $d\mathcal{GT}(D_i) = \sum_{\pi \in P_i} \pm q^{\bullet} \pi(D_i)$. Now the reduction paths P for $D_1 \otimes D_2$ are exactly those of the form $\pi_1 \cup \pi_2$ for some $\pi_1 \in P_1$, $\pi_2 \in P_2$ (remember tensor product of diagrams is side-by-side disjoint union). Thus

$$\begin{aligned} d\mathcal{GT}(D_1 \otimes D_2) &= d\mathcal{GT}(D_1) \otimes d\mathcal{GT}(D_2) \\ &= \left(\sum_{\pi_1 \in P_1} (-1)^{\bullet} q^{\bullet} \pi_1(D_1) \right) \otimes \left(\sum_{\pi_2 \in P_2} (-1)^{\bullet} q^{\bullet} \pi_2(D_2) \right) \\ &= \sum_{\substack{\pi_1 \in P_1 \\ \pi_2 \in P_2}} (-1)^{\bullet} q^{\bullet} (\pi_1 \cup \pi_2)(D_1 \otimes D_2) \\ &= \sum_{\pi \in P} (-1)^{\bullet} q^{\bullet} \pi(D_1 \otimes D_2). \end{aligned}$$

The argument for compositions is a tiny bit more complicated. Supposing D_1 and D_2 are composable, the reduction paths P for $D_1 \circ D_2$ are those of the form $\pi_1 \cup \pi_2$, for some $\pi_1 \in P_1$, $\pi_2 \in P_2$ such that $s(\pi_1) = t(\pi_2)$. Thus

$$\begin{aligned} d\mathcal{GT}(D_1 \circ D_2) &= d\mathcal{GT}(D_1) \circ d\mathcal{GT}(D_2) \\ &= \left(\sum_{\pi_1 \in P_1} \pm q^{\bullet} \pi_1(D_1) \right) \circ \left(\sum_{\pi_2 \in P_2} \pm q^{\bullet} \pi_2(D_2) \right) \\ &= \sum_{\substack{\pi_1 \in P_1 \\ \pi_2 \in P_2}} \pm q^{\bullet} \delta_{s(\pi_1), t(\pi_2)} (\pi_1 \cup \pi_2)(D_1 \circ D_2) \\ &= \sum_{\pi \in P} \pm q^{\bullet} \pi(D_1 \otimes D_2). \end{aligned}$$

□

We'll now calculate $d\mathcal{GT}$ on the $\mathcal{P}$- and $\mathcal{Q}$-polygons. On a $\mathcal{P}$-polygon, there are two reduction paths which do not traverse any boundary points; the empty reduction path, and the path around the perimeter of the polygon. We'll call these $\pi_{\mathcal{P};\emptyset}$ and $\pi_{\mathcal{P};\circ}$. Otherwise, there is a unique reduction path for each (cyclic) subsequence of boundary points which alternately includes incoming and outgoing edges. We'll write $\pi_{\mathcal{P};s}$ for this, where s is the subset of $\{1, \ldots, 2k\}$ corresponding to the points on boundary traversed by the path components. The reduction path has one component for each such pair of incoming and outgoing edges, and traverses the perimeter of the polygon counterclockwise between them. On a $\mathcal{Q}$-polygon, there are two reductions paths which traverse every boundary point; in one, each component enters at an incoming edge, and traverse counterclockwise, departing at the nearest outgoing edge, while in the other each component traverses clockwise. We'll call these $\pi_{\mathcal{Q};\circlearrowleft}$ and $\pi_{\mathcal{Q};\circlearrowright}$. Otherwise, there is a unique reduction path for each disjoint collection of adjacent pairs of boundary edges; the path connects each pair, traversing a single edge of the polygon between them, and again, we'll write $\pi_{\mathcal{Q};s}$, where s is the subset of $\{1, \ldots, 2k\}$ corresponding to the subset of the boundary consisting of the union of all the pairs. All these statements about reduction paths should be immediately obvious, looking at the definition of the $\mathcal{P}$- and $\mathcal{Q}$-polygons in Equations (2.4.1) and (2.4.2).

Now,

$$\pi_{\mathcal{P};\emptyset}(\mathcal{P}^n_{a,b;l}) = \mathcal{P}^{n-1}_{a,b;l}$$

and

$$\pi_{\mathcal{P};\circ}(\mathcal{P}^n_{a,b;l}) = \mathcal{P}^{n-1}_{a+\overrightarrow{1},b+\overrightarrow{1};l} = \mathcal{P}^n_{a,b;l-1}.$$

Further

$$\pi_{\mathcal{P};s}(\mathcal{P}^n_{a,b;l}) = \mathcal{P}^{n-1}_{a+a',b+b';l},$$

where $a', b' \in \{0,1\}^k$ is determined from s by $2i-1 \in s$ iff $a'_i = 1$ and $b'_i = 0$, and $2i \in s$ iff $a'_{i+1} = 1$ and $b'_i = 0$. The condition that s contains alternately incoming and outgoing edges translates to $b'_i \leq a'_i, a'_{i+1}$; we'll call such pairs a', b' $\mathcal{P}$-admissible. Note that $\pi_{\mathcal{P};\emptyset}$ and $\pi_{\mathcal{P};\circ}$ correspond to two exceptional pairs, $a' = b' = \overrightarrow{0}$ and $a' = b' = \overrightarrow{1}$ respectively.

On the $\mathcal{Q}$-polygons, we have $\pi_{\mathcal{Q};\circlearrowleft}(\mathcal{Q}^n_{a,b;l}) = \mathcal{Q}^n_{a,b+\overrightarrow{-1};l} = \mathcal{Q}^n_{a+\overrightarrow{1},b;l+1}$, $\pi_{\mathcal{Q};\circlearrowright}(\mathcal{Q}^n_{a,b;l}) = \mathcal{Q}^n_{a+\overrightarrow{1},b;l}$ and $\pi_{\mathcal{Q};s}(\mathcal{Q}^n_{a,b;l}) = \mathcal{Q}^{n-1}_{a+a',b+b';l}$, where $a' \in \{0,1,\}^k$ and $b' \in \{-1,0\}^k$ are determined by $2i-1 \in s$ iff $a_i = 1$ or $b_i = -1$, and $2i \in s$ iff $a_{i+1} = 1$ or $b_i = -1$. The condition that the points in s come in disjoint adjacent pairs translates to the condition $a'_i b'_i = 0 = a'_{i+1} b'_i$; such pairs a', b' are called $\mathcal{Q}$-admissible. Note that $\pi_{\mathcal{Q};\circlearrowleft}$ and $\pi_{\mathcal{Q};\circlearrowright}$ correspond to two exceptional pairs, $a' = \overrightarrow{0}, b' = \overrightarrow{-1}$ and $a' = \overrightarrow{1}, b' = \overrightarrow{0}$ respectively. Also, note that a pair a', b' is both $\mathcal{P}$- and $\mathcal{Q}$-admissible exactly if $b' = \overrightarrow{0}$; any $a' \in \{0,1\}^k$ is allowed.

Thus we have

$$d\mathcal{GT}\left(\mathcal{P}^n_{a,b;l}\right) = \sum_{a',b'\ \mathcal{P}\text{-admissible}} d\mathcal{GT}_{a',b'}\left(\mathcal{P}^n_{a,b;l}\right)$$

and

$$d\mathcal{GT}\left(\mathcal{Q}^n_{a,b;l}\right) = \sum_{a',b'\ \mathcal{Q}\text{-admissible}} d\mathcal{GT}_{a',b'}\left(\mathcal{Q}^n_{a,b;l}\right)$$

where

$$d\mathcal{GT}_{a',b'}\left(\mathcal{P}^n_{a,b;l}\right) = (-1)^{b'\cdot a + \mathrm{rotl}(a)} q^{l(\Sigma b' - \Sigma a' + 1)} q^{\mathrm{rotl}(a')\cdot b - b'\cdot a - a_1 - n a'_1} \mathcal{P}^{n-1}_{a+a',b+b';l}, \tag{4.4.2}$$

and

$$d\mathcal{GT}_{a',b'}\left(\mathcal{Q}^n_{a,b;l}\right) = (-1)^{b'\cdot a + \mathrm{rotl}(a)} q^{l(\Sigma a' - \Sigma b' - \frac{|\partial|}{2} + 1)} q^{\Sigma b - n\Sigma b' + b'\cdot\mathrm{rotl}(a) - a'\cdot b - a_1 - n a'_1} \mathcal{Q}^{n-1}_{a+a',b+b';l}. \tag{4.4.3}$$

The coefficients here are products of the coefficients appearing in the formulas for $d\mathcal{GT}$ on the two vertex webs in Equations (4.3.1) and (4.3.2). We'll often also use the notation $d\mathcal{GT}_\emptyset$ for the terms corresponding to reduction paths not traversing any boundary edges, and $d\mathcal{GT}_\partial$ for the terms corresponding to reduction paths traversing all boundary edges. Thus

$$\begin{aligned} d\mathcal{GT}_\emptyset\left(\mathcal{P}^n_{a,b;l}\right) &= d\mathcal{GT}_{\overrightarrow{0},\overrightarrow{0}}\left(\mathcal{P}^n_{a,b;l}\right) + d\mathcal{GT}_{\overrightarrow{1},\overrightarrow{1}}\left(\mathcal{P}^n_{a,b;l}\right) \\ &= q^{l-a_1}\mathcal{P}^{n-1}_{a,b;l} + q^{l-a_1+\Sigma b - \Sigma a - n}\mathcal{P}^{n-1}_{a,b;l-1}, \end{aligned} \tag{4.4.4}$$

$$d\mathcal{GT}_{\emptyset}\left(\mathcal{Q}^{n}_{a,b;l}\right) = d\mathcal{GT}_{\overrightarrow{0},\overrightarrow{0}}\left(\mathcal{Q}^{n}_{a,b;l}\right) = q^{l-a_1} q^{\Sigma b - \frac{l|\partial|}{2}} \mathcal{Q}^{n-1}_{a,b;l}, \tag{4.4.5}$$

$$d\mathcal{GT}_{\partial}\left(\mathcal{P}^{n}_{a,b;l}\right) = d\mathcal{GT}_{\overrightarrow{1},\overrightarrow{0}}\left(\mathcal{P}^{n}_{a,b;l}\right) = q^{l-a_1-n} q^{\Sigma b - \frac{l|\partial|}{2}} \mathcal{P}^{n-1}_{a+\overrightarrow{1},b;l}, \tag{4.4.6}$$

and

$$\begin{aligned} d\mathcal{GT}_{\partial}\left(\mathcal{Q}^{n}_{a,b;l}\right) &= d\mathcal{GT}_{\overrightarrow{0},\overrightarrow{-1}}\left(\mathcal{Q}^{n}_{a,b;l}\right) + d\mathcal{GT}_{\overrightarrow{1},\overrightarrow{0}}\left(\mathcal{Q}^{n}_{a,b;l}\right) \\ &= q^{l-a_1+\Sigma b - \Sigma a - \frac{n|\partial|}{2}} \mathcal{Q}^{n-1}_{a+\overrightarrow{1},b;l+1} + q^{l-a_1-n} \mathcal{Q}^{n-1}_{a+\overrightarrow{1},b;l}. \end{aligned} \tag{4.4.7}$$

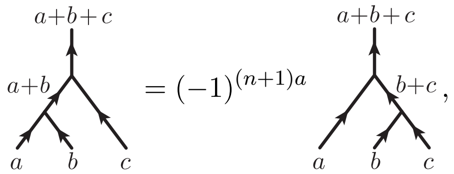
Chapter 5

Describing the kernel

Finally, in this chapter we'll use the methods developed to this point to try to pin down the kernel of the representation functor. In particular, Theorems 5.1.1, 5.2.1 and 5.3.2, stated in the three subsequent sections, describe particular classes of relations amongst diagrams. Theorems 5.2.1 and 5.3.2 additionally rule out any other similar relations.

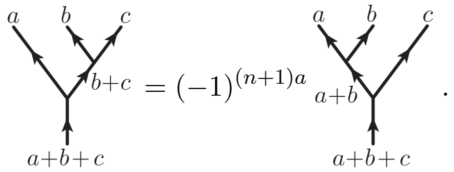
5.1 The $I = H$ relations

Theorem 5.1.1. *For each $a, b, c \in \{0, \ldots, n\}$ with $a + b + c \leq n$, there's an identity*



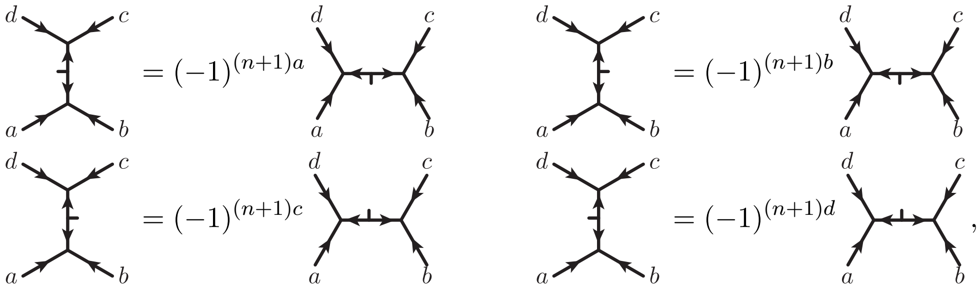
(5.1.1)

and another



(5.1.2)

The proof appears in §5.7. *Remark.* It's somewhat unfortunate that there's a sign here in the first place, and that it depends unsymmetrically on the parameters. (See also §6.1 and §6.3, for a discussion of sign differences between my setup and previous work.) Replacing the vertices in Equation (5.1.1) with standard ones, we can equivalently write this identity in the following four ways (obtaining each by flipping tags, per Equation (2.2.9)):



(5.1.3)

where here $a + b + c + d = n$. These signs are special cases of the $6 - j$ symbols for $U_q(\mathfrak{sl}_n)$.

5.2 The square-switch relations

We can now state the 'square switch' relations, which essentially say that every $\mathcal{P}$-type square can be written as a linear combination of $\mathcal{Q}$-type squares, as vice versa. (Refer back to §2.4 for the definitions of $\mathcal{P}$- and $\mathcal{Q}$-polygons.)

Theorem 5.2.1. *The subspace ($\mathcal{SS}$ for 'square switch')*

$$\mathcal{SS}^n_{a,b} = \begin{cases} \operatorname{span}_{\mathcal{A}} \left\{ \mathcal{P}^n_{a,b;l} - \sum_{m=\max a}^{\min b} \begin{bmatrix} n + \Sigma a - \Sigma b \\ m + l - \Sigma b \end{bmatrix}_q \mathcal{Q}^n_{a,b;m} \right\}_{l=\max b}^{\min a + n} & \text{if } n + \Sigma a - \Sigma b \geq 0 \\ \operatorname{span}_{\mathcal{A}} \left\{ \mathcal{Q}^n_{a,b;l} - \sum_{m=\max b}^{n+\min a} \begin{bmatrix} \Sigma b - n - \Sigma a \\ m + l - \Sigma a - n \end{bmatrix}_q \mathcal{P}^n_{a,b;m} \right\}_{l=\max a}^{\min b} & \text{if } n + \Sigma a - \Sigma b \leq 0 \end{cases}$$

of $\mathcal{AP}^n_{a,b} + \mathcal{AQ}^n_{a,b} \subset \operatorname{Hom}_{\mathcal{S}ym^n}(\emptyset, \mathcal{L}(a,b))$ *lies in the kernel of the representation functor.*

The proof appears in §5.7. Notice that we're stating this result for all tuples a and b, not just those for which $\mathcal{SS}$ is non-trivial. This is a minor point, except that it makes the base cases of our inductive proof easy.
Remark. In the case $n + \Sigma a - \Sigma b = 0$, this becomes simply

$$SS^n_{a,b} = \operatorname{span}_{\mathcal{A}} \left\{ \mathcal{P}^n_{a,b;l} - \mathcal{Q}^n_{a,b;\Sigma b - l} \right\}_{l=\max b}^{\min a+n}.$$

Remark. For $n + \Sigma a - \Sigma b \leq 0$, at the maximal or minimal values of l, we simply get the span of $\mathcal{P}^n_{a,b;\max b} - \mathcal{Q}^n_{a,b;\min b}$ (since the only terms for which the q-binomial is nonzero are $m \geq \min b$) and $\mathcal{P}^n_{a,b;\min a+n} - \mathcal{Q}^n_{a,b;\max a}$ (again, the q-binomials vanish unless $m \leq \max a$). We already knew these were identically zero in $\mathcal{S}ym_n$, by the discussion at the end of §2.4 on relations between $\mathcal{P}$- and $\mathcal{Q}$-polygons. A similar statement applies when $n + \Sigma a - \Sigma b \leq 0$.

Loops, and let bigons be bygones

Over on the representation theory side, any loops or bigon must become a multiple of the identity, by Schur's lemma. We can see this as a special case of the $\mathcal{SS}$ relations. (Alternatively, we'd be able to derive the same formulas from Theorem 5.3.2 in the next section.)

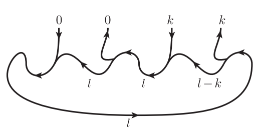
Consider the boundary flow labels $a = (0,0), b = (k,0)$, so

$$\mathcal{L}(a,b) = ((0,-),(0,+),(k,-),(k,+)).$$

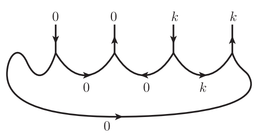
Then $\mathcal{AP}^n_{a,b} = \operatorname{span}_{\mathcal{A}} \left\{ \mathcal{P}^n_{a,b;l} \right\}_{l=k}^n$, while $\mathcal{AQ}^n_{a,b} = \operatorname{span}_{\mathcal{A}} \left\{ \mathcal{Q}^n_{a,b;0} \right\}$. By 5.2.1, we have

$$\mathcal{SS}^n_{a,b} = \operatorname{span}_{\mathcal{A}} \left\{ \mathcal{P}^n_{a,b;l} - \begin{bmatrix} n-k \\ l-k \end{bmatrix}_q \mathcal{Q}^n_{a,b;0} \right\}.$$

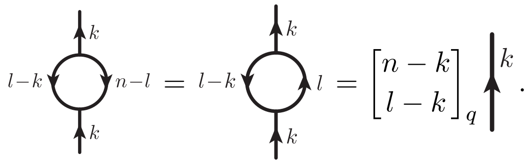
Diagrammatically, this says that in the quotient by $\ker \mathbf{Rep}$,



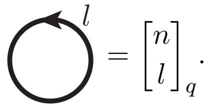
$$= \begin{bmatrix} n-k \\ l-k \end{bmatrix}_q$$



or, removing trivial edges, and cancelling tags

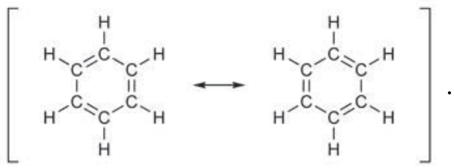


The special case $k = 0$ evaluates loops:

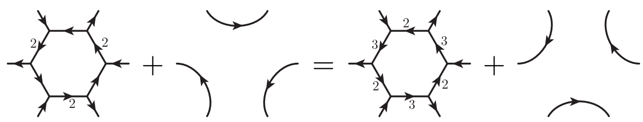


5.3 The Kekulé relations

Friedrich August Kekulé is generally thought to have been the first to suggest the cyclic structure of the benzene molecule [38]. A simplistic description of the benzene molecule is as the quantum superposition of two mesomers, each with alternating double bonds



The similarity to the relation between the two hexagonal diagrams in $\mathcal{S}ym_4/\ker$,



(5.3.1)

discovered by Kim [21], prompted Kuperberg to suggest the name 'Kekulé relation', even though in my classification all of the relations for $U_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_3)$ are also of this type!

In the following, we'll use the convenient notation $\overline{\sum} x = \Sigma x - \max x$, $\underline{\sum} x = \Sigma x - \min x$. (With the convention that $\overline{\sum}\emptyset = \underline{\sum}\emptyset = 0$.)

Definition 5.3.1. We'll say a set of flow labels (a, b) is *n-hexagonal* if there's a sequence of six elements of $\mathcal{L}(a, b)$, each non-trivial for n, i.e. between 1 and $n-1$ inclusive, which are alternately incoming and outgoing edges.

