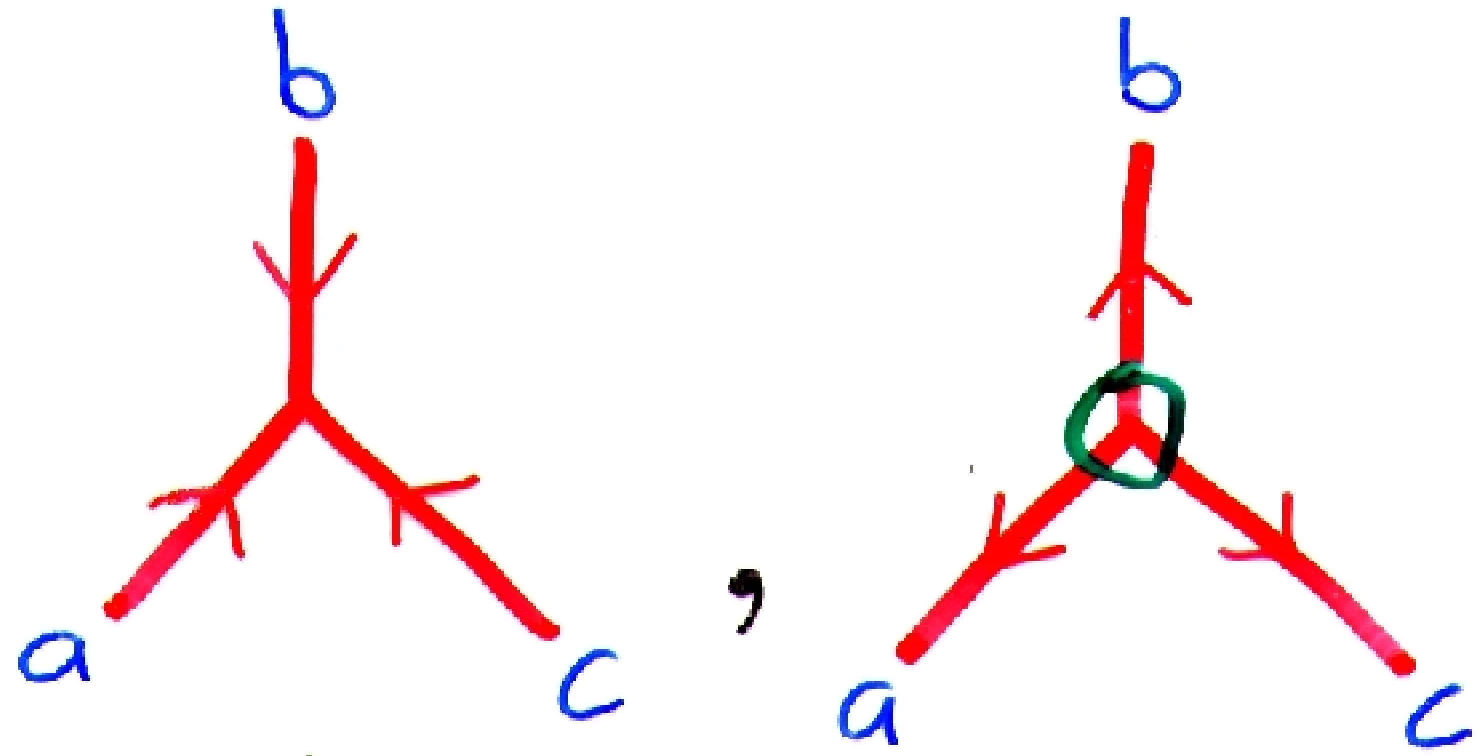
Generators and relations for $\mathrm{Rep}\, U_q \mathfrak{sl}_n$ as a pivotal category.