Remark. It's not enough to simply say there are 6 non-trivial edges. For example, the boundary labels

$$((5, +), (1, -), (0, +), (1, -), (0, +), (1, -), (0, +), (1, -), (0, +), (1, -), (0, +), (1, -))$$

seem to allow hexagonal webs, but modulo the $I = H$ relations, the $\mathcal{P}$-polygons are all actually bigons in disguise.

Theorem 5.3.2. *The subspaces $\mathcal{APR}^n_{a,b} \subset \mathcal{AP}^n_{a,b} \subset \mathrm{Hom}_{\mathcal{S}ym^n}(\emptyset, \mathcal{L}(a,b))$ and $\mathcal{AQR}^n_{a,b} \subset \mathcal{AQ}^n_{a,b} \subset \mathrm{Hom}_{\mathcal{S}ym^n}(\emptyset, \mathcal{L}(a,b))$ defined by*

$$\mathcal{APR}^n_{a,b} = \mathrm{span}_{\mathcal{A}} \left\{ \sum_{k=-\overline{\sum} b}^{-\underline{\sum} a+1} (-1)^{j+k} \begin{bmatrix} j+k-\max b \\ j - \Sigma b \end{bmatrix}_q \begin{bmatrix} \min a + n - j - k \\ \Sigma a + n - 1 - j \end{bmatrix}_q \mathcal{P}^n_{a,b;j+k} \right\}_{j=\Sigma b}^{\Sigma a + n - 1} \tag{5.3.2}$$

$$\mathcal{AQR}^n_{a,b} = \mathrm{span}_{\mathcal{A}} \left\{ \sum_{k=-\underline{\sum} a}^{-\overline{\sum} b + \frac{n|\partial|}{2}+1} (-1)^{j+k} \begin{bmatrix} j+k-\max a \\ j - \Sigma a \end{bmatrix}_q \times \right.$$
$$\left. \times \begin{bmatrix} \min b - j - k \\ \Sigma b - n(\frac{|\partial|}{2} - 1) - 1 - j \end{bmatrix}_q \mathcal{Q}^n_{a,b;j+k} \right\}_{j=\Sigma a}^{\Sigma b - n(\frac{|\partial|}{2}-1)-1} \tag{5.3.3}$$

are in the kernel of the representation functor. Moreover, when the representation functor is restricted to $\mathcal{AP}^n_{a,b}$, its kernel is exactly $\mathcal{APR}^n_{a,b}$, and similarly for $\mathcal{AQ}^n_{a,b}$ and $\mathcal{AQR}^n_{a,b}$. Further, in the case that (a,b) is n-hexagonal, when the representation functor is restricted to $\mathcal{AP}^n_{a,b} + \mathcal{AQ}^n_{a,b}$, its kernel is exactly $\mathcal{APR}^n_{a,b} + \mathcal{AQR}^n_{a,b}$.

The proof appears in §5.7.

Remark. It's worth being a little careful here; remember from Lemma 2.4.1 that sometimes $\mathcal{AP}^n_{a,b} \cap \mathcal{AQ}^n_{a,b} \neq 0$; when a or b is constant, the intersection is 1 dimensional, and when both are constant, but not equal, it's 2 dimensional. Besides this, however, you can understand the second half of the above Theorem as saying that there are no relations between $\mathcal{P}$-type polygons, and $\mathcal{Q}$-type polygons, when those polygons are at least hexagons. In particular, this contradicts the conjecture of [11], which expected to find relations looking like the 'square-switch' relations described in Theorem 5.2.1 amongst larger polygons.

The elements of $\mathcal{APR}^n_{a,b}$ described above have 'breadth' $\overline{\sum} b - \underline{\sum} a + 2$; that is, they have that many terms. We can generalise the definition of n-hexagonal to say that a pair (a,b) has n-circumference at least $2k$ if there's a sequence of $2k$ edges in $\mathcal{L}(a,b)$, which are nontrivial for n and alternately incoming and outgoing. Then

Lemma 5.3.3. *If (a,b) has n-circumference at least $2k$, the breadth of the relations in $\mathcal{APR}^n_{a,b}$ is at least $k+1$. In particular, the only place that relations of breadth 2 appear is when (a,b) has n-circumference no more than 2.*

The proof appears in §5.7, after Lemma 5.7.7.

5.4 More about squares

Theorem 5.4.1. *Further, when $n + \Sigma a - \Sigma b \geq 0$, the space $\mathcal{AQR}^n_{a,b}$ of relations amongst the $\mathcal{Q}$-squares becomes 0, and in fact $\mathcal{APR}^n_{a,b} \subset \mathcal{SS}^n_{a,b}$. (Conversely, when $n + \Sigma a - \Sigma b \leq 0$, $\mathcal{APR}^n_{a,b} = 0$ and $\mathcal{AQR}^n_{a,b} \subset \mathcal{SS}^n_{a,b}$.) When $n + \Sigma a - \Sigma b \geq 0$, defining*

$$\mathcal{SS}'^n_{a,b} = \mathrm{span}_{\mathcal{A}} \left\{ \mathcal{Q}^n_{a,b;m} - \sum_{l=n+\Sigma a - m}^{n+\min a} (-1)^{m+l+n+\Sigma a} \begin{bmatrix} m+l-1-\Sigma b \\ m+l-n-\Sigma a \end{bmatrix}_q \mathcal{P}^n_{a,b;l} \right\}_{m=\max a}^{\min b}$$

we find that $SS^n_{a,b} = \mathcal{APR}^n_{a,b} \oplus SS'^n_{a,b}$*. When* $n + \Sigma a - \Sigma b \leq 0$*, we define instead*

$$\mathcal{SS}'^n_{a,b} = \operatorname{span}_{\mathcal{A}} \left\{ \mathcal{P}^n_{a,b;m} - \sum_{l=\Sigma b - m}^{\min b} (-1)^{m+l+\Sigma b} \begin{bmatrix} m+l-n-1-\Sigma a \\ m+l-\Sigma b \end{bmatrix}_q \mathcal{Q}^n_{a,b;l} \right\}_{m=\max b}^{\min a+n}$$

and find that $\mathcal{SS}^n_{a,b} = \mathcal{AQR}^n_{a,b} \oplus SS'^n_{a,b}$*.*

Corollary 5.4.2. *The kernel of* $\mathbf{Rep}$ *on* $\mathcal{AP}^n_{a,b} + \mathcal{AQ}^n_{a,b}$*, in the case of squares, is exactly* $SS^n_{a,b}$*.*

Proof. This follows from Theorems 5.3.2 and 5.4.1. Suppose $x \in \ker \mathbf{Rep}_n \cap (\mathcal{AP}^n_{a,b} + \mathcal{AQ}^n_{a,b})$, we'll show that it's zero as an element of $(\mathcal{AP}^n_{a,b} + \mathcal{AQ}^n_{a,b})/SS^n_{a,b}$. In fact, $(\mathcal{AP}^n_{a,b} + \mathcal{AQ}^n_{a,b})/SS^n_{a,b} = \mathcal{AP}^n_{a,b}/SS^n_{a,b} = \mathcal{AP}^n_{a,b}/\mathcal{APR}^n_{a,b}$, by Theorem 5.4.1, and by Theorem 5.3.2,

$$(\ker \mathbf{Rep}_n \cap \mathcal{AP}^n_{a,b})/\mathcal{APR}^n_{a,b} = 0.$$

□

5.5 A conjecture

I wish I could prove the following

Conjecture. *The kernel of the representation functor is generated, as a pivotal category ideal, by the elements described in Theorems 5.1.1, 5.2.1 and 5.3.2.*

Evidence. First of all, we have Corollary 5.4.2, which confirms that the conjecture holds for diagrams with at most four boundary points.

Second, Theorem 5.3.2 tells us more than simply that certain elements are in the kernel; it tells us exactly the kernel of the representation functor when restricted to diagrams with a single (hexagonal or bigger) polygonal face. Any relations not generated by those discovered here must then be 'more nonlocal', in the sense that they involve diagrams with multiple polygons.

Finally, there's some amount of computer evidence. I have a `Mathematica` package, sadly unreleased at this point, which can explicitly produce intertwiners between representations of arbitrary quantum groups $U_q\mathfrak{g}$. A program which can translate diagrams into instructions for taking tensor products and compositions of elementary morphisms (i.e. an implementation of the representation functor described in §3.6) can then automatically look for linear dependencies amongst diagrams. For small values of n (up to 8 or 9), I used such a program to look for relations involving two adjacent hexagons, but found nothing besides relations associated, via Theorem 5.3.2, to one of the individual hexagons. □

5.6 Examples: $U_q(\mathfrak{sl}_n)$, for $n = 2, 3, 4$ and 5.

For each n, we can enumerate the finite list of examples of each of the three families of relations. To enumerate $\mathcal{APR}^n$ relations, we find each set of flow labels (a, b) such that every element of $\mathcal{L}(a, b)$ is between 1 and $n-1$ inclusive, and $\dim \mathcal{APR}^n_{a,b} = n + \Sigma a - \Sigma b > 0$. When sets of flow labels (a, b) and $(\operatorname{rotl}^k(a), \operatorname{rotl}^k(b))$ differ by a cyclic permutation, we'll only discuss the lexicographically smaller one. There's no need to separately enumerate the $\mathcal{AQR}^n$ relations; they're just rotations of the $\mathcal{APR}^n$ relations. To enumerate $\mathcal{SS}^n$ relations, we'll do this same, but weakening the inequality to $n + \Sigma a - \Sigma b \geq 0$. Recall that we don't need to look at the extreme values of l in the $\mathcal{SS}$ relations in Theorem 5.2.1; those relations automatically hold already in $\mathcal{S}ym_n$.

For $n = 2$ and $n = 3$, there are no $\mathcal{APR}^n_{a,b}$ relations with a, b of length 3 or more; the relations of length 2 or less follow from the $\mathcal{SS}^n$ relations. We recover exactly the Temperley-Lieb loop relation for $n = 2$, and Kuperberg's loop, bigon and square relations for $n = 3$, from $\mathcal{SS}^3_{(0,0),(0,0)}$, $\mathcal{SS}^3_{(0,0),(0,1)}$ and $\mathcal{SS}^3_{(0,0),(1,1)}$ respectively.

For $n = 4$, we have non-trivial $\mathcal{APR}^4_{a,b}$ spaces for

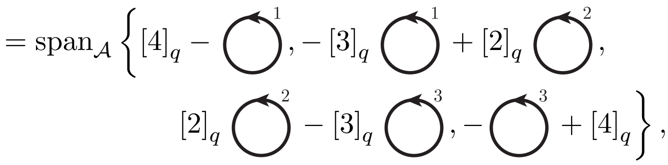
$$(a, b) = \begin{cases} (\emptyset, \emptyset) & \text{when } k = 0 \\ ((0),(1)), ((0),(2)), \text{ or } ((0),(3)) & \text{when } k = 1 \\ ((0,0),(1,1)), ((0,0),(1,2)), \text{ or } ((0,0),(2,2)) & \text{when } k = 2 \end{cases}$$

and for $k = 3$

$$(a, b) = ((0, 0, 0), (1, 1, 1)).$$

Thus we have relations involving loops:

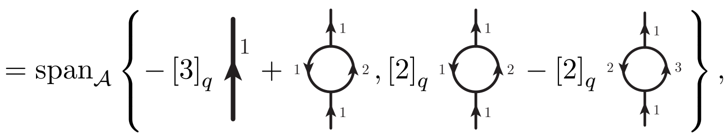
$$\mathcal{APR}^4_{\emptyset,\emptyset} = \operatorname{span}_{\mathcal{A}} \Big\{ [4]_q \, \mathcal{P}^4_{\emptyset,\emptyset;0} - \mathcal{P}^4_{\emptyset,\emptyset;1}, - [3]_q \, \mathcal{P}^4_{\emptyset,\emptyset;1} + [2]_q \, \mathcal{P}^4_{\emptyset,\emptyset;2}, [2]_q \, \mathcal{P}^4_{\emptyset,\emptyset;2} - [3]_q \, \mathcal{P}^4_{\emptyset,\emptyset;3}, -\mathcal{P}^3_{\emptyset,\emptyset;3} + [4]_q \, \mathcal{P}^4_{\emptyset,\emptyset;4} \Big\}$$



(5.6.1)

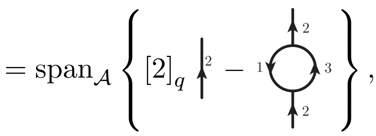
and bigons:

$$\mathcal{APR}^4_{(0),(1)} = \operatorname{span}_{\mathcal{A}} \Big\{ - [3]_q \, \mathcal{P}^4_{(0),(1);1} + \mathcal{P}^4_{(0),(1);2}, [2]_q \, \mathcal{P}^4_{(0),(1);2} - [2]_q \, \mathcal{P}^4_{(0),(1);3}, -\mathcal{P}^4_{(0),(1);3} + [3]_q \, \mathcal{P}^4_{(0),(1);4} \Big\}$$



(5.6.2)

$$\mathcal{APR}^4_{(0),(2)} = \operatorname{span}_{\mathcal{A}} \Big\{ [2]_q \, \mathcal{P}^4_{(0),(2);2} - \mathcal{P}^4_{(0),(2);3}, -\mathcal{P}^4_{(0),(2);3} + [2]_q \, \mathcal{P}^4_{(0),(2);4} \Big\}$$



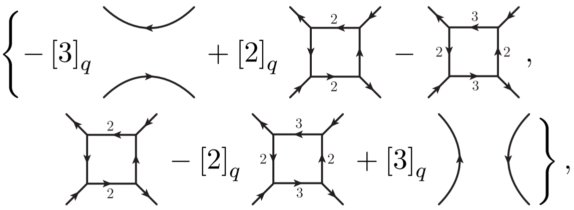
(5.6.3)

and

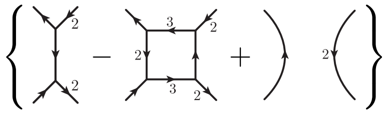
$$\mathcal{APR}^4_{(0),(3)} = \operatorname{span}_{\mathcal{A}} \Big\{ -\mathcal{P}^4_{(0),(3);3} + \mathcal{P}^4_{(0),(3);4} \Big\} = 0, \tag{5.6.4}$$

(there's a redundancy in each $APR^4_{(0),(k)}$ space, since $\mathcal{P}^4_{(0),(k);k} = \mathcal{P}^4_{(0),(k);4}$) and then relations involving squares:

$$\mathcal{APR}^4_{(0,0),(1,1)} = \operatorname{span}_{\mathcal{A}} \Big\{ -[3]_q\, \mathcal{P}^4_{(0,0),(1,1);1} + [2]_q\, \mathcal{P}^4_{(0,0),(1,1);2} - \mathcal{P}^4_{(0,0),(1,1);3}, \; \mathcal{P}^4_{(0,0),(1,1);2} - [2]_q\, \mathcal{P}^4_{(0,0),(1,1);3} + [3]_q\, \mathcal{P}^4_{(0,0),(1,1);4} \Big\}$$

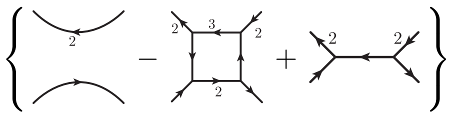
$= \operatorname{span}_{\mathcal{A}}$  (5.6.5)

$$\mathcal{APR}^4_{(0,0),(1,2)} = \operatorname{span}_{\mathcal{A}} \Big\{ \mathcal{P}^4_{(0,0),(1,2);2} - \mathcal{P}^4_{(0,0),(1,2);3} + \mathcal{P}^4_{(0,0),(1,2);4} \Big\}$$

$= \operatorname{span}_{\mathcal{A}}$  (5.6.6)

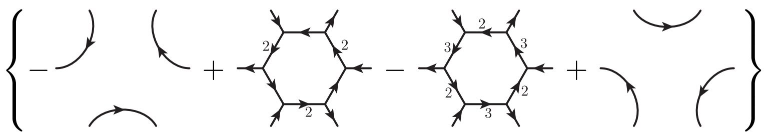
and

$$\mathcal{APR}^4_{(0,1),(2,2)} = \operatorname{span}_{\mathcal{A}} \Big\{ \mathcal{P}^4_{(0,1),(2,2);2} - \mathcal{P}^4_{(0,1),(2,2);3} + \mathcal{P}^4_{(0,1),(2,2);4} \Big\}$$

$= \operatorname{span}_{\mathcal{A}}$  (5.6.7)

and finally the eponymous Kekulé relation, involving hexagons:

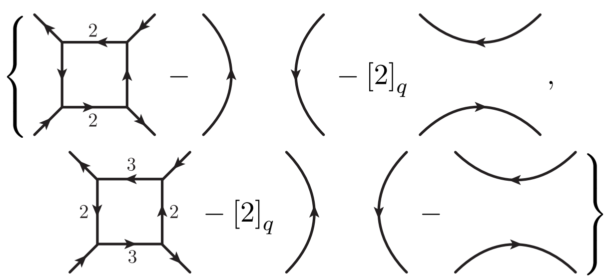
$$\mathcal{APR}^4_{(0,0,0),(1,1,1)} = \operatorname{span}_{\mathcal{A}} \Big\{ -\mathcal{P}^4_{(0,0,0),(1,1,1);1} + \mathcal{P}^4_{(0,0,0),(1,1,1);2} - \; -\mathcal{P}^4_{(0,0,0),(1,1,1);3} + \mathcal{P}^4_{(0,0,0),(1,1,1);4} \Big\}$$

$= \operatorname{span}_{\mathcal{A}}$  . (5.6.8)

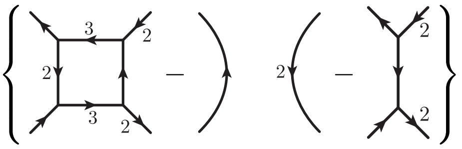
Here we're telling some small lies; $\mathcal{APR}^4_{a,b}$ is always a subspace of $\operatorname{Hom}_{\mathcal{S}ym^4}(\emptyset, \mathcal{L}(a,b))$, but I've drawn some of these diagrams as elements of other Hom spaces, via rotations.

The interesting $\mathcal{SS}^4$ spaces are

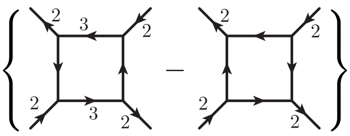
$$\mathcal{SS}^4_{(0,0),(1,1)} = \operatorname{span}_{\mathcal{A}} \Big\{ \mathcal{P}^4_{(0,0),(1,1);2} - \mathcal{Q}^4_{(0,0),(1,1);0} - [2]_q\, \mathcal{Q}^4_{(0,0),(1,1);1}, \; \mathcal{P}^4_{(0,0),(1,1);3} - [2]_q\, \mathcal{Q}^4_{(0,0),(1,1);0} - \mathcal{Q}^4_{(0,0),(1,1);1} \Big\}$$

$= \operatorname{span}_{\mathcal{A}}$  ,

$$\mathcal{SS}^4_{(0,0),(1,2)} = \text{span}_{\mathcal{A}} \left\{ \mathcal{P}^4_{(0,0),(1,2);3} - \mathcal{Q}^4_{(0,0),(1,2);0} - \mathcal{Q}^4_{(0,0),(1,2);1} \right\}$$

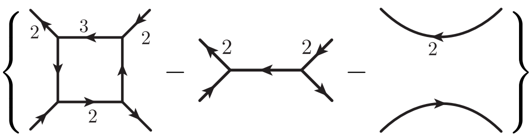
$= \text{span}_{\mathcal{A}}$  ,

$$\mathcal{SS}^4_{(0,0),(2,2)} = \text{span}_{\mathcal{A}} \left\{ \mathcal{P}^4_{(0,0),(2,2);3} - \mathcal{Q}^4_{(0,0),(2,2);1} \right\}$$

$= \text{span}_{\mathcal{A}}$  ,

and

$$\mathcal{SS}^4_{(0,1),(2,2)} = \text{span}_{\mathcal{A}} \left\{ \mathcal{P}^4_{(0,1),(2,2);3} - \mathcal{Q}^4_{(0,1),(2,2);1} - \mathcal{Q}^4_{(0,1),(2,2);2} \right\}$$

$= \text{span}_{\mathcal{A}}$  .

Notice the redundancies here: $\mathcal{SS}^4_{(0,0),(1,1)} = \mathcal{APR}^4_{(0,0),(1,1)}$, $\mathcal{SS}^4_{(0,0),(1,2)} = \mathcal{APR}^4_{(0,0),(1,2)}$ and $\mathcal{SS}^4_{(0,1),(2,2)} = \mathcal{APR}^4_{(0,1),(2,2)}$, but $\mathcal{SS}^4_{(0,0),(2,2)}$ is independent of any of the $\mathcal{APR}$ relations.

For $n = 5$, we'll just look at non-trivial $\mathcal{APR}^5_{a,b}$ spaces where a and b are each of length at least 3; we know what the length 0 and 1 relations look like (loops and bigons), and the length 2 relations all follow from $\mathcal{SS}$ relations. Thus we have

$$(a, b) = ((0,0,0),(1,1,1)), ((0,0,0),(1,1,2)), ((0,0,1),(1,2,2)), ((0,1,1),(2,2,2))$$

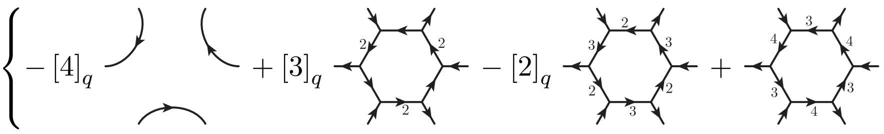
or

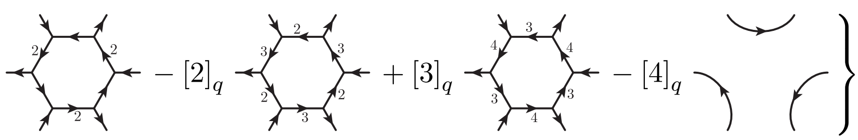
$$= ((0,0,0,0),(1,1,1,1)).$$

Then

$$\mathcal{APR}^5{}_{(0,0,0),(1,1,1)} =$$

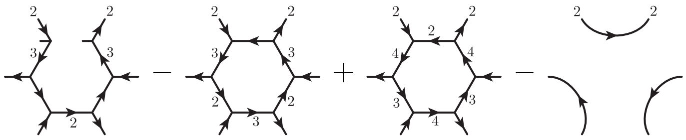
$$= \text{span}_{\mathcal{A}} \Big\{ -[4]_q \mathcal{P}^5_{\vec{0},\vec{1};1} + [3]_q \mathcal{P}^5_{\vec{0},\vec{1};2} - [2]_q \mathcal{P}^5_{\vec{0},\vec{1};3} + \mathcal{P}^5_{\vec{0},\vec{1};4},$$
$$\mathcal{P}^5_{\vec{0},\vec{1};2} - [2]_q \mathcal{P}^5_{\vec{0},\vec{1};3} + [3]_q \mathcal{P}^5_{\vec{0},\vec{1};4} - [4]_q \mathcal{P}^5_{\vec{0},\vec{1};5} \Big\}$$

$= \text{span}_{\mathcal{A}}$ 

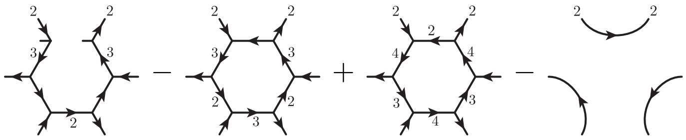


(5.6.9)

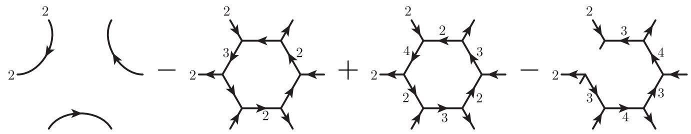
$$\mathcal{APR}^5_{(0,0,0),(1,1,2)} = \mathrm{span}_{\mathcal{A}} \Big\{ \mathcal{P}^5_{(0,0,0),(1,1,2);2} - \mathcal{P}^5_{(0,0,0),(1,1,2);3} + \mathcal{P}^5_{(0,0,0),(1,1,2);4} - \mathcal{P}^5_{(0,0,0),(1,1,2);5} \Big\}$$

$= \mathrm{span}_{\mathcal{A}} \Big\{$  $\Big\}$ (5.6.10)

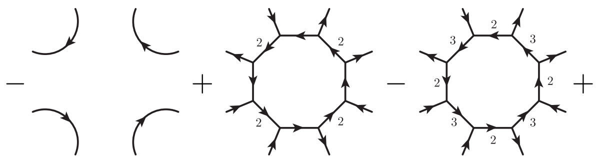
$$\mathcal{APR}^5_{(0,0,1),(1,2,2)} = \mathrm{span}_{\mathcal{A}} \Big\{ \mathcal{P}^5_{(0,0,1),(1,2,2);2} - \mathcal{P}^5_{(0,0,1),(1,2,2);3} + \mathcal{P}^5_{(0,0,1),(1,2,2);4} - \mathcal{P}^5_{(0,0,1),(1,2,2);5} \Big\}$$

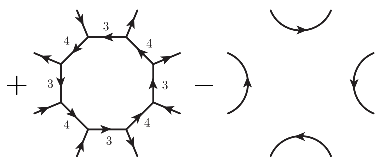
$= \mathrm{span}_{\mathcal{A}} \Big\{$  $\Big\}$ (5.6.11)

$$\mathcal{APR}^5_{(0,1,1),(2,2,2)} = \mathrm{span}_{\mathcal{A}} \Big\{ \mathcal{P}^5_{(0,1,1),(2,2,2);2} - \mathcal{P}^5_{(0,1,1),(2,2,2);3} + \mathcal{P}^5_{(0,1,1),(2,2,2);4} - \mathcal{P}^5_{(0,1,1),(2,2,2);5} \Big\}$$

$= \mathrm{span}_{\mathcal{A}} \Big\{$  $\Big\}$ (5.6.12)

$$\mathcal{APR}^5_{(0,0,0,0),(1,1,1,1)} = \mathrm{span}_{\mathcal{A}} \Big\{ -\mathcal{P}^5_{(0,0,0,0),(1,1,1,1);1} + \mathcal{P}^5_{(0,0,0,0),(1,1,1,1);2} - \mathcal{P}^5_{(0,0,0,0),(1,1,1,1);3} + \mathcal{P}^5_{(0,0,0,0),(1,1,1,1);4} - \mathcal{P}^5_{(0,0,0,0),(1,1,1,1);5} \Big\}$$

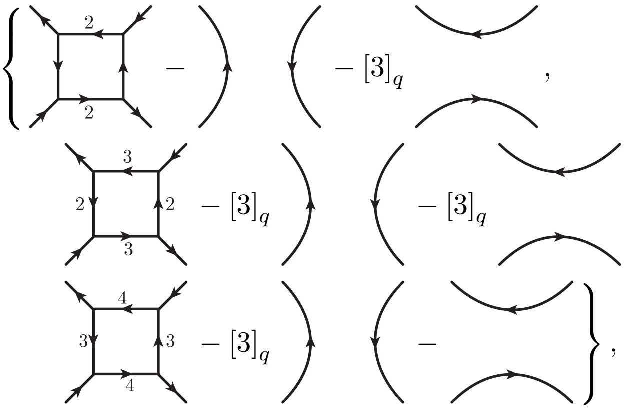
$= \mathrm{span}_{\mathcal{A}} \Big\{$ 

 $\Big\}$ (5.6.13)

The interesting $\mathcal{SS}^5$ relations are

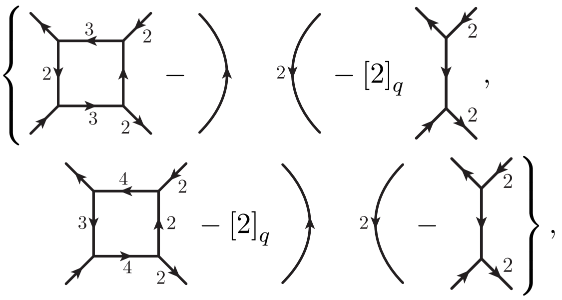
$$\mathcal{SS}^5_{(0,0),(1,1)} = \mathrm{span}_{\mathcal{A}} \Big\{ \mathcal{P}^5_{(0,0),(1,1);2} - \mathcal{Q}^5_{(0,0),(1,1);0} - [3]_q\, \mathcal{Q}^5_{(0,0),(1,1);1},\; \mathcal{P}^5_{(0,0),(1,1);3} - [3]_q\, \mathcal{Q}^5_{(0,0),(1,1);0} - [3]_q\, \mathcal{Q}^5_{(0,0),(1,1);1},\; \mathcal{P}^5_{(0,0),(1,1);4} - [3]_q\, \mathcal{Q}^5_{(0,0),(1,1);0} - \mathcal{Q}^5_{(0,0),(1,1);1} \Big\}$$

$= \mathrm{span}_{\mathcal{A}}$



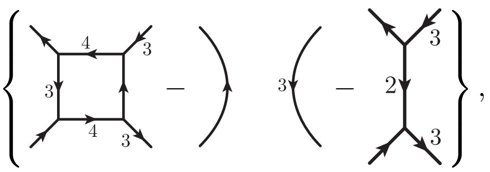
$$\mathcal{SS}^5_{(0,0),(1,2)} = \mathrm{span}_{\mathcal{A}} \Big\{ \mathcal{P}^5_{(0,0),(1,2);3} - \mathcal{Q}^5_{(0,0),(1,2);0} - [2]_q\, \mathcal{Q}^5_{(0,0),(1,2);1},\; \mathcal{P}^5_{(0,0),(1,2);4} - [2]_q\, \mathcal{Q}^5_{(0,0),(1,2);0} - \mathcal{Q}^5_{(0,0),(1,2);1} \Big\}$$

$= \mathrm{span}_{\mathcal{A}}$



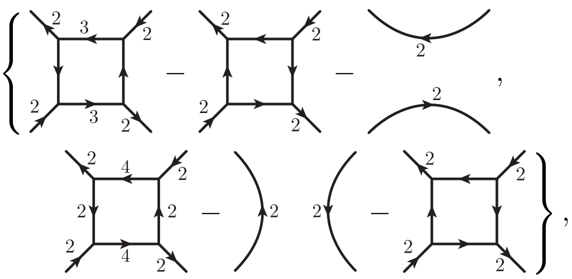
$$\mathcal{SS}^5_{(0,0),(1,3)} = \mathrm{span}_{\mathcal{A}} \Big\{ \mathcal{P}^5_{(0,0),(1,3);4} - \mathcal{Q}^5_{(0,0),(1,3);0} - \mathcal{Q}^5_{(0,0),(1,3);1} \Big\}$$

$= \mathrm{span}_{\mathcal{A}}$



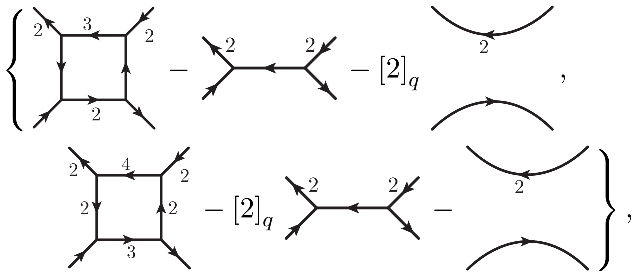
$$\mathcal{SS}^5_{(0,0),(2,2)} = \mathrm{span}_{\mathcal{A}} \Big\{ \mathcal{P}^5_{(0,0),(2,2);3} - \mathcal{Q}^5_{(0,0),(2,2);1} - \mathcal{Q}^5_{(0,0),(2,2);2},\; \mathcal{P}^5_{(0,0),(2,2);4} - \mathcal{Q}^5_{(0,0),(2,2);0} - \mathcal{Q}^5_{(0,0),(2,2);1} \Big\}$$

$= \mathrm{span}_{\mathcal{A}}$



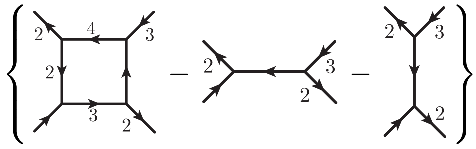
$$\mathcal{SS}^5_{(0,1),(2,2)} = \mathrm{span}_{\mathcal{A}} \Big\{ \mathcal{P}^5_{(0,1),(2,2);3} - \mathcal{Q}^5_{(0,1),(2,2);1} - [2]_q \, \mathcal{Q}^5_{(0,1),(2,2);2}, \\ \mathcal{P}^5_{(0,1),(2,2);4} - [2]_q \, \mathcal{Q}^5_{(0,1),(2,2);1} - \mathcal{Q}^5_{(0,1),(2,2);2} \Big\}$$

$= \mathrm{span}_{\mathcal{A}}$



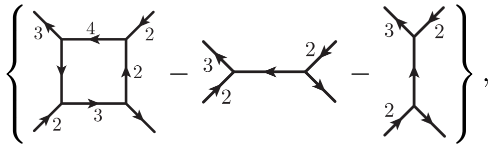
$$\mathcal{SS}^5_{(0,1),(2,3)} = \mathrm{span}_{\mathcal{A}} \Big\{ \mathcal{P}^5_{(0,1),(2,3);4} - \mathcal{Q}^5_{(0,1),(2,3);1} - \mathcal{Q}^5_{(0,1),(2,3);2} \Big\}$$

$= \mathrm{span}_{\mathcal{A}}$



$$\mathcal{SS}^5_{(0,1),(3,2)} = \mathrm{span}_{\mathcal{A}} \Big\{ \mathcal{P}^5_{(0,1),(3,2);4} - \mathcal{Q}^5_{(0,1),(3,2);1} - \mathcal{Q}^5_{(0,1),(3,2);2} \Big\}$$

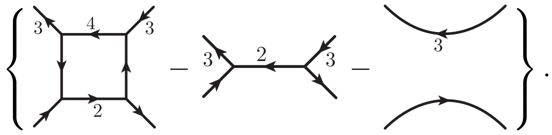
$= \mathrm{span}_{\mathcal{A}}$



and

$$\mathcal{SS}^5_{(0,2),(3,3)} = \mathrm{span}_{\mathcal{A}} \Big\{ \mathcal{P}^5_{(0,2),(3,3);4} - \mathcal{Q}^5_{(0,2),(3,3);2} - \mathcal{Q}^5_{(0,2),(3,3);3} \Big\}$$

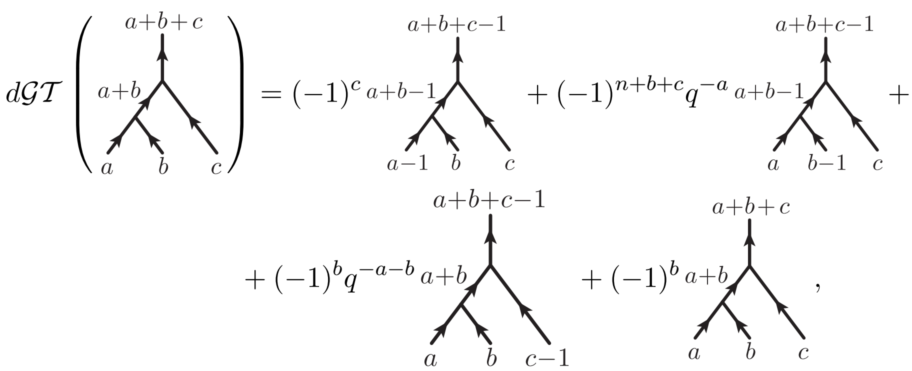
$= \mathrm{span}_{\mathcal{A}}$



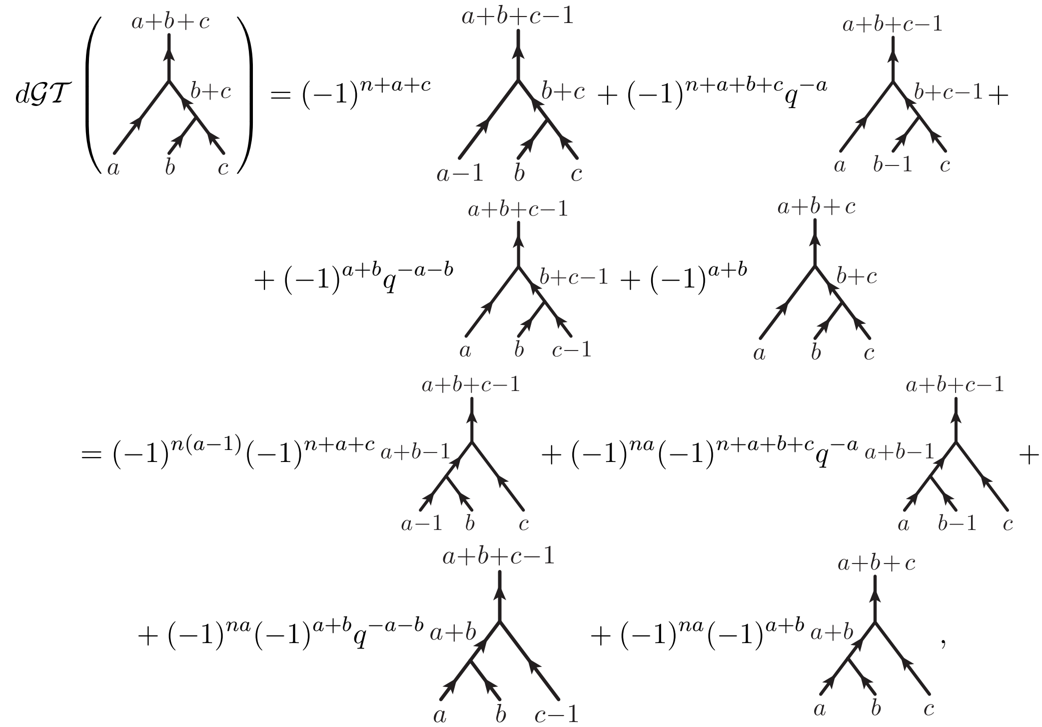
5.7 Proofs of Theorems 5.1.1, 5.2.1 and 5.3.2

5.7.1 The $I = H$ relations

Proof of Theorem 5.1.1. We'll just show the calculation for Equation (5.1.1), with 'merging' vertices; the calculation for Equation (5.1.2) is identical. It's an easy induction, beginning by calculating $d\mathcal{GT}$ of both sides, using the calculations from §4.3.

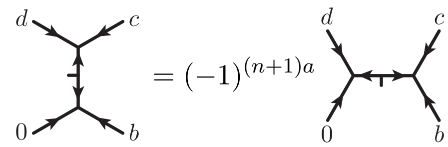


while



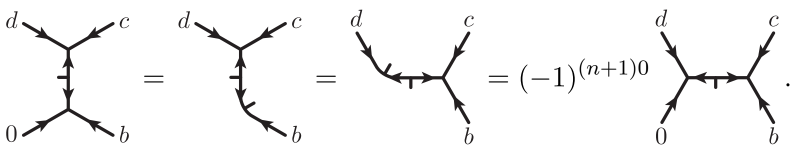
which differs by exactly an overall factor of $(-1)^{(n+1)a}$.

For the base cases of the induction, we need consider the possibility that a, b or c is 0 or $a+b+c = n$. It's actually more convenient to consider one of the equivalent relations in Equation (5.1.3), where the corresponding base cases are a, b, c or $d = 0$. Then I claim



(5.7.1)

in $\mathcal{S}ym_n$ (and similarly in the other 3 cases when instead b, c or d is zero), and so the difference is automatically in the kernel of **Rep**. Each of the four variations of Equation (5.7.1) follows from the 'degeneration' relations in $\mathcal{S}ym_n$, given in Equation (2.2.10). If $a = 0$, we have



(5.7.2)

The middle equality here uses 'tag cancellation' from Equation (2.2.8). The other cases are similar, but use 'tag flipping' from Equation (2.2.9), producing the desired signs. □

5.7.2 The square-switch relations

Proof of Theorem 5.2.1. This is quite straightforward, by induction. We'll just do the $n + \Sigma a - \Sigma b \geq 0$ case; the other follows immediately by rotation, by §2.4.1.

The base cases are easy; since either each $b_i \geq a_i$, or $\mathcal{AP}^n_{a,b} = \mathcal{AQ}^n_{a,b} = 0$, when we're at $n = 0$ the only interesting case is $a = b = (0, 0)$. Then $\mathcal{SS}^0_{(0,0),(0,0)}$ is simply the span of

$\mathcal{P}^0_{(0,0),(0,0);0} - \mathcal{Q}^0_{(0,0),(0,0);0}$; this element is actually zero in $\mathcal{S}ym_0$, by Lemma 2.4.1, and so automatically in the kernel of $\mathbf{Rep}$.