Scott Morrison, June 2005
UC Berkeley

objects: $1, 2, \dots, n-1$ $\quad k^* = n-k$

$$FS_n = \Big\langle \quad \cdots \quad \Big\rangle$$



$a+b+c=n$ $\qquad$ $a+b+c=n$

free pivotal category

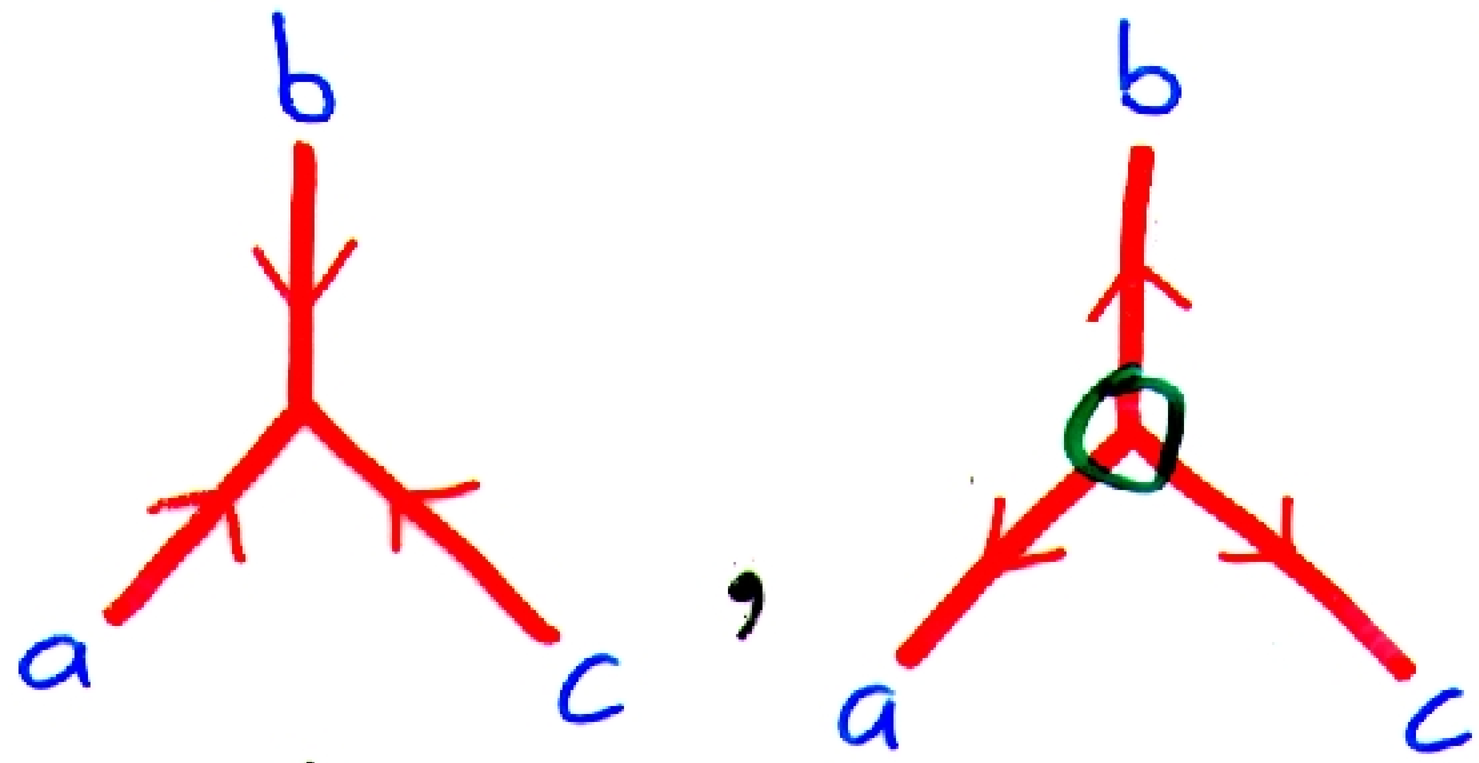
$$\Big\downarrow R$$

$\mathrm{FundRep}\,U_q\mathfrak{sl}_n = \Big\{$

objects —
tensor products of fundamental representations of $U_q\mathfrak{sl}_n$

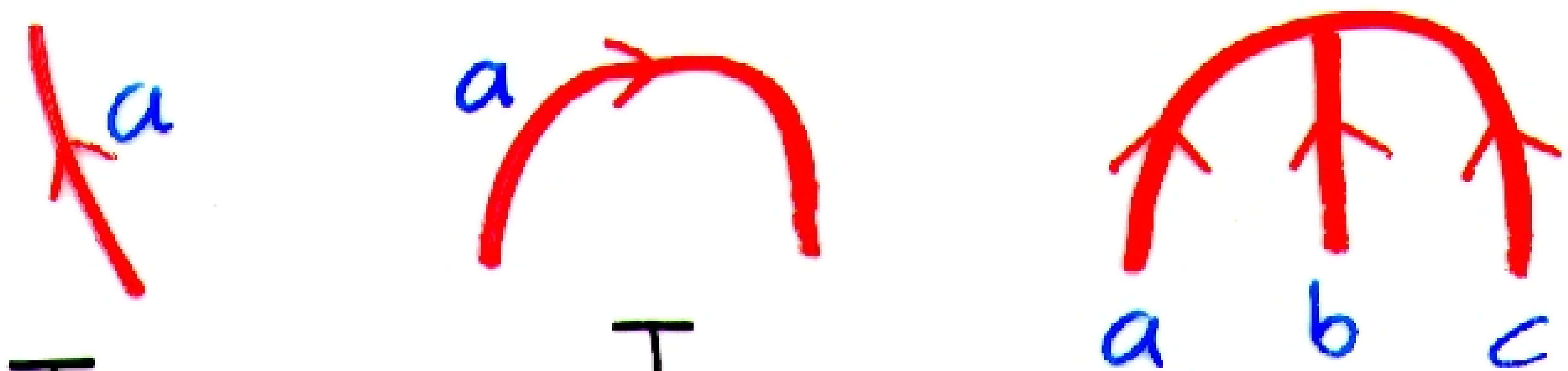
morphisms —
maps between representations $\Big\}$