For the inductive step, we simply show that for $l = \max b, \ldots, \min a + n$ the component of

$$d\mathcal{GT}\left(\mathcal{P}^n_{a,b;l} - \sum_{m=\max a}^{\min b} \begin{bmatrix} n + \Sigma a - \Sigma b \\ m + l - \Sigma b \end{bmatrix}_q \mathcal{Q}^n_{a,b;m}\right) \tag{5.7.3}$$

in $\mathrm{Hom}_{\mathcal{S}ym^{n-1}}(\emptyset, \mathcal{L}(a + a', b + b'))$ lies in $\mathcal{SS}^{n-1}_{a+a',b+b'}$ for each a', b'. The expression in Equation (5.7.3) has one matrix for each subset of the four boundary points. The only nonzero matrix entries correspond to those subsets for which there is a reduction path with endpoints coinciding with the subset. Thus, numbering the four boundary points $1, 2, 3$ and 4, we have $d\mathcal{GT} = d\mathcal{GT}_\emptyset + d\mathcal{GT}_{\{1,2\}} + d\mathcal{GT}_{\{1,4\}} + d\mathcal{GT}_{\{3,4\}} + d\mathcal{GT}_{\{3,2\}} + d\mathcal{GT}_\partial$. In the notation of §4.4,

$$\begin{aligned} d\mathcal{GT}_{\{1,2\}|\mathcal{AP}} &= d\mathcal{GT}_{(1,1),(0,1)}, & d\mathcal{GT}_{\{1,2\}|\mathcal{AQ}} &= d\mathcal{GT}_{(0,0),(-1,0)}, \\ d\mathcal{GT}_{\{1,4\}|\mathcal{AP}} &= d\mathcal{GT}_{(1,0),(0,0)}, & d\mathcal{GT}_{\{1,4\}|\mathcal{AQ}} &= d\mathcal{GT}_{(1,0),(0,0)}, \\ d\mathcal{GT}_{\{3,4\}|\mathcal{AP}} &= d\mathcal{GT}_{(1,1),(1,0)}, & d\mathcal{GT}_{\{3,4\}|\mathcal{AQ}} &= d\mathcal{GT}_{(0,0),(0,-1)}, \\ d\mathcal{GT}_{\{3,2\}|\mathcal{AP}} &= d\mathcal{GT}_{(0,1),(0,0)}, & d\mathcal{GT}_{\{3,2\}|\mathcal{AQ}} &= d\mathcal{GT}_{(0,1),(0,0)}. \end{aligned}$$

We now compute all of the components of Equation (5.7.3). First, use Equations (4.4.2) and (4.4.3) to write down

$$\begin{aligned} d\mathcal{GT}_{\{1,2\}}\left(\mathcal{P}^n_{a,b;l}\right) &= (-1)^{\Sigma a} q^{-n-\Sigma a+\Sigma b} \mathcal{P}^{n-1}_{a+(1,1),b+(0,1);l}, \\ d\mathcal{GT}_{\{1,2\}}\left(\mathcal{Q}^n_{a,b;l}\right) &= (-1)^{\Sigma a} q^{-n-\Sigma a+\Sigma b} \mathcal{Q}^{n-1}_{a+(0,0),b+(-1,0);l} \\ &= (-1)^{\Sigma a} q^{-n-\Sigma a+\Sigma b} \mathcal{Q}^{n-1}_{a+(1,1),b+(0,1);l+1}, \\ d\mathcal{GT}_{\{1,4\}}\left(\mathcal{P}^n_{a,b;l}\right) &= q^{-n-a_1+b_2} \mathcal{P}^{n-1}_{a+(1,0),b+(0,0);l}, \\ d\mathcal{GT}_{\{1,4\}}\left(\mathcal{Q}^n_{a,b;l}\right) &= q^{-n-a_1+b_2} \mathcal{Q}^{n-1}_{a+(1,0),b+(0,0);l}, \\ d\mathcal{GT}_{\{3,4\}}\left(\mathcal{P}^n_{a,b;l}\right) &= (-1)^{\Sigma a} q^{-n-\Sigma a+\Sigma b} \mathcal{P}^{n-1}_{a+(1,1),b+(1,0);l}, \\ d\mathcal{GT}_{\{3,4\}}\left(\mathcal{Q}^n_{a,b;l}\right) &= (-1)^{\Sigma a} q^{-n-\Sigma a+\Sigma b} \mathcal{Q}^{n-1}_{a+(0,0),b+(0,-1);l} \\ &= (-1)^{\Sigma a} q^{-n-\Sigma a+\Sigma b} \mathcal{Q}^{n-1}_{a+(1,1),b+(1,0);l+1}, \\ d\mathcal{GT}_{\{3,2\}}\left(\mathcal{P}^n_{a,b;l}\right) &= q^{-a_1+b_1} \mathcal{P}^{n-1}_{a+(0,1),b+(0,0);l}, \end{aligned}$$

and

$$d\mathcal{GT}_{\{3,2\}}\left(\mathcal{Q}^n_{a,b;l}\right) = q^{-a_1+b_1} \mathcal{Q}^{n-1}_{a+(0,1),b+(0,0);l}$$

and notice that the coefficients appearing are in each case independent of whether we're acting on $\mathcal{AP}$ or $\mathcal{AQ}$, and moreover that the coefficients are independent of the internal flow labels l. We can use this to show that $d\mathcal{GT}_{\{i,j\}}\left(\mathcal{P}^n_{a,b;l} - \sum_{m=\max a}^{\min b} \begin{bmatrix} n+\Sigma a-\Sigma b \\ m+l-\Sigma b \end{bmatrix}_q \mathcal{Q}^n_{a,b;m}\right)$ lies in $\mathcal{SS}^{n-1}$ for each of the four pairs $\{i,j\}$. For example,

$$d\mathcal{GT}_{\{1,2\}}\left(\mathcal{P}^n_{a,b;l}-\sum_{m=\max a}^{\min b}\begin{bmatrix}n+\Sigma a-\Sigma b\\ m+l-\Sigma b\end{bmatrix}_q\mathcal{Q}^n_{a,b;m}\right)=$$
$$=(-1)^{\Sigma a}q^{-n-\Sigma a+\Sigma b}\times$$
$$\times\left(\mathcal{P}^{n-1}_{a+(1,1),b+(0,1);l}-\sum_{m=\max a}^{\min b}\begin{bmatrix}n+\Sigma a-\Sigma b\\ m+l-\Sigma b\end{bmatrix}_q\mathcal{Q}^{n-1}_{a+(1,1),b+(0,1);m+1}\right)$$
$$=(-1)^{\Sigma a}q^{-n-\Sigma a+\Sigma b}\times$$
$$\times\Big(\mathcal{P}^{n-1}_{a+(1,1),b+(0,1);l}-$$
$$-\sum_{m=\max a}^{\min b}\begin{bmatrix}n-1+\Sigma(a+(1,1))-\Sigma(b+(0,1))\\ m+1+l-\Sigma(b+(0,1))\end{bmatrix}_q\mathcal{Q}^{n-1}_{a+(1,1),b+(0,1);m+1}\Big)$$

which becomes, upon reindexing the summation,

$$(-1)^{\Sigma a}q^{-n-\Sigma a+\Sigma b}\times$$
$$\times\Big(\mathcal{P}^{n-1}_{a+(1,1),b+(0,1);l}-$$
$$-\sum_{m=\max a+1}^{\min b+1}\begin{bmatrix}n-1+\Sigma(a+(1,1))-\Sigma(b+(0,1))\\ m+l-\Sigma(b+(0,1))\end{bmatrix}_q\mathcal{Q}^{n-1}_{a+(1,1),b+(0,1);m}\Big)$$

Now, making use of the fact that if $\min b+1>\min(b+(0,1))$, the last term vanishes since $\mathcal{Q}^{n-1}_{a+(1,1),b+(0,1);\min b+1}=0$, we can rewrite the summation limits and obtain

$$d\mathcal{GT}_{\{1,2\}}\left(\mathcal{P}^n_{a,b;l}-\sum_{m=\max a}^{\min b}\begin{bmatrix}n+\Sigma a-\Sigma b\\ m+l-\Sigma b\end{bmatrix}_q\mathcal{Q}^n_{a,b;m}\right)=$$
$$=(-1)^{\Sigma a}q^{-n-\Sigma a+\Sigma b}\times\Big(\mathcal{P}^{n-1}_{a+(1,1),b+(0,1);l}-$$
$$-\sum_{m=\max(a+(1,1))+1}^{\min(b+(0,1))}\begin{bmatrix}n-1+\Sigma(a+(1,1))-\Sigma(b+(0,1))\\ m+l-\Sigma(b+(0,1))\end{bmatrix}_q\mathcal{Q}^{n-1}_{a+(1,1),b+(0,1);m}\Big).$$

The parenthesised expression is exactly an element of the spanning set for $\mathcal{SS}^{n-1}_{a+(1,1),b+(0,1)}$. It remains to show that $d\mathcal{GT}_\emptyset$ and $d\mathcal{GT}_\partial$ carry $\mathcal{SS}^n$ into $\mathcal{SS}^{n-1}$. Both are similar; here's the calculation establishing the first.

$$d\mathcal{GT}_\emptyset\left(\mathcal{P}^n_{a,b;l}-\sum_{m=\max a}^{\min b}\begin{bmatrix}n+\Sigma a-\Sigma b\\ m+l-\Sigma b\end{bmatrix}_q\mathcal{Q}^n_{a,b;m}\right)=$$
$$=q^{-a_1}\left(q^l\mathcal{P}^{n-1}_{a,b;l}+q^{l-n-\Sigma a+\Sigma b}\mathcal{P}^{n-1}_{a,b;l-1}-\sum_{m=\max a}^{\min b}q^{-m}q^{\Sigma b}\begin{bmatrix}n+\Sigma a-\Sigma b\\ m+l-\Sigma b\end{bmatrix}_q\mathcal{Q}^{n-1}_{a,b;m}\right),$$

which, using the q-binomial identity $\begin{bmatrix}s\\ t\end{bmatrix}_q=q^t\begin{bmatrix}s-1\\ t\end{bmatrix}_q+q^{t-s}\begin{bmatrix}s-1\\ t-1\end{bmatrix}_q$, we can rewrite as

$$=q^{-a_1}\Big(q^l\mathcal{P}^{n-1}_{a,b;l}-\sum_{m=\max a}^{\min b}q^l\begin{bmatrix}n-1+\Sigma a-\Sigma b\\ m+l-\Sigma b\end{bmatrix}_q\mathcal{Q}^{n-1}_{a,b;m}+$$
$$q^{l-n-\Sigma a+\Sigma b}\mathcal{P}^{n-1}_{a,b;l-1}-\sum_{m=\max a}^{\min b}q^{l-n-\Sigma a+\Sigma b}\begin{bmatrix}n-1+\Sigma a-\Sigma b\\ m+l-1-\Sigma b\end{bmatrix}_q\mathcal{Q}^{n-1}_{a,b;m}\Big),$$

which is in $\mathcal{SS}^{n-1}_{a,b}$. ☐

5.7.3 The Kekulé relations

We now prove Theorem 5.3.2, describing the relations amongst polygonal diagrams. We begin with a slight reformulation of the goal; it turns out that the orthogonal complement $\mathcal{APR}^\perp$ of the relations in the dual space $\mathcal{AP}^*$ is easier to describe.

Lemma 5.7.1. *The orthogonal complement ${\mathcal{APR}^n_{a,b}}^\perp$ in ${\mathcal{AP}^n_{a,b}}^*$ is spanned by the elements*

$$e^{\mathcal{P},n}_{a,b;j^*} = \sum_{k^*=\Sigma b}^{n+\Sigma a} \begin{bmatrix} n+\Sigma a - \Sigma b \\ k^* - \Sigma b \end{bmatrix}_q (\mathcal{P}^n_{a,b;k^*+j^*})^*$$

for $j^* = -\overline{\sum} b, \ldots, -\underline{\sum} a$.

Proof. First note that $\mathcal{APR}^n_{a,b} \subset \mathcal{AP}^n_{a,b}$ is an $(n+\Sigma a - \Sigma b)$-dimensional subspace of $\mathcal{AP}^n_{a,b}$; the spanning set we've described for it really is linearly independent in $\mathcal{S}ym_n$. It's a subspace of a $(\min a - \max b + n + 1)$-dimensional vector space, so we expect the orthogonal complement to be $(1 + \overline{\sum} b - \underline{\sum} a)$-dimensional, as claimed in the lemma. We thus need only check that each $e^{\mathcal{P},n}_{a,b;j^*}$ annihilates each $d^{\mathcal{P},n}_{a,b;j}$, for $j = \Sigma b, \ldots, \Sigma a + n - 1$. Thus we calculate

$$e^{\mathcal{P},n}_{j^*}(d^{\mathcal{P},n}_j) = \sum_{k=\max(-\overline{\sum} b, \Sigma b + j^* - j)}^{\min(-\underline{\sum} a + 1, j^* - j + n + \Sigma a)} (-1)^{j+k} \begin{bmatrix} j+k-\max b \\ j - \Sigma b \end{bmatrix}_q \times \\ \times \begin{bmatrix} \min a + n - j - k \\ \Sigma a + n - 1 - j \end{bmatrix}_q \begin{bmatrix} n + \Sigma a - \Sigma b \\ j - j^* + k - \Sigma b \end{bmatrix}_q$$

and observe that the limits of the summation are actually irrelevant; outside the limits at least one of the three quantum binomial coefficients is zero.

Lemma A.1.1 in the appendix §A.1 shows that this q-binomial sum is zero. ☐

Corollary 5.7.2. *The orthogonal complement ${\mathcal{AQR}^n_{a,b}}^\perp$ in ${\mathcal{AQ}^n_{a,b}}^*$ is spanned by the elements*

$$e^{\mathcal{Q},n}_{a,b;j^*} = \sum_{k^*=\Sigma a}^{\Sigma b - n\left(\frac{|\partial|}{2}-1\right)} \begin{bmatrix} \Sigma b - \Sigma a - n\left(\frac{|\partial|}{2} - 1\right) \\ k^* - \Sigma a \end{bmatrix}_q {\mathcal{Q}^n_{a,b;j^*+k^*}}^\perp$$

for $j^* = -\underline{\sum} a, \ldots, -\overline{\sum} b + n\left(\frac{|\partial|}{2} - 1\right)$.

Proof. This follows immediately from the previous lemma, by rotation, as in §2.4.1. ☐

The proof of Theorem 5.3.2 now proceeds inductively. We'll assume that

$$\ker \mathbf{Rep} \cap \mathcal{AP}^{n-1}_{a,b} = \mathcal{APR}^{n-1}_{a,b}$$

for all flows a, b. We'll look at $d\mathcal{GT}_\emptyset$ first; we know that anything in $\ker \mathbf{Rep} \cap \mathcal{AP}^n_{a,b}$ must lie inside $d\mathcal{GT}^{-1}_\emptyset(\mathcal{APR}^{n-1}_{a,b})$, and we'll discover that in fact $d\mathcal{GT}^{-1}_\emptyset(\mathcal{APR}^{n-1}_{a,b}) = \mathcal{APR}^n_{a,b}$, in Lemma 5.7.3. Subsequently, we need to check, in Lemma 5.7.4, that $d\mathcal{GT}_{a',b'}(\mathcal{APR}^n_{a,b}) \subset \mathcal{APR}^{n-1}_{a+a',b+b'}$ for each allowed pair of boundary flow reduction patterns a', b'. Modulo these lemmas, that proves the first part of Theorem 5.3.2 (at least the statements about $\mathcal{APR}$; Lemma 5.7.5 deals with the statements about $\mathcal{AQR}$). Later, Lemma 5.7.8 deals with the other parts of Theorem 5.3.2.

Explicit formulas for $d\mathcal{GT}_\emptyset$ and $d\mathcal{GT}_{a',b'}$ are easy to come by, from §4.4:

$$d\mathcal{GT}_\emptyset\left(\mathcal{P}^n_{a,b;l}\right) = q^{l-a_1}\left(\mathcal{P}^{n-1}_{a,b;l} + q^{\Sigma b - \Sigma a - n}\mathcal{P}^{n-1}_{a,b;l-1}\right) \tag{5.7.4}$$

and

$$d\mathcal{GT}_{a',b'}(\mathcal{P}^n_{a,b;l}) = \kappa_{n,a,b,a',b'} q^{-l(m(a',b')-1)}\mathcal{P}^{n-1}_{a+a',b+b';l} \tag{5.7.5}$$

where here $\kappa_{n,a,b,a',b'}$ is some constant not depending on l, which for now we don't need to keep track of, but can be reconstructed from Equation (4.4.2), and $m(a',b')$ is the number of components of the reduction path indexed by a', b', which is just $\Sigma a' - \Sigma b'$.

It's worth noting at this point that the $\mathcal{P}$-webs appearing in the right hand sides of Equations (5.7.4) and (5.7.5) are sometimes zero; the internal flow value might be out of the allowable range.

In particular, the allowed values of l for $\mathcal{P}^n_{a,b;l} \in \mathcal{AP}^n_{a,b}$ are $l = \max b, \ldots, \min a + n$, and so the target of $d\mathcal{GT}_\emptyset$ is one dimension smaller than its source. Moreover, it's easy to see from Equation (5.7.4) that the kernel of $d\mathcal{GT}_\emptyset$ is exactly one dimensional. Life is a little more complicated for $d\mathcal{GT}_{a',b'}$; if $b'_i = 1$ for any i where $b_i = \max b$, then

$$d\mathcal{GT}_{a',b'}\left(\mathcal{P}^n_{a,b;\max b}\right) = 0$$

and if $a'_i = 1$ for every i where $a_i = \min a$, then

$$d\mathcal{GT}_{a',b'}\left(\mathcal{P}^n_{a,b;\min a+n}\right) = 0$$

and otherwise $d\mathcal{GT}_{a',b'}$ is actually nonzero on each $\mathcal{P}$-web. Thus the kernel of $d\mathcal{GT}_{a',b'}$ may be $0, 1$ or 2 dimensional.

Lemma 5.7.3. *The only candidates for the kernel of the representation functor on $\mathcal{AP}^n_{a,b}$ are in $\mathcal{APR}^n_{a,b}$, since*

$$d\mathcal{GT}^{-1}_\emptyset(\mathcal{APR}^{n-1}_{a,b}) = \mathcal{APR}^n_{a,b}.$$

Proof. We've already noticed that the kernel of $d\mathcal{GT}_\emptyset$ is 1-dimensional, so we actually only need to check that $d\mathcal{GT}_\emptyset\left(\mathcal{APR}^n_{a,b}\right) \subset \mathcal{APR}^{n-1}_{a,b}$, or, equivalently, that

$$d\mathcal{GT}^*_\emptyset({\mathcal{APR}^{n-1}_{a,b}}^{\perp}) \subset {\mathcal{APR}^n_{a,b}}^{\perp}.$$

Easily, we obtain

$$d\mathcal{GT}^*_\emptyset({\mathcal{P}^{n-1}_{a,b;l^*}}^*) = q^{l^*-a_1}\left({\mathcal{P}^n_{a,b;l^*}}^* + q^{\Sigma b - \Sigma a - n+1}{\mathcal{P}^n_{a,b;l^*+1}}^*\right),$$

simply by taking duals in Equation (4.4.4). Then

$$\begin{aligned} d\mathcal{GT}^*_\emptyset(e^{\mathcal{P},n-1}_{a,b;j^*}) &= d\mathcal{GT}^*_\emptyset\left(\sum_{k^*=\Sigma b}^{n-1+\Sigma a}\begin{bmatrix} n-1+\Sigma a - \Sigma b \\ k^* - \Sigma b\end{bmatrix}_q (\mathcal{P}^{n-1}_{a,b;k^*+j^*})^*\right) \\ &= q^{-a_1}\sum_{k^*=\Sigma b}^{n-1+\Sigma a} q^{k^*+j^*}\begin{bmatrix} n-1+\Sigma a - \Sigma b \\ k^* - \Sigma b\end{bmatrix}_q \left({\mathcal{P}^n_{a,b;k^*+j^*}}^* + q^{\Sigma b - \Sigma a - n+1}{\mathcal{P}^n_{a,b;k^*+j^*+1}}^*\right)\end{aligned}$$

and by reindexing the summation of the second terms, we can rewrite this as

$$dGT^*_\emptyset(e^{\mathcal{P},n-1}_{a,b;j^*}) = q^{j^*-a_1+\Sigma b} \sum_{k^*=\Sigma b}^{n+\Sigma a} \left(q^{k^*-\Sigma b} \begin{bmatrix} n-1+\Sigma a-\Sigma b \\ k^*-\Sigma b \end{bmatrix}_q + q^{k^*-n-\Sigma a} \begin{bmatrix} n-1+\Sigma a-\Sigma b \\ k^*-1-\Sigma b \end{bmatrix}_q \right) {\mathcal{P}^n_{a,b;k^*+j^*}}^*$$

$$= q^{j^*-a_1+\Sigma b} \sum_{k^*=\Sigma b}^{n+\Sigma a} \begin{bmatrix} n+\Sigma a-\Sigma b \\ k^*-\Sigma b \end{bmatrix}_q {\mathcal{P}^n_{a,b;k^*+j^*}}^*$$

$$= q^{j^*-a_1+\Sigma b} e^{\mathcal{P},n}_{a,b;j^*},$$

tidily completing the proof. ☐

Remark. By rotation, we can also claim that

$$dGT^{-1}_\partial(\mathcal{AQR}^{n-1}_{a,b}) = \mathcal{AQR}^n_{a,b},$$

and that if you're in the kernel of **Rep**, and in $\mathcal{AQ}^n_{a,b}$, you must also be in $\mathcal{AQR}^n_{a,b}$.

Lemma 5.7.4. *The candidate relations $\mathcal{APR}^n_{a,b}$ really are in the kernel of the representation functor, since*

$$dGT_{a',b'}(\mathcal{APR}^n_{a,b}) \subset \mathcal{APR}^{n-1}_{a+a',b+b'}$$

for each pair a', b' (other than the pairs $a' = b' = \overrightarrow{0}$ and $a' = b' = \overrightarrow{1}$; the previous Lemma dealt with that matrix entry of dGT).

Proof. For this we need the quantum Vandermonde identity [39],

$$\begin{bmatrix} x+y \\ z \end{bmatrix}_q = q^{yz} \sum_{i=0}^{y} q^{-(x+y)i} \begin{bmatrix} y \\ i \end{bmatrix}_q \begin{bmatrix} x \\ z-i \end{bmatrix}_q.$$

Throughout, we'll write $m = \Sigma a' - \Sigma b'$ for the number of components of the reduction path. We want to show

$$dGT^*_{a',b'}(e^{\mathcal{P},n-1}_{a+a',b+b';j^*}) = \kappa_{n,a+a',b+b',a',b'} \times \times \sum_{k^*=\Sigma b}^{n+\Sigma a+m-1} q^{-(j^*+k^*+\Sigma b')(m-1)} \begin{bmatrix} n-\Sigma a-\Sigma b+m-1 \\ k^*-\Sigma b \end{bmatrix}_q {\mathcal{P}^n_{a,b;k^*+j^*+\Sigma b}}^* \tag{5.7.6}$$

is equal to

$$\sum_{i=0}^{m-1} X_i e^{\mathcal{P},n}_{a,b,j^*+\Sigma b+i} = \sum_{i=0}^{m-1} X_i \sum_{k_*=\Sigma b}^{n+\Sigma a} \begin{bmatrix} n+\Sigma a-\Sigma b \\ k_*-\Sigma b \end{bmatrix}_q {\mathcal{P}^n_{a,b;k_*+j^*+i+\Sigma b'}}^* \in {\mathcal{APR}^n_{a,b}}^\perp$$

for some coefficients X_i. Looking at the coefficient of ${\mathcal{P}^n_{a,b;k^*+j^*+\Sigma b'}}^n$ in Equation (5.7.6) and applying the quantum Vandermonde identity with $x = n+\Sigma a-\Sigma b$, $y = m-1$, and $z = k^*-\Sigma b$, we obtain

$$\kappa_{n,a+a',b+b',a',b'} q^{-(j^*+k^*+\Sigma b')(m-1)} q^{(m-1)(k^*-\Sigma b)} \times \times \sum_{i=0}^{m-1} q^{-(n+\Sigma a-\Sigma b+m-1)} \begin{bmatrix} m-1 \\ i \end{bmatrix}_q \begin{bmatrix} n+\Sigma a-\Sigma b \\ k^*-\Sigma b-i \end{bmatrix}_q$$

Choosing $X_i = \kappa_{n,a+a',b+b',a',b'} q^{(m-1)(-j^*-\Sigma b-\Sigma b')} q^{-(n+\Sigma a-\Sigma b+m-1)i}$, (which, note, is independent of k^*, the index of the term we're looking at) this is exactly the coefficient of ${\mathcal{P}^n_{a,b;k^*+j^*+\Sigma b'}}^n$ in $\sum_{i=0}^{m-1} X_i e^{\mathcal{P},n}_{a,b,j^*+\Sigma b+i}$. ☐

Lemma 5.7.5. *The intersection of* $\ker \mathbf{Rep}$ *and* $\mathcal{AQ}$ *is exactly* $\mathcal{AQR}$.

Proof. Again, this is just by rotation, see §2.4.1. □

The next two Lemmas, and the subsequent proof of Lemma 5.3.3, are kinda hairy. Hold on tight!

Lemma 5.7.6. *For any* n*-hexagonal flows* (a, b) *of length* $k \geq 3$ *such that* $\mathcal{AP}^n_{a,b} \neq 0$ *one of the following must hold:*

1. *The flows* (a, b) *are also* $(n-1)$*-hexagonal.*
2. *The flows* $(a + \overrightarrow{1}, b)$ *are* $(n-1)$*-hexagonal.*
3. *There is some* $(a', b') \in (\{0, 1\}^k)^2$*, so* $(a + a', b + b')$ *is* $(n-1)$*-hexagonal and moreover* $d\mathcal{GT}_{a',b'}$ *maps* $\mathcal{AP}^n_{a,b}$ *faithfully into* $\mathcal{AP}^{n-1}_{a+a',b+b'}$.
4. $n \leq 3$.

Proof. For (a, b) not to be $(n-1)$-hexagonal, at least $k-5$ edge labels must be $(n-1)$, and for $(a + \overrightarrow{1}, b)$ not to be $(n-1)$-hexagonal, at least $k-5$ edge labels must be 1. Already, this establishes the lemma for $k > 10$ and $n > 2$ (and it's trivially true for $n \leq 2$).

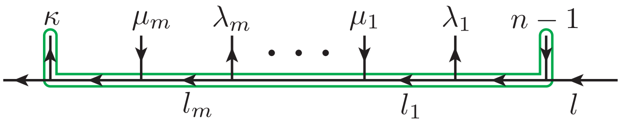
We thus need to deal with $k = 6, 8$ or 10. If $k = 10$, one of the first two alternatives hold, unless $\mathcal{L}(a, b)$ is some permutation of $(1, 1, 1, 1, 1, n-1, n-1, n-1, n-1, n-1)$. It's easy enough to see that no permutation is possible, given the condition that the incoming edges and outgoing edges have the same sum. For $k = 6$ or 8, we can be sure that both 1 and $n-1 \in \mathcal{L}(a, b)$. Notice that this implies $\mathcal{APR}^n_{a,b} = \mathcal{AQR}^n_{a,b} = 0$, and both $\mathcal{AP}^n_{a,b}$ and $\mathcal{AQ}^n_{a,b}$ are at most 2-dimensional, with $\mathcal{AP}^n_{a,b}$ spanned by $\mathcal{P}^n_{a,b;\max b}$ and $\mathcal{P}^n_{a,b;\min a+n}$ and $\mathcal{AQ}^n_{a,b}$ spanned by $\mathcal{Q}^n_{a,b;\max a}$ and $\mathcal{Q}^n_{a,b;\min b}$. Let's now assume that we never see (not even 'cyclically') as contiguous sequences of boundary edges labels either

$$((n-1,-),(* \leq 1,+),(*,-),(* \leq 1,+),(*,-),\ldots,(*,-),(* \leq 1,+),(n-1,-)) \tag{5.7.7a}$$

or

$$((n-1,+),(* \leq 1,-),(*,+),(* \leq 1,-),(*,+),\ldots,(*,+),(* \leq 1,-),(n-1,+)). \tag{5.7.7b}$$

We can then prescribe a reduction path on the $\mathcal{P}$-polygons, which corresponds to a pair (a', b') establishing the third alternative above. Consider the reduction path which starts at each incoming edge labeled $n-1$, and ends at the next (heading counterclockwise) outgoing edge with a label greater than 1, and which additionally ends at each outgoing edge labeled $n-1$, having started at the previous incoming edge with a label greater than 1. That this forms a valid reduction path (i.e. there are no overlaps between the components described) follows immediately from the assumption of the previous paragraph, that certain sequences do not appear. The pair (a', b') corresponding (via the correspondence discussed in §4.4) to this reduction path then satisfies the first condition of the third alternative above, that $(a + a', b + b')$ be $(n-1)$-hexagonal. However, it doesn't obviously satisfy the second condition. The only way that $d\mathcal{GT}_{a',b'}$ might not be faithful on $\mathcal{AP}^n_{a,b}$ is if it kills $\mathcal{P}^n_{a,b;\max b}$ or $\mathcal{P}^n_{a,b;\min a+n}$ (remember $\max b = \min a + n - 1$ or $\min a + n$, since $0 < \dim \mathcal{AP}^n_{a,b} \leq 2$). To kill $\mathcal{P}^n_{a,b;\max b}$, the reduction path we've described would have to traverse an internal edge labeled 0. However, it turns out there's 'no room' for this. Consider some component of the reduction path starting at an incoming edge labeled $n-1$. It then looks like:



for some $0 \leq l \leq n$, $1 < \kappa \leq n$, $\lambda_i = 0$ or 1 and $0 \leq \mu_i \leq n$. If any internal 0 edge gets reduced, there must be some i so that the internal edge l_i also gets reduced. However, $l_1 = l+n-1-\lambda_1 \geq l+n-2$, and generally $l_i = l+n-1+\sum_{j=1}^{i-1}(-\lambda_j+\mu_i)-\lambda_i \geq l+n+\sharp(\mu_i \geq 1)-i-1$. Thus for l_i to be 0, we must have

$$m \geq i \geq n-1+\sharp(\mu_i \geq 1). \tag{5.7.8}$$

Now, if $k=6$, none of the external edges can be zero, since (a,b) is n-hexagonal. Thus $\sharp(\mu_i \geq 1) = m$, so as long as $n \geq 2$, the inequality of Equation (5.7.8) cannot be satisfied. If, on the other hand $k=8$, we can only say $\sharp(\mu_i \geq 1) \geq m-1$ (since any non-adjacent pair of external edges being zero makes it impossible to satisfy the 'alternating' part of the condition for being n-hexagon), so $m \geq n-1+m-1$, or $n \leq 2$. The argument preventing $d\mathcal{GT}_{a',b'}$ from killing $\mathcal{P}^n_{a,b;\min a+n}$ is much the same.

Finally, we need to show that the appearance of one of the 'forbidden sequences' from Equation (5.7.7) forces $n \leq 3$, so that the last alternative holds. Suppose that, for some $m \geq 0$, $((n-1,-),(\lambda_1,+),(\kappa_1,-),\ldots,(\kappa_m,-),(\lambda_{m+1},+),(n-1,-)$ with $\lambda_i = 0$ or 1 appears as a contiguous subsequence of $\mathcal{L}(a,b)$. The internal edge labels on either side of this subsequence then differ by $2n-2+\sum_{i=1}^m \kappa_i - \sum_{i=1}^{m+1}\lambda_i$. We know this quantity must be no more than n, since $\mathcal{AP}^n_{a,b} \neq 0$, i.e., there is some internal flow value l so $\mathcal{P}^n_{a,b;l} \neq 0$. However, $n-2+\sum_{i=1}^m \kappa_i - \sum_{i=1}^{m+1}\lambda_i \geq n-2+\sharp(\kappa_i \neq 0)-\sharp(\lambda_i = 1) \geq n+\sharp(\kappa_i \neq 0)-m-3$. Thus we must have $n \leq m+3-\sharp(\kappa_i \neq 0)$. If such a subsequence appears with $m=0$, this immediately forces $n \leq 3$. If such a subsequence appears with $m>0$, either $n \leq 3$ or $\sharp(\kappa_i \neq 0) < m$, so there's at least one $\kappa_i = 0$. This certainly isn't possible for $k=3$; (a,b) couldn't be n-hexagonal.

We're now left with a fairly finite list of cases to check for the $k=4$ case; for each one, even in the presence of a 'forbidden sequence' we'll explicitly describe an ad hoc reduction path, so that the corresponding (a',b') pair satisfies the third alternative. We need to consider both $m=1$ and $m=2$; there's no room for $m \geq 3$ if $k=4$. For $m=1$, the boundary edges (up to a rotation) are

$$((n-1,-),(\lambda_1,+),(0,-),(\lambda_2,+),(n-1,-),(\mu_1,+),(\mu_2,-),(\mu_3,+))$$

with $\lambda_i = 0$ or 1 and some μ_i. In order for this to be n-hexagon, we can't have $\mu_1 = n$, $\mu_2 = 0$, $\mu_3 = n$ or both $\lambda_i = 0$. Subject then to the condition that incoming and outgoing labels have the same sum, we must have one of

$$\begin{array}{lllll}
\mu_2 = 2 & \lambda_1 = 1 & \lambda_2 = 1 & \mu_1 = n-1 & \mu_3 = n-1, \\
\mu_2 = 1 & \lambda_1 = 1 & \lambda_2 = 1 & \mu_1 = n-2 & \mu_3 = n-1, \\
\mu_2 = 1 & \lambda_1 = 1 & \lambda_2 = 1 & \mu_1 = n-1 & \mu_3 = n-2, \\
\mu_2 = 1 & \lambda_1 = 0 & \lambda_2 = 1 & \mu_1 = n-1 & \mu_3 = n-1,
\end{array}$$

or

$$\begin{array}{lllll}
\mu_2 = 1 & \lambda_1 = 1 & \lambda_2 = 0 & \mu_1 = n-1 & \mu_3 = n-1.
\end{array}$$

For the first three, however, there's an (a',b') pair:

$$((1,0,1,1),(0,0,0,0)),$$
$$((1,1,1,0),(0,1,0,0)),$$

or

$$((1,0,1,1),(0,0,0,1)),$$

and for the last two, we find that $\mathcal{AP}^n_{a,b} = 0$. For $m = 2$, the boundary edges are

$$((n-1,-),(\lambda_1,+),(\kappa_1,-),(\lambda_2,+),(\kappa_2,-),(\lambda_3,+),(n-1,-),(\mu,+))$$

with $\lambda_i = 0$ or 1 and at least one of the $\kappa_i = 0$. If both $\kappa_i = 0$, or if $\mu = n$, we can't achieve n-hexagon-ness. But then the sum of the incoming labels is $2n-2+\kappa_1+\kappa_2 \geq 2n-1$, while the sum of the outgoing edges is $\lambda_1+\lambda_2+\lambda_3+\mu \leq n+2$. If $2n-1 \leq n+2$, $n \leq 3$ anyway, and we're done.

The argument for the forbidden sequence with opposite orientations is almost identical. □

Lemma 5.7.7. *The intersection* $\mathcal{APR}^n_{a,b} \cap \operatorname{span}_{\mathcal{A}} \left\{ \mathcal{P}^n_{a,b;\max b}, \mathcal{P}^n_{a,b;\max b+1}, \ldots, \mathcal{P}^n_{a,b;\Sigma b - \underline{\sum} a} \right\}$ *is zero.*

Proof. Say $x = \sum_{i=0}^{\overline{\sum} b - \underline{\sum} a} x_i \mathcal{P}^n_{a,b;\max b+i}$, write $i_{\max}$ for the greatest i so $x_i \neq 0$. Define $j^* = i_{\max} - \Sigma b$, and note that since $\max b \leq i_{\max} \leq \Sigma b - \underline{\sum} a$ we must have $-\overline{\sum} b \leq j^* \leq -\underline{\sum} a$. Then apply $e^{\mathcal{P},n}_{a,b;j^*}$, as defined in Lemma 5.7.1 to x, obtaining:

$$e^{\mathcal{P},n}_{a,b;j^*}(x) = x_{i_{\max}} \neq 0,$$

so $x \notin \mathcal{APR}^n_{a,b}$. □

We now give the proof of an earlier lemma, as we're about to need it.

Proof of Lemma 5.3.3. We can just prove the ∞-circumference case. Moreover, if we're assuming (a,b) has ∞-circumference $2k$, we can further assume that a and b each actually have length k. (Otherwise, there's some i so $b_i = a_i$, or $b_i = a_{i+1}$; in that case, 'snip out' those entries from a and b.) Now, pick some $i_{\min}$ so $a_{i_{\min}} = \min a$, and then pick as $i_{\max}$ the first value of $i \geq i_{\min}$ (in the cyclic sense) so that $b_{i_{\max}} = \max b$. If $i_{\max} = i_{\min}$, we're done; $\overline{\sum} b - \underline{\sum} a = \sum_{i \neq i_{\max}} b_i - a_i \geq k-1$. Otherwise, we'll show how to modify a and b, preserving the value of $\overline{\sum} b - \underline{\sum} a$, to make $i_{\max} - i_{\min}$ (again, measured cyclically) smaller. Specifically, for each $i_{\min} \leq i < i_{\max}$, increase b_i by $b_{i_{\max}} - b_{i_{\max}-1}$, and for each $i_{\max} < i \leq i_{\max}$, increase a_i by the same. The resulting sequence is still valid, in the sense that $b_i \geq a_i, a_{i+1}$ for each i. Clearly, this doesn't change any of $\max b$, $\min a$, Σb or Σa, but it reduces the minimal value of $i_{\max} - i_{\min}$ by one. □

We the previous three monstrosities tamed, we can finish of Theorem 5.3.2.

Lemma 5.7.8. *If (a,b) is a pair of flow labels of length at least 3, and all the external edge labels in $\mathcal{L}(a,b)$ are between 1 and $n-1$ inclusive, the intersection of* $\ker \mathbf{Rep}$ *and* $\mathcal{AP}^n_{a,b} + \mathcal{AQ}^n_{a,b}$ *is* $\mathcal{APR}^n_{a,b} + \mathcal{AQR}^n_{a,b}$.

Proof. If $\mathcal{AQ}^n_{a,b} \neq 0$ but $\mathcal{AP}^n_{a,b} = 0$, first apply a rotation, as described in §2.4.1. We need to deal with the four alternatives of the previous lemma.

Suppose $n-1 \notin \mathcal{L}(a,b)$, so we're in the first alternative. Then after applying $d\mathcal{GT}_\emptyset$, we're in a situation where 5.3.2 holds in full; the kernel of $\mathbf{Rep}$ is just $\mathcal{APR}^{n-1} + \mathcal{AQR}^{n-1}$. (The argument below works also if $1 \notin \mathcal{L}(a,b)$ and we're in the second alternative, mutatis mutandis, exchanging the roles of $d\mathcal{GT}_\emptyset$ and $d\mathcal{GT}_\partial$.) Thus suppose $X \in \mathcal{AP}^n + \mathcal{AQ}^n$ is in $\ker \mathbf{Rep}$. Then $d\mathcal{GT}_\emptyset(X) = X_{\emptyset;\mathcal{P}} + X_{\emptyset;\mathcal{Q}}$, for some $X_{\emptyset;\mathcal{P}} \in \mathcal{APR}^{n-1}$ and $X_{\emptyset;\mathcal{Q}} \in \mathcal{AQR}^{n-1}$. Pick some $X_{\mathcal{P}} \in d\mathcal{GT}_\emptyset^{-1}(X_{\emptyset;\mathcal{P}})$, which we know by Lemma 5.7.3 must also be an element of $\mathcal{APR}^n$. Now $d\mathcal{GT}_\emptyset(X - X_{\mathcal{P}}) = X_{\emptyset;\mathcal{Q}} \in \mathcal{AQR}^{n-1}$, so $X - X_{\mathcal{P}}$ must lie in $\mathcal{AQ}^n$. We want to show it's actually in $\mathcal{AQR}^n$. We know (via Lemma 5.7.4), that $X - X_{\mathcal{P}} \in \ker \mathbf{Rep}$, so $d\mathcal{GT}_\partial(X - X_{\mathcal{P}}) \in \mathcal{AQR}^{n-1}$, by the remark following Lemma 5.7.3. Thus $X - X_{\mathcal{P}} \in \mathcal{AQR}^n$. The decomposition $X = X_{\mathcal{P}} + (X - X_{\mathcal{P}})$ establishes the desired result.