$FS_n = \Big\langle$ objects: $1, 2, \ldots, n-1$ $k^* = n-k$



$a+b+c=n$, $a+b+c=n$ $\Big\rangle$ free pivotal category

$\Big\downarrow R$



$\mapsto$ id_{V_a}

$\mapsto$ the duality pairing $V_a \otimes V_{n-a} \to \mathbb{C}(q)$

$\mapsto$ the 'q-determinant' $V_a \otimes V_b \otimes V_c \to \Lambda^{a+b+c=n} V_1 \cong \mathbb{C}(q)$

$\mathrm{FundRep}\, U_q \underline{\underline{\mathfrak{sl}}}_n = \Big\{$ objects — tensor products of fundamental representations of $U_q \underline{\underline{\mathfrak{sl}}}_n$; morphisms — maps between representations $\Big\}$

$\underline{Q}$ What is the kernel of $\mathcal{R}$?

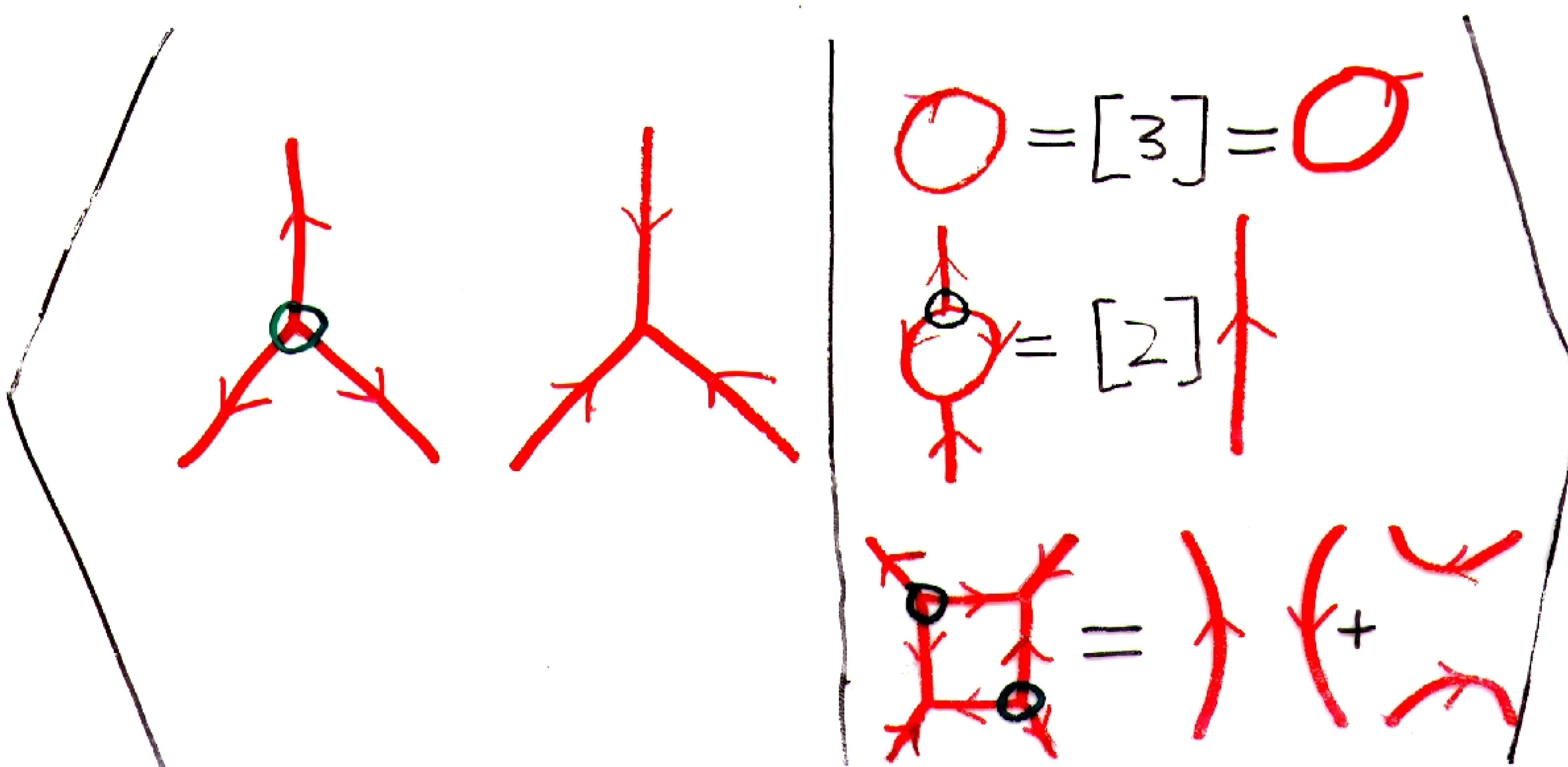
For $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$, the answer has been known for a while:

$\mathbf{FundRep}\, U_q \mathfrak{sl