Suppose that there's some (a', b') with $d\mathcal{GT}_{a',b'|\mathcal{AP}^n_{a,b}}$ faithful and $(a + a', b + b')$ $(n-1)$-hexagonal. Now, if $1, n-1 \notin \mathcal{L}(a, b)$, as in the last lemma we know $\mathcal{AP}^n_{a,b}$ is spanned by $\mathcal{P}^n_{a,b;\max b}$ and $\mathcal{P}^n_{a,b;\min a+n}$ and $\mathcal{AQ}^n_{a,b}$ is spanned by $\mathcal{Q}^n_{a,b;\max a}$ and $\mathcal{Q}^n_{a,b;\min b}$ (remember that in each case the pairs of elements might actually coincide). Thus

$$\begin{aligned} d\mathcal{GT}_{a',b'}\left(\mathcal{P}^n_{a,b;\max b}\right) &= (-1)^{b'\cdot a+\mathrm{rotl}(a)} q^{\max b(\Sigma b'-\Sigma a'+1)}\times \\ &\quad \times q^{\mathrm{rotl}(a')\cdot b-b'\cdot a-a_1-na'_1}\mathcal{P}^{n-1}_{a+a',b+b';\max b} \neq 0, \\ d\mathcal{GT}_{a',b'}\left(\mathcal{P}^n_{a,b;\min a+n}\right) &= (-1)^{b'\cdot a+\mathrm{rotl}(a)} q^{(\min a+n)(\Sigma b'-\Sigma a'+1)}\times \\ &\quad \times q^{\mathrm{rotl}(a')\cdot b-b'\cdot a-a_1-na'_1}\mathcal{P}^{n-1}_{a+a',b+b';\min a+n} \neq 0, \\ d\mathcal{GT}_{a',b'}\left(\mathcal{Q}^n_{a,b;\max a}\right) &= (-1)^{b'\cdot a+\mathrm{rotl}(a)} q^{\max a(\Sigma a'-\Sigma b'-\frac{|\partial|}{2}+1)}\times \\ &\quad \times q^{\Sigma b-n\Sigma b'+b'\cdot\mathrm{rotl}(a)-a'\cdot b-a_1-na'_1}\mathcal{Q}^{n-1}_{a+a',b+b';\max a}, \end{aligned}$$

and

$$\begin{aligned} d\mathcal{GT}_{a',b'}\left(\mathcal{Q}^n_{a,b;\min b}\right) &= (-1)^{b'\cdot a+\mathrm{rotl}(a)} q^{\min b(\Sigma a'-\Sigma b'-\frac{|\partial|}{2}+1)}\times \\ &\quad \times q^{\Sigma b-n\Sigma b'+b'\cdot\mathrm{rotl}(a)-a'\cdot b-a_1-na'_1}\mathcal{Q}^{n-1}_{a+a',b+b';\min b}. \end{aligned}$$

Thus suppose $x_1\mathcal{P}^n_{a,b;\max b} + x_2\mathcal{P}^n_{a,b;\min a+n} + y_1\mathcal{Q}^n_{a,b;\max a} + y_2\mathcal{Q}^n_{a,b;\min b}$ is in $\ker \mathbf{Rep}_n$. Then

$$x'_1\mathcal{P}^{n-1}_{a+a',b+b';\max b} + x'_2\mathcal{P}^{n-1}_{a+a',b+b';\min a+n} + y'_1\mathcal{Q}^{n-1}_{a+a',b+b';\max a} + y'_2\mathcal{Q}^{n-1}_{a+a',b+b';\min b}$$

is in $\ker \mathbf{Rep}_{n-1}$, where x'_1 is a nonzero multiple of x_1, and x'_2 is a nonzero multiple of x_2, by the formulas above. Since $(a+a', b+b')$ is $(n-1)$-hexagonal, we can inductively assume $\ker \mathbf{Rep}_{n-1} = \mathcal{APR}^{n-1}_{a+a',b+b'} + \mathcal{AQR}^{n-1}_{a+a',b+b'}$, so $x'_1\mathcal{P}^{n-1}_{a+a',b+b';\max b} + x'_2\mathcal{P}^{n-1}_{a+a',b+b';\min a+n} \in \mathcal{APR}^{n-1}_{a+a',b+b'}$. However, by Lemma 5.7.7, along with the knowledge that the relations in $\mathcal{APR}^{n-1}_{a+a',b+b'}$ have 'breadth' at least 3, as per Lemma 5.3.3, this implies $x'_1 = x'_2 = 0$, and thus $x_1 = x_2 = 0$. Now we know $y_1\mathcal{Q}^n_{a,b;\max a} + y_2\mathcal{Q}^n_{a,b;\min b}$ is in $\ker \mathbf{Rep}_n$, which when restricted to $\mathcal{AQ}^n_{a,b}$ is just $\mathcal{AQR}^n_{a,b}$, by Lemma 5.7.5. Again by Lemma 5.7.7, $y_1 = y_2 = 0$.

The final alternative is $n \leq 3$, where the lemma is obvious from the known complete descriptions of $\ker \mathbf{Rep}_2$ and $\ker \mathbf{Rep}_3$. □

5.7.4 More about squares

To prove Theorem 5.4.1, that $\mathcal{SS} = \mathcal{APR} \oplus \mathcal{SS}'$, we need to establish the following three lemmas. (Trivially, $\mathcal{APR} \cap \mathcal{SS}' = 0$.) We'll in fact only deal with the $n + \Sigma a - \Sigma b > 0$ case of Theorem 5.4.1; the $n + \Sigma a - \Sigma b = 0$ is trivial, and, as usual, the $n + \Sigma a - \Sigma b < 0$ case follows by rotation.

Lemma 5.7.9. *$\mathcal{APR} \subset \mathcal{SS}$.*

Proof. In the quotient $(\mathcal{AP} + \mathcal{AQ})/\mathcal{SS}$,

$$\sum_{k=-\sum b}^{\sum a+1} (-1)^{j+k} \begin{bmatrix} j+k-\max b \\ j-\Sigma b \end{bmatrix}_q \begin{bmatrix} \min a+n-j-k \\ \Sigma a+n-1-j \end{bmatrix}_q \mathcal{P}^n_{a,b;j+k} =$$

$$\sum_{k=-\sum b}^{\sum a+1} (-1)^{j+k} \begin{bmatrix} j+k-\max b \\ j-\Sigma b \end{bmatrix}_q \begin{bmatrix} \min a+n-j-k \\ \Sigma a+n-1-j \end{bmatrix}_q \sum_{m=\max a}^{\min b} \begin{bmatrix} n+\Sigma a-\Sigma b \\ m+j+k-\Sigma b \end{bmatrix}_q \mathcal{Q}^n_{a,b;m}.$$

The coefficient of $\mathcal{Q}^n_{a,b;m}$ in this is

$$\sum_{k=-\sum b}^{\sum a+1} \sum_{m=\max a}^{\min b} (-1)^{j+k} \begin{bmatrix} j+k-\max b \\ j-\Sigma b \end{bmatrix}_q \begin{bmatrix} \min a+n-j-k \\ \Sigma a+n-1-j \end{bmatrix}_q \begin{bmatrix} n+\Sigma a-\Sigma b \\ m+j+k-\Sigma b \end{bmatrix}_q.$$

Not only does this vanish, but each term indexed by a particular value of m vanishes separately: replacing m with $-j^*$, this is precisely the q-binomial identity of Lemma A.1.1. □

Lemma 5.7.10. *$\mathcal{SS}' \subset \mathcal{SS}$.*

Proof. We need to show that each element of the spanning set of $\mathcal{SS}'$ presented in Theorem 5.4.1 is in $\mathcal{SS}$. In $\mathcal{SS}'/(\mathcal{SS} \cap \mathcal{SS}')$,

$$\begin{aligned} \mathcal{Q}^n_{a,b;m} - \sum_{l=n+\Sigma a-m}^{n+\min a} (-1)^{m+l+n+\Sigma a} \begin{bmatrix} m+l-1-\Sigma b \\ m+l-n-\Sigma a \end{bmatrix}_q \mathcal{P}^n_{a,b;l} = \\ = \mathcal{Q}^n_{a,b;m} - \sum_{l=n+\Sigma a-m}^{n+\min a} (-1)^{m+l+n+\Sigma a} \begin{bmatrix} m+l-1-\Sigma b \\ m+l-n-\Sigma a \end{bmatrix}_q \sum_{m'=\max a}^{\min b} \begin{bmatrix} n+\Sigma a-\Sigma b \\ m+l-\Sigma b \end{bmatrix}_q \mathcal{Q}^n_{a,b;m'}. \end{aligned}$$

The coefficient of $\mathcal{Q}^n_{a,b;m'}$ here is

$$\delta_{mm'} - (-1)^{m+n+\Sigma a} \sum_{l=n+\Sigma a-\min b}^{n+\min a} (-1)^l \begin{bmatrix} m+l-1-\Sigma b \\ m+l-n-\Sigma a \end{bmatrix}_q \begin{bmatrix} n+\Sigma a-\Sigma b \\ m'+l-\Sigma b \end{bmatrix}_q.$$

That this is zero follows from the q-binomial identity proved in §A.1 as Lemma A.1.2. □

Lemma 5.7.11. *$\mathcal{SS} \subset \mathcal{SS}' + \mathcal{APR}$.*

Proof. We take an element of the spanning set described for $\mathcal{SS}$, considered as an element of $(\mathcal{AP} + \mathcal{AQ})/\mathcal{SS}'$. Using the relations from $\mathcal{SS}'$, we write this as an element of $\mathcal{AP}/\mathcal{SS}'$, and check that this element is annihilated by each element of the spanning set for $\mathcal{APR}^\perp$ described in Lemma 5.7.1. Thus,

$$\begin{aligned} \mathcal{P}^n_{a,b;l} - \sum_{m=\max a}^{\min b} \begin{bmatrix} n+\Sigma a-\Sigma b \\ m+l-\Sigma b \end{bmatrix}_q \mathcal{Q}^n_{a,b;m} = \\ = \mathcal{P}^n_{a,b;l} - \sum_{m=\max a}^{\min b} \begin{bmatrix} n+\Sigma a-\Sigma b \\ m+l-\Sigma b \end{bmatrix}_q \sum_{l'=n+\Sigma a-m}^{n+\min a} (-1)^{m+l'+n+\Sigma a} \begin{bmatrix} m+l'-1-\Sigma b \\ m+l'-n-\Sigma a \end{bmatrix}_q \mathcal{P}^n_{a,b;l'}. \end{aligned}$$

Applying $\sum_{k^*=\Sigma b}^{n+\Sigma a} \begin{bmatrix} n+\Sigma a-\Sigma b \\ k^*-\Sigma b \end{bmatrix}_q (\mathcal{P}^n_{a,b;k^*+j^*})^*$ to this we get

$$\begin{aligned} \begin{bmatrix} n+\Sigma a-\Sigma b \\ k^*-\Sigma b \end{bmatrix}_q \delta_{l,k^*+j^*} - \\ \sum_{m=\max a}^{\min b} \sum_{l'=n+\Sigma a-m}^{n+\min a} \sum_{k^*=\Sigma b}^{n+\Sigma a} (-1)^{m+l'+n+\Sigma a} \begin{bmatrix} n+\Sigma a-\Sigma b \\ m+l-\Sigma b \end{bmatrix}_q \\ \begin{bmatrix} m+l'-1-\Sigma b \\ m+l'-n-\Sigma a \end{bmatrix}_q \begin{bmatrix} n+\Sigma a-\Sigma b \\ k^*-\Sigma b \end{bmatrix}_q \delta_{l',k^*+j^*}. \end{aligned}$$

First noticing that the lower limit for l' is redundant, a slight variation Lemma A.1.2 lets us evaluate the sum over m, obtaining

$$\begin{aligned}
&\begin{bmatrix} n+\Sigma a-\Sigma b \\ k^*-\Sigma b \end{bmatrix}_q \delta_{l,k^*+j^*} - \sum_{l'=-\infty}^{n+\min a} \sum_{k^*=\Sigma b}^{n+\Sigma a} \delta_{l,l'} \begin{bmatrix} n+\Sigma a-\Sigma b \\ k^*-\Sigma b \end{bmatrix}_q \delta_{l',k^*+j^*} \\
&= \begin{bmatrix} n+\Sigma a-\Sigma b \\ k^*-\Sigma b \end{bmatrix}_q \delta_{l,k^*+j^*} - \sum_{k^*=\Sigma b}^{n+\Sigma a} \begin{bmatrix} n+\Sigma a-\Sigma b \\ k^*-\Sigma b \end{bmatrix}_q \delta_{l,k^*+j^*} \\
&= \begin{bmatrix} n+\Sigma a-\Sigma b \\ k^*-\Sigma b \end{bmatrix}_q \delta_{l,k^*+j^*} - \begin{bmatrix} n+\Sigma a-\Sigma b \\ k^*-\Sigma b \end{bmatrix}_q \delta_{l,k^*+j^*} \\
&= 0. \qquad \square
\end{aligned}$$

Chapter 6

Relationships with previous work

6.1 The Temperley-Lieb category

The usual Temperley-Lieb category (equivalently, the Kauffman bracket skein module) has unoriented strands, and a 'loop value' of $-[2]_q$. The category for $U_q(\mathfrak{sl}_2)$ described here has oriented strands, and a 'loop value' of $[2]_q$, along with orientation reversing 'tags', which themselves can be flipped for a sign. Nevertheless, the categories are equivalent, as described below (in parallel with a description of the relationship between my category for $U_q(\mathfrak{sl}_4)$ and Kim's previous conjectured one) in §6.4. The modification of Khovanov homology described by myself and Kevin Walker in [26], producing a fully functorial invariant, is based on a categorification of the quantum group skein module, rather than the Kauffman skein module.

6.2 Kuperberg's spider for $U_q(\mathfrak{sl}_3)$

Kuperberg's work on $U_q(\mathfrak{sl}_3)$ is stated in the language of spiders. These are simply (strict) pivotal categories, with an alternative set of operations emphasised; instead of composition and tensor product, 'stitch' and 'join'. Further, in a spider the isomorphisms such as $\mathrm{Hom}(a, b \otimes c) \cong \mathrm{Hom}(a \otimes c^*, b)$ are replaced with identifications.In this and the following sections, I'll somewhat freely mix vocabularies.

In his work, the only edges that appear are edges labelled by 1, and there are no tags. As previously pointed out, the only trivalent vertices in my construction at $n = 3$ involve three edges labelled by 1 (here I'm not counting the flow vertices; remember they're secretly defined in terms of the original vertices and tags). This means that in any diagram, edges labelled by 2 only appear either at the boundary, connected to a tag, or in the middle of an internal edge, with tags on either sides. Since tags cancel without signs in the $n = 3$ theory, we can ignore all the internal tags.

There's thus an easy equivalence between Kuperberg's $n = 3$ pivotal category (defined implicitly by his spider) and mine. (In fact, this equivalence holds before or after quotienting out by the appropriate relations.) In one direction, to his category, we send $(1, \pm) \to (1, \pm)$ and $(2, \pm) \to (1, \mp)$ at the level of objects, and 'chop off' any external 2-edges, and their associated tag, at the level of morphisms.[1] The functor the other direction is just the 'inclusion'. Of the two composition functors, the one on his category is actually the identity; the one on mine is

[1]This picturesque description needs a patch for the identity 2-edge; there we create a pair of tags first, so we have something to chop off at either end.

naturally isomorphic to the identity, via the tensor natural transformation defined by

6.3 Kim's proposed spider for $U_q(\mathfrak{sl}_4)$

In the final chapter of his Ph.D. thesis, Kim [21] conjectured relations for the $U_q(\mathfrak{sl}_4)$ spider. These relations agree exactly with mine at $n = 4$. He discovered his relations by calculating the dimensions of some small $U_q(\mathfrak{sl}_4)$ invariant spaces. As soon as it's possible to write down more diagrams with a given boundary than the dimension of the corresponding space in the representation theory, there must be linear combinations of these diagrams in the kernel. Consistency conditions coming from composing with fixed diagrams which reduce the size of the boundary enabled him to pin down all the coefficients, although, as with my work, he's unable to show that he's found generators for the kernel.

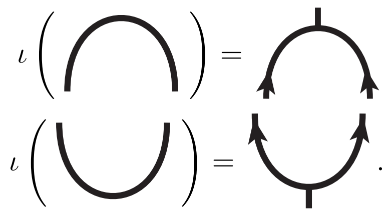
6.4 Tags and orientations

In this section I'll describe an equivalence between my categories and the categories previously described for $n = 2$, in §6.1 and for $n = 4$, in §6.3. The description will also encompass the equivalence described for $n = 3$ in §6.2; I'm doing these separately because further complications arise when n is even.

The usual spider for $U_q(\mathfrak{sl}_2)$, the Temperley-Lieb category, has unoriented edges. Similarly, Kim's proposed spider for $U_q(\mathfrak{sl}_4)$ does not specify orientations on the 'thick' edges, that is, those edges labelled by 2. Moreover, as in Kuperberg's $U_q(\mathfrak{sl}_3)$ spider, only a subset of the edge labels I use appear; he only has edges labelled 1 and 2.

Nevertheless, those spiders are equivalent to the spiders described here. First, the issue of the edge label 3 being disallowed is treated exactly as above in §6.2; we notice that at $n = 4$, there are no vertices involving edges labelled 3, so we can remove any internal 3-edges by cancelling tags, and, at the cost of keeping track of a natural isomorphism, remove external 3-edges too. There's a second issue, however, caused by the unoriented edges. The category equivalences we define will have to add and remove orientation data, and tags.

Thus we define two functors, ι, which decorates spider diagrams with 'complete orientation data', and π, which forgets orientations and tags (entirely for $n = 2$, and only on the 2 edges for $n = 4$). The forgetful functor also 'chops off' any 3 edges (in the $n = 4$ case), just like the functor in §6.2. The decorating functor ι simply fixes an up-to-isotopy representative of a diagram, and orients each unoriented edge up the page, placing tags at critical points of unoriented edges as follows:



It's well defined because opposite tags cancel. The composition $\pi \circ \iota$ is clearly the identity. The other composition $\iota \circ \pi$ isn't quite, but is natural isomorphic to the identity, via the tag maps.

6.5 Murakami, Ohtsuki and Yamada's trivalent graph invariant

In [27], Murakami, Ohtsuki and Yamada (MOY, hereafter) define an invariant of closed knotted trivalent graphs, which includes as a special case the HOMFLYPT polynomial.[2] Their graphs carry oriented edge labels, and the trivalent vertices are exactly as my 'flow vertices'. (They don't have 'tags'.) They don't make any explicit connection with $U_q(\mathfrak{sl}_n)$ representation theory, although this is certainly their motivation. In fact, they say:

> We also note that our graph invariant may be obtained (not checked yet) by direct computations of the universal R-matrix. But the advantage of our definition is that it does not require any knowledge of quantum groups nor representation theory.

One of their closed graphs can be interpreted in my diagrammatic category $\mathcal{S}ym_n$; pushing it over into the representation theory $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$, it must then evaluate to a number. Presumably, this number must be a multiple of their evaluation (modulo replacing their q with my q^2), with the coefficient depending on the vertices appearing in the diagram, but not their connectivity. Thus given a suitable closed trivalent graph D, the two evaluations would be related via

$$\langle D \rangle_{\mathrm{MOY}} = \lambda(D)\, \mathbf{Rep}_n\,(D)\,. \tag{6.5.1}$$

Knowing this evaluation coefficient $\lambda(D)$ explicitly would be nice. You might approach it by either 'localising' the MOY formulation[3], or deriving a recursive version of the MOY evaluation function, writing the evaluation of a graph for n in terms of the evaluation of slightly modified graphs for $n-1$. In particular, there's an evaluation function in my category, obtained by branching all the way down to $n = 0$, which is of almost the same form as their evaluation function. It's a sum over multiple reduction paths, such that each edge is traversed by as many reductions paths as its label. Moreover, each reduction path comes with an index, indicating which step of the branching process it is 'applied' to the diagram at. This index corresponds exactly to the labels in $\mathcal{N}$ in MOY, after a linear change of variable. Writing down the details of this should produce a formula for $\lambda(D)$.

Modulo the translation described in the previous paragraph, their (unproved) Proposition A.10 is presumably equivalent to the $n + \Sigma a - \Sigma b \geq 0$ case of Theorem 5.2.1.

[2] There's also a 'cheat sheet', containing a terse summary of their construction, at `http://katlas.math.toronto.edu/drorbn/index.php?title=Image:The_MOY_Invariant_-_Feb_2006.jpg`.

[3] If you're interested in trying, perhaps ask Dror Bar-Natan or myself, although we don't have that much to say; the localising step is easy enough.

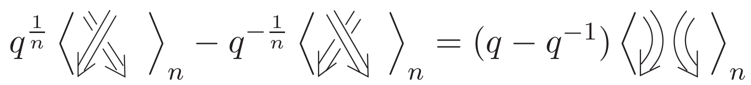
6.6 Jeong and Kim on $U_q(\mathfrak{sl}_n)$

In [11], Jeong and Kim independently discovered a result analogous to both cases of Theorem 5.2.1, using a dimension counting argument to show that there must be such relations, and finding coefficients by gluing on other small webs. (Of course, they published first, and have priority on that theorem.) They never describe an explicit map from trivalent webs to the representation theory of $U_q(\mathfrak{sl}_n)$, however; they posit relations for loops and bigons, and $I = H$ relations, which differ from ours up to signs, and use these to show that the relations of Theorem 5.2.1 must hold in order to get the dimensions of spaces of diagrams right.

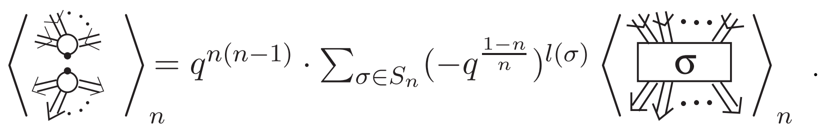
They later make a conjecture which says (in my language) that even for $2k$-gons with $k \geq 3$, each $\mathcal{P}$-$2k$-gon can be written in terms of $\mathcal{Q}$-$2k$-gons and smaller polygons. My results show this conjecture is false.

6.7 Other work

Sikora, in [34], defines an invariant of oriented n-valent braided ribbon graphs. His graphs do not have labels on the edges, and allow braidings of edges. The invariant is defined by some local relations, including some normalisations, a 'traditional skein relation' for the braiding,



and a relation expressing a pair of n-valent vertices as a linear combination of braids



Easily, every closed graph evaluates to a number: the second relation above lets you remove all vertices, resulting in a linear combination of links, which can be evaluated via the first relation. He further explains the connection with Murakami, Ohtsuki and Yamada's work, giving a formula for their evaluation in terms of his invariant.

Although his work does not expressly use the language of a category of diagrams, it's straightforward to make the translation. He implicitly defines a braided tensor category, with objects generated by a single object, the unlabeled strand, and morphisms generated by caps, cups, crossings, and the n-valent vertices.

This category, with no more relations than he explicitly gives, ought to be equivalent to the full subcategory of $\mathbf{Rep}\, U_q(\mathfrak{sl}_n)$ generated by tensor powers of the standard representation. Note that this isn't the same as the category $\mathbf{FundRep} U_q(\mathfrak{sl}_n)$ used here; it has even fewer objects, although again its Karoubi envelope is the entire representation category. However, I don't think that this equivalence is obvious, at least with the currently available results. Certainly he proves that there is a functor to this representation category (by explicitly construct $U_q(\mathfrak{sl}_n)$ equivariant tensors for the braiding, coming from the R-matrix, and for the n-valent vertices). Even though he additionally proves that there are no nontrivial quotients of his invariant, this does not prove that the functor to the representation theory is faithful. Essentially, there's no reason why 'open' diagrams, such as appear in the Hom spaces of the category, shouldn't have further relations amongst them. Any way of closing up such a relation would have to result in a linear combination of closed diagrams which evaluated to zero simply using the initially specified relations. Alternatively, we could think of this as a question about'nondegeneracy': there's a pairing on diagrams, giving by gluing together pairs of diagrams with the same, but oppositely oriented, boundaries. It's $\mathbb{C}(q)$-valued, since every closed diagram can be evaluated, but it may be degenerate. That is, there might be elements of the kernel of this pairing which do not follow from any of the local relations. See [25, §3.3] for a discussion of a similar issue in $\mathfrak{sl}_3$ Khovanov homology.

Perhaps modulo some normalisation issues, one can write down functors between his category and mine, at least before imposing relations. In one direction, send an edge labeled k in my category to k parallel ribbons in Sikora's category, and send each vertex, with edges labeled a, b and c to either the incoming or outgoing n-valent vertex in Sikora's category. In the other direction, send a ribbon to an edge labeled 1, an n-valent vertex to a tree of trivalent vertices, with n leaves labeled 1 (up to a sign it doesn't matter, by the $I = H$ relations, which tree we use), and a crossing to the appropriate linear combination of the two irreducible diagrams with boundary $((1,+),(1,+),(1,-),(1,-))$.

This suggests an obvious question: are the elements of the kernel of the representation functor I've described generated by Sikora's relations? They need not be, if the diagrammatic pairing on Sikora's category is degenerate.

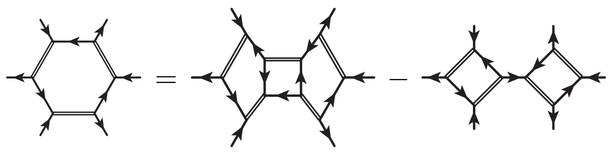
Yokota's work in [40] is along similar lines as Sikora's, but directed towards giving an explicit construction of quantum $SU(n)$ invariants of 3-manifolds.

Chapter 7

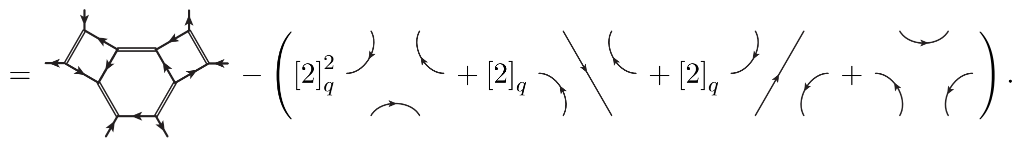
Future directions

Sad though it is to say, I've barely scratched the surface here. There are a number of obvious future directions.

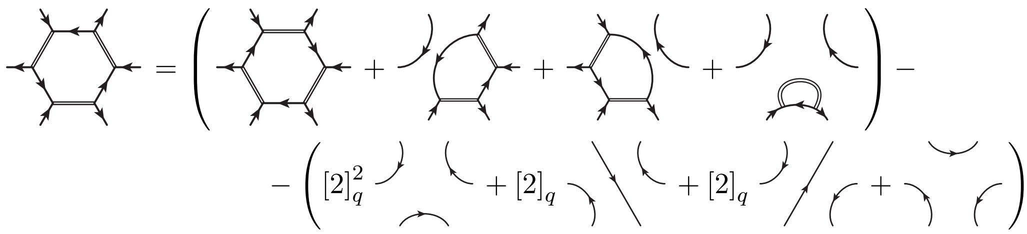
- Prove Conjecture 5.5; that the whole kernel of **Rep** is generated by the relations I've described via tensor product and composition.
- Try to find a 'confluent' set of relations, according to the definition of [35]. There's no obvious basis of diagrams modulo the relations I've given, for $n \geq 4$; this is essentially just saying that the relations I've presented are not confluent.
- Find an evaluation algorithm; that is, describe how to use the relations given here to evaluate any closed web. The authors of [11] claim to do so, using only the $I = H$ and 'square-switch' relations, although I have to admit not being able to follow their proof.
- Complete the discussion from §6.5, explicitly giving the relationship between the evaluation of a closed diagram in my category, and its evaluation in the MOY theory.
- In fact, the first Kekulé relation, appearing in Equation (5.3.1) at $n = 4$, follows from the 'square switch' and 'bigon' relations, as follows. First, applying Equation (5.6.6) across the middle of a hexagon, we see



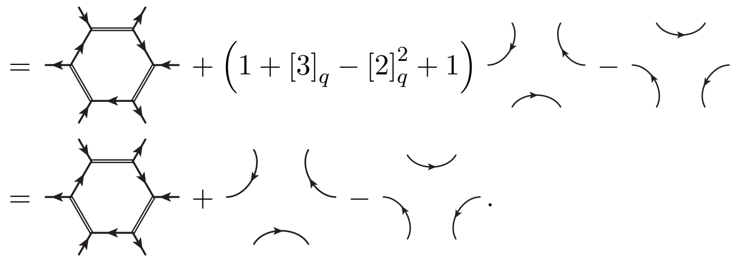
and then, using the $I = H$ relation to switch two of the edges in the first term, and Equation (5.6.5) to resolve each of the squares in the second term,



Next, we resolve the remaining squares using Equation (5.6.6), obtaining



and then resolve the newly created squares using Equation (5.6.5) again, and resolve the newly created bigons using Equation (5.6.2)



Is there any evidence that this continues to happen? Perhaps all the Kekulé relations follow in a similar manner from 'square-switch' relations?

- Try to do the same thing for the other simple Lie algebras. Each Lie algebra has a set of fundamental representations, corresponding to the nodes of its Dynkin diagram. You should first look for intertwiners amongst these fundamental representations, until you're sure (probably by a variation of the Schur-Weyl duality proof given here) that they generate the entire representation theory. After that, describe some relations, and perhaps prove that you have all of them. I have a quite useful computer program for discovering relations, which is general enough to attempt this problem for any simple Lie algebra. On the other hand, the methods of proof used here, based on the multiplicity free branching rule for $\mathfrak{sl}_n$, will need considerable modification. For the notion of Gel'fand-Tsetlin basis in the orthogonal and symplectic groups, see [36].

- Describe the representation theory of the restricted quantum group at a root unity in terms of diagrams; in particular find a diagrammatic expression for the idempotent in $\otimes_a V_{a<n}^{m_a}$ which picks out the irrep of high weight $\sum_a m_a \lambda_a$, for each high weight. Kim solved this problem for $U_q(\mathfrak{sl}_3)$ in his Ph.D. thesis [22]. It would be nice to start simply by describing the 'lowest root of unity' quotients, in which only the fundamental representations survive. After this, one might learn how to use these categories of trivalent graphs to give a discussion, parallel to that in [3], of the modular tensor categories appearing at roots of unity.

- Categorify everything in sight! Bar-Natan's work [1] on local Khovanov homology provides a categorification of the $U_q(\mathfrak{sl}_2)$ theory. Khovanov's work [18] on a foam model for $\mathfrak{sl}_3$ link homology categorifies the $U_q(\mathfrak{sl}_3)$ theory, although it's only made explicitly local in later work by Ari Nieh and myself [25]. Finding an alternative to the matrix factorisation method [19, 20] of categorifying the $U_q(\mathfrak{sl}_n)$ knot invariants, based explicitly on a categorification of the $U_q(\mathfrak{sl}_n)$ spiders, is a very tempting, and perhaps achievable, goal!

If you have answers to any of these problems, I'd love to hear about them; if you have partial answers, or even just enthusiasm for the problems, I'd love to work on them with you!

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Appendix A

Appendices

A.1 Boring q-binomial identities

In this section we'll prove some q-binomial identities needed in various proofs in the body of the thesis. The identities are sufficiently complicated that I've been unable to find combinatorial arguments for them. Instead, we'll make use of the Zeilberger algorithm, described in [31]. In particular, we'll use the `Mathematica'` implementation in [30, 33], and the implementation of the q-analogue of the Zeilberger algorithm described in [29, 32].

If you're unhappy with the prospect of these identities being proved with computer help, you should read Donald Knuth's foreword in [31]

> Science is what we understand well enough to explain to a computer. Art is everything else we do. During the past several years an important part of mathematics has been transformed from an Art to a Science: No longer do we need to get a brilliant insight in order to evaluate sums of binomial coefficients, and many similar formulas that arise frequently in practice; we can now follow a mechanical procedure and discover the answers quite systematically.

For each use made of the the q-Zeilberger algorithm, I've included a `Mathematica` notebook showing the calculation. These are available in the `/code/` subdirectory, after you've downloaded the source of this thesis from the arXiv.

Unfortunately, the conventions I've used for q-binomials don't quite agree with those used in the implementation of the q-Zeilberger algorithm. They are related via

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2}^{\mathrm{qZeil}},$$

although in my uses of their algorithms, I've replaced q^2 with q throughout; this apparently produces cleaner looking results.

Lemma A.1.1. *For $\Sigma b \leq j \leq \Sigma a + n - 1$ and $-\overline{\sum} b \leq j^* \leq -\underline{\sum} a$*

$$\sum_{k=\max(-\overline{\sum} b, \Sigma b + j^* - j)}^{\min(-\underline{\sum} a + 1, j^* - j + n + \Sigma a)} (-1)^{j+k} \begin{bmatrix} j + k - \max b \\ j - \Sigma b \end{bmatrix}_q \times \\ \times \begin{bmatrix} \min a + n - j - k \\ \Sigma a + n - 1 - j \end{bmatrix}_q \begin{bmatrix} n + \Sigma a - \Sigma b \\ j - j^* + k - \Sigma b \end{bmatrix}_q = 0. \quad \text{(A.1.1)}$$

Proof. Writing Σ_n for the sum above (suppressing the dependence on $\Sigma a, \Sigma b, \min a, \max b, j$ and j^*), the q-Zeilberger algorithm reports (see `/code/triple-identity.nb/`) that

$$\Sigma_n = \frac{q^{(\underline{\sum} a + \Sigma b - j + j^* - 1)}(1 - q^{(2+2n+2\min a - 2\max b)})}{1 - q^{(2\Sigma a + 2\overline{\sum} b - 2j + 2j^* + 2n)}} \Sigma_{n-1}$$

as long as

$$n + \Sigma a - \Sigma b \neq 0 \qquad \text{and} \qquad n + \Sigma a + \overline{\sum} b - j + j^* \neq 0. \tag{A.1.2}$$

In the case that $n + \Sigma a - \Sigma b = 0$, the top of the third q-binomial in Equation (A.1.1) is zero, so the sum is automatically zero unless $k = \Sigma b + j^* - j$, in which case the sum collapses to

$$(-1)^{\Sigma b + j^*} \begin{bmatrix} \Sigma b + j^* \\ j - \Sigma b \end{bmatrix}_q \begin{bmatrix} -\underline{\sum} a - j^* \\ \Sigma b - 1 - j \end{bmatrix}_q.$$

However, since $j \geq \Sigma b$, the bottom of the second q-binomial here is less than zero, so the whole expression vanishes.

In the case that $n + \Sigma a + \overline{\sum} b - j + j^* = 0$, we can replace n everywhere in Equation (A.1.1), obtaining

$$\sum_k (-1)^{j+k} \begin{bmatrix} j + k - \max b \\ j - \Sigma b \end{bmatrix}_q \begin{bmatrix} -\underline{\sum} a - \overline{\sum} b - j^* - k \\ -\overline{\sum} b - j^* - 1 \end{bmatrix}_q \begin{bmatrix} -\Sigma b - \overline{\sum} b + j - j^* \\ j - j^* - k - \Sigma b \end{bmatrix}_q.$$

Again, the second q-binomial vanishes, since $j^* \geq -\overline{\sum} b$.

Now that we're sure the identity holds when either of the inequalities of Equation (A.1.2) are broken, we can easily establish the identity when they're not; if it holds for some value of n, it must hold for $n + 1$ (notice that the restriction on j in the hypothesis of the lemma doesn't depend on n). Amusingly, the easiest starting case for the induction is not $n = 0$, but n sufficiently large and negative, even though this makes no sense in the original representation theoretic context! In particular, at $n \leq -\Sigma a + \Sigma b - 1$, the third binomial in Equation (A.1.1) is always zero, so the sum vanishes. □

For the next lemma, we'll need q-Pochhammer symbols,

$$(a; q)_k = \begin{cases} \prod_{j=0}^{k-1}(1 - aq^j) & \text{if } k > 0, \\ 1 & \text{if } k = 0, \\ \prod_{j=1}^{|k|}(1 - aq^{-j})^{-1} & \text{if } k < 0. \end{cases}$$

In particular, note that $(q;q)_{k<0}^{-1} = 0$, and $(q;q)_{k>0} \neq 0$.

Lemma A.1.2. *For any $n, m, m' \in \mathbb{Z}$, and $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{Z}^2$, with $n + \Sigma a - \Sigma b > 0$*

$$\delta_{mm'} - (-1)^{m+n+\Sigma a} \sum_{l=n+\Sigma a - \min b}^{n + \min a} (-1)^l \begin{bmatrix} m + l - 1 - \Sigma b \\ m + l - n - \Sigma a \end{bmatrix}_q \begin{bmatrix} n + \Sigma a - \Sigma b \\ m' + l - \Sigma b \end{bmatrix}_q = 0.$$

Proof. First, notice that the expression is invariant under the transformation adding some integer to each of $m, m', l, a_1, a_2, b_1, b_2$. We'll take advantage of this to assume $\Sigma b + 2m - 1$, $\Sigma b + m + m' - 1$, $2\Sigma b + 2m - 1 - n - \Sigma a$, $\min a + m + n$, $\min a + m' + n$, and $\Sigma b - \max a + m$ are all positive, and $-2\Sigma b - m - m' + n - \Sigma a$ and $\max a - \Sigma b - m' - 1$ are both negative.

Again, writing Σ_n for the left hand side of the above expression, the q-Zeilberger algorithm reports (see `/code/SS-identity.nb/`) that as long as $m \neq m'$,

$$\begin{aligned}\Sigma_n = (-1)^\bullet q^\bullet &\frac{(q;q)_{\Sigma b+2m-1}}{\left(q^m - q^{m'}\right)(q;q)_{\Sigma b+m+m'-1}\,(q;q)_{2\Sigma b+2m-1-n-\Sigma a}\,(q;q)_{-2\Sigma b-m-m'+n-\Sigma a}} + \\ + (-1)^\bullet q^\bullet &\frac{(q;q)_{\min a+m+n}}{\left(q^m - q^{m'}\right)(q;q)_{\min a+m'+n}\,(q;q)_{\Sigma b-\max a+m}\,(q;q)_{\max a-\Sigma b-m'-1}}.\end{aligned}$$

This is exactly zero, using the inequalities described in the previous paragraph, and the definition of the q-Pochhammer symbol.

When $m = m'$, the q-Zeilberger algorithm reports that

$$\Sigma_n = (-1)^\bullet q^\bullet \frac{\left(q^{n+\Sigma a-\Sigma b} - 1\right)}{\left(q^{\min a+m+n} - 1\right)(q;q)_{\Sigma b-\max a+m}\,(q;q)_{-\Sigma b+\max a-m}} + \Sigma_{n-1}. \tag{A.1.3}$$

The first term is zero, since $-\Sigma b + \max a - m < 0$, so $(q;q)^{-1}_{-\Sigma b+\max a-m} = 0$.

Thus we just need to finish off the case $m = m'$, $n + \Sigma a - \Sigma b = 1$, where the left hand side of Equation (A.1.2) reduces to

$$\delta_{mm'} - (-1)^{m+\Sigma b+1} \sum_{l=\underline{\sum} b+1}^{\Sigma b-\underline{\sum} a+1} (-1)^l \begin{bmatrix} m+l-1-\Sigma b \\ m+l-1-\Sigma b \end{bmatrix}_q \begin{bmatrix} 1 \\ m'+l-\Sigma b \end{bmatrix}_q.$$

There are then two terms in the summation, $l = \Sigma b - m'$ and $l = \Sigma b - m' + 1$, so we obtain

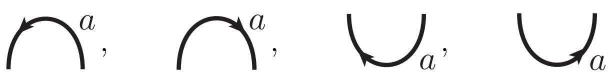
$$\delta_{mm'} - (-1)^{m-m'+1} \begin{bmatrix} m-m'-1 \\ m-m'-1 \end{bmatrix}_q - (-1)^{m-m'} \begin{bmatrix} m-m' \\ m-m' \end{bmatrix}_q.$$

If $m = m'$, the first q-binomial vanishes, but the second is 1, while if $m > m'$, both q-binomials are equal to 1, and cancel, and if $m < m'$, both q-binomials vanish. □

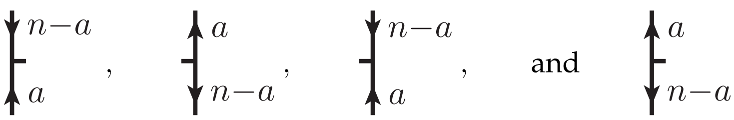
A.2 The $U_q(\mathfrak{sl}_n)$ spider cheat sheet.

A.2.1 The diagrammatic category, §2.2.

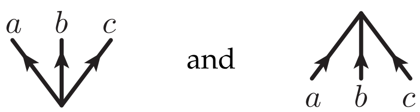
Generators.



for each $a = 1, \ldots, n-1$,

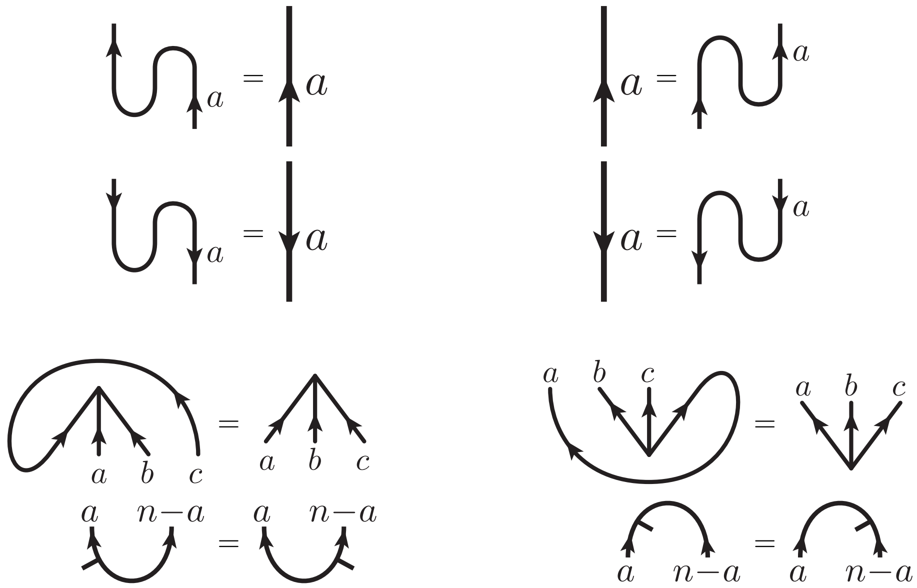


for each $a = 1, \ldots, n-1$, and



for $0 \leq a, b, c \leq n$, with $a + b + c = n$.

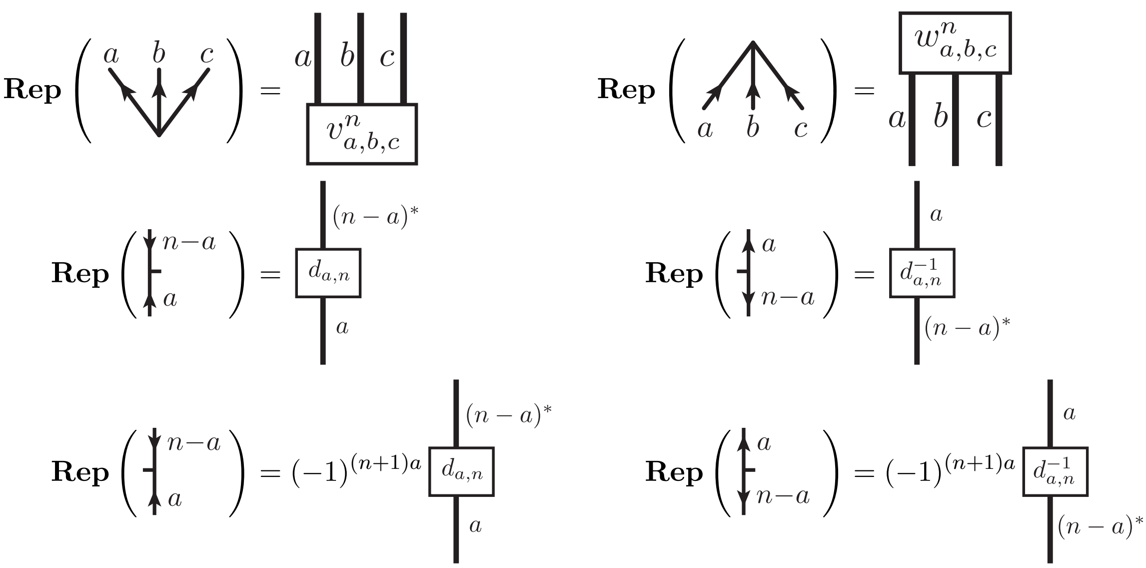
Relations: planar isotopy.



Relations: (anti-)symmetric duality.

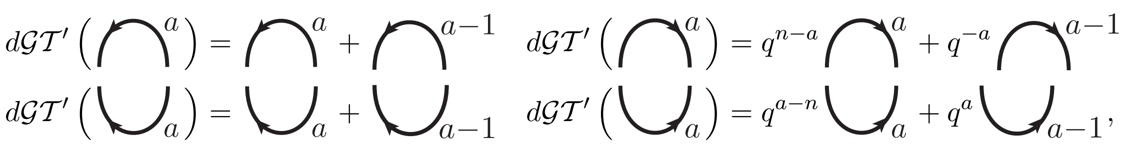
Relations: degeneration.

A.2.2 The map from diagrams to representation theory, §3.6.



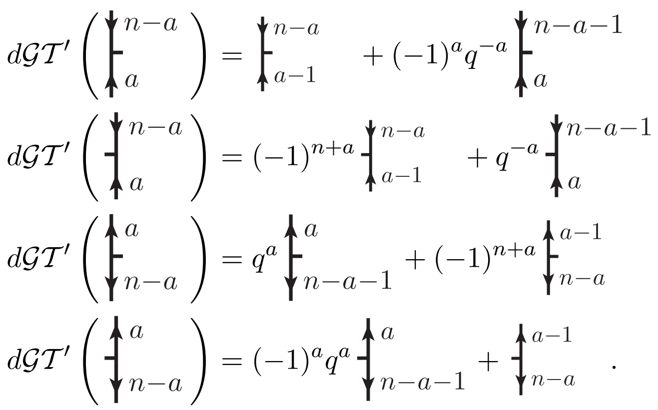
A.2.3 The diagrammatic Gel‘fand-Tsetlin functor, §4.

Pairings and copairings.



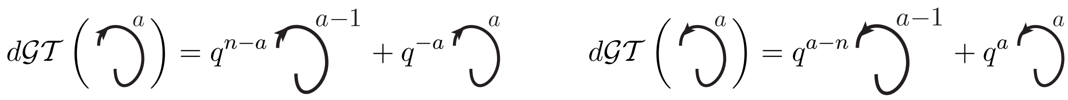
(4.1.2)

Identifications with duals.

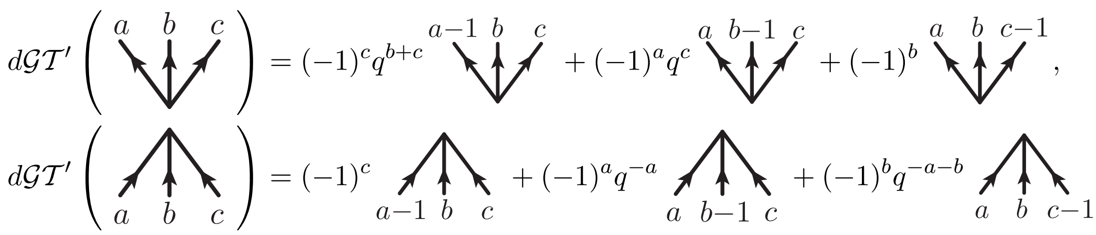


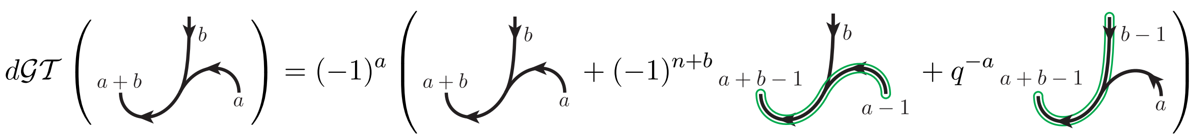
(4.1.3)

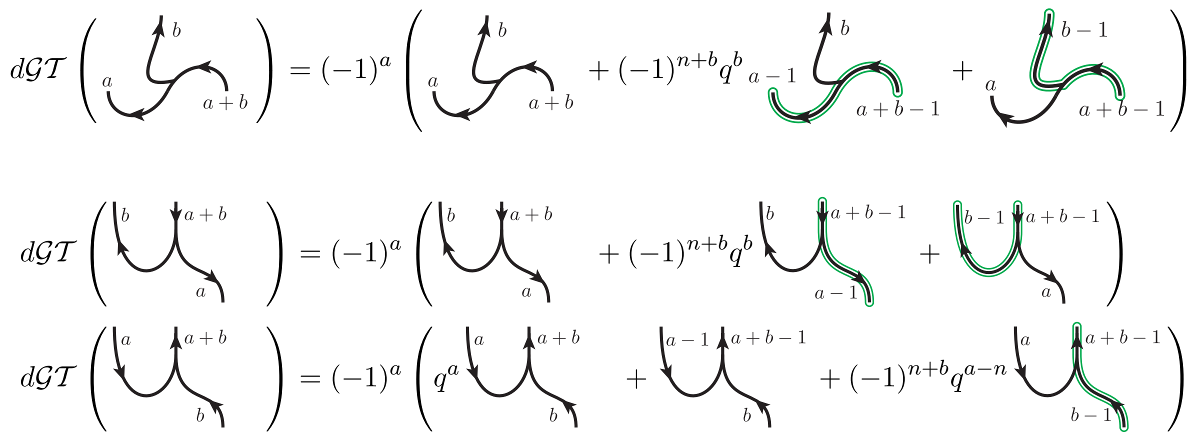
Curls.



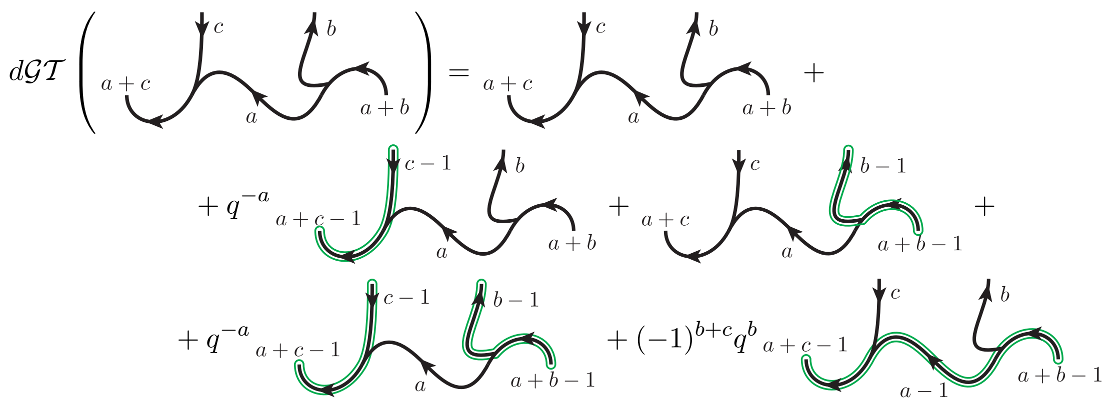
Trivalent vertices.



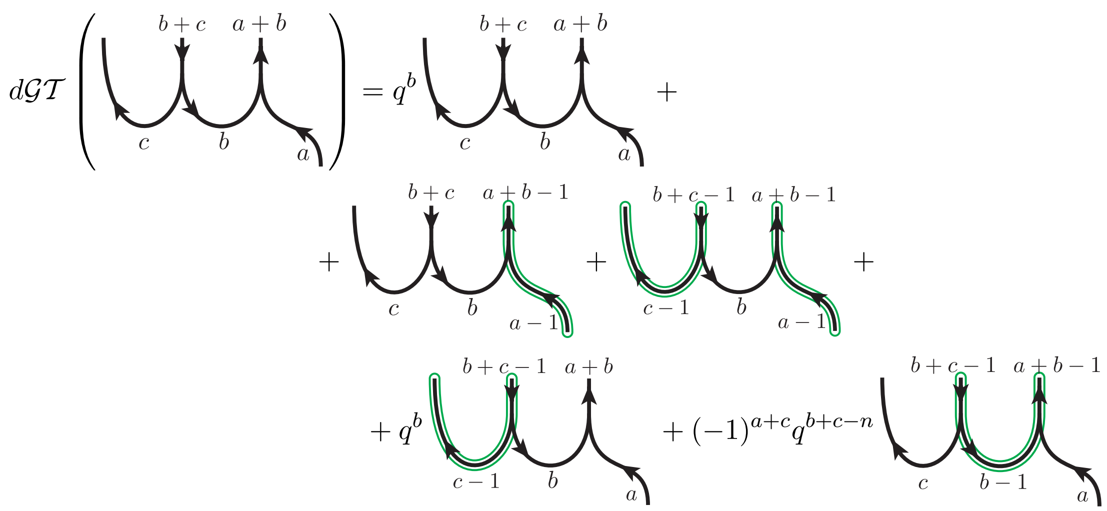




Larger webs.



(4.3.1)



(4.3.2)

$\mathcal{P}$'s and $\mathcal{Q}$'s, §2.4.

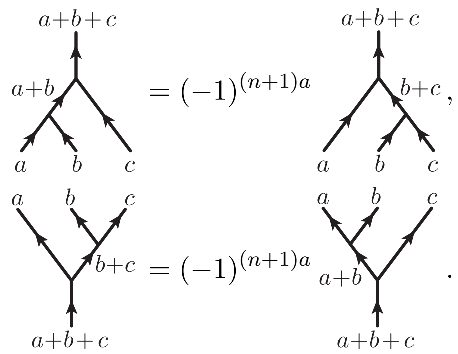
$$d\mathcal{GT}_{a',b'}(\mathcal{P}^n_{a,b;l}) = (-1)^{b'\cdot(a+\operatorname{rotl} a)} q^{l(\Sigma b'-\Sigma a'+1)} q^{\operatorname{rotl} a'\cdot b - b'\cdot a - a_1 - na'_1} \mathcal{P}^{n-1}_{a+a',b+b';l}$$
$$d\mathcal{GT}_{a',b'}(\mathcal{Q}^n_{a,b;l}) = (-1)^{b'\cdot(a+\operatorname{rotl} a)} q^{l(\Sigma a'-\Sigma b'-\frac{|\partial|}{2}+1)} q^{\Sigma b + n\Sigma b' + b'\cdot\operatorname{rotl} a - a'\cdot b - a_1 - na'_1} \mathcal{Q}^{n-1}_{a+a',b+b';l}$$

and

$$\begin{aligned}
d\mathcal{GT}_{\emptyset}(\mathcal{P}^n_{a,b;l}) &= d\mathcal{GT}_{\vec{0},\vec{0}}(\mathcal{P}^n_{a,b;l}) + d\mathcal{GT}_{\vec{1},\vec{1}}(\mathcal{P}^n_{a,b;l}) \\
&= q^{l-a_1}\left(\mathcal{P}^{n-1}_{a,b;l} + q^{-n-\Sigma a+\Sigma b}\mathcal{P}^{n-1}_{a,b;l-1}\right) \\
d\mathcal{GT}_{\emptyset}(\mathcal{Q}^n_{a,b;l}) &= d\mathcal{GT}_{\vec{0},\vec{0}}(\mathcal{Q}^n_{a,b;l}) \\
&= q^{l-a_1} q^{\Sigma b-\frac{l|\partial|}{2}}\mathcal{Q}^{n-1}_{a,b;l} \\
d\mathcal{GT}_{\partial}(\mathcal{P}^n_{a,b;l}) &= d\mathcal{GT}_{\vec{1},\vec{0}}(\mathcal{P}^n_{a,b;l}) \\
&= q^{l-a_1} q^{\Sigma b-\frac{l|\partial|}{2}-n}\mathcal{P}^{n-1}_{a+\vec{1},b;l} \\
d\mathcal{GT}_{\partial}(\mathcal{Q}^n_{a,b;l}) &= d\mathcal{GT}_{\vec{0},-\vec{1}}(\mathcal{Q}^n_{a,b;l}) + d\mathcal{GT}_{\vec{1},\vec{0}}(\mathcal{Q}^n_{a,b;l}) \\
&= q^{l-a_1}\left(q^{\Sigma b-\Sigma a-\frac{n|\partial|}{2}}\mathcal{Q}^{n-1}_{a+\vec{1},b;l+1} + q^{-n}\mathcal{Q}^{n-1}_{a+\vec{1},b;l}\right)
\end{aligned}$$

A.2.4 Conjectural complete set of relations, §5.

The $I = H$ relations



The 'square-switching' relations

$$\mathcal{SS}^n_{a,b} = \begin{cases} \operatorname{span}_{\mathcal{A}}\left\{\mathcal{P}^n_{a,b;l} - \sum_{m=\max a}^{\min b}\begin{bmatrix} n+\Sigma a-\Sigma b \\ m+l-\Sigma b\end{bmatrix}_q \mathcal{Q}^n_{a,b;m}\right\}_{l=\max b}^{\min a+n} & \text{if } n+\Sigma a-\Sigma b \geq 0 \\ \operatorname{span}_{\mathcal{A}}\left\{\mathcal{Q}^n_{a,b;l} - \sum_{m=\max b}^{n+\min a}\begin{bmatrix} \Sigma b-n-\Sigma a \\ m+l-\Sigma a-n\end{bmatrix}_q \mathcal{P}^n_{a,b;m}\right\}_{l=\max a}^{\min b} & \text{if } n+\Sigma a-\Sigma b \leq 0 \end{cases}$$

The Kekulé relations

$$\mathcal{APR}^n_{a,b} = \operatorname{span}_{\mathcal{A}}\left\{ d^{\mathcal{P},n}_{a,b;j} = \sum_{k=-\overline{\sum} b}^{-\underline{\sum} a+1}(-1)^{j+k}\begin{bmatrix} j+k-\max b \\ j-\Sigma b\end{bmatrix}_q \times \right.$$
$$\left. \times \begin{bmatrix} \min a+n-j-k \\ \Sigma a+n-1-j\end{bmatrix}_q \mathcal{P}^n_{a,b;j+k}\right\}_{j=\Sigma b}^{\Sigma a+n-1} \tag{A.2.1}$$

$$\mathcal{AQR}^n_{a,b} = \operatorname{span}_{\mathcal{A}} \left\{ d^{\mathcal{P},n}_{a,b;j} = \sum_{k=-\underline{\sum} a}^{-\overline{\sum} b + \frac{n|\partial|}{2}+1} (-1)^{j+k} \begin{bmatrix} j+k-\max a \\ j-\Sigma a \end{bmatrix}_q \times \right.$$

$$\left. \times \begin{bmatrix} \min b - j - k \\ \Sigma b - n(\frac{|\partial|}{2}-1) - 1 - j \end{bmatrix}_q \mathcal{Q}^n_{a,b;j+k} \right\}_{j=\Sigma a}^{\Sigma b - n(\frac{|\partial|}{2}-1)-1} \tag{A.2.2}$$

with orthogonal complements

$$\mathcal{APR}^{n\ \perp}_{a,b} = \operatorname{span}_{\mathcal{A}} \left\{ e^{\mathcal{P},n}_{a,b;j^*} = \sum_{k^*=\Sigma b}^{n+\Sigma a} \begin{bmatrix} n+\Sigma a - \Sigma b \\ k^* - \Sigma b \end{bmatrix}_q \mathcal{P}^{n\ \perp}_{a,b;j^*+k^*} \right\}_{j^*=-\overline{\sum} b}^{-\underline{\sum} a}$$

$$\mathcal{AQR}^{n\ \perp}_{a,b} = \operatorname{span}_{\mathcal{A}} \left\{ e^{\mathcal{Q},n}_{a,b;j^*} = \sum_{k^*=\Sigma a}^{\Sigma b - n\left(\frac{|\partial|}{2}-1\right)} \begin{bmatrix} \Sigma b - \Sigma a - n\left(\frac{|\partial|}{2}-1\right) \\ k^* - \Sigma a \end{bmatrix}_q \times \right.$$

$$\left. \times \mathcal{Q}^{n\ \perp}_{a,b;j^*+k^*} \right\}_{j^*=-\underline{\sum} a}^{-\overline{\sum} b + n\left(\frac{|\partial|}{2}-1\right)} \tag{A.2.3}$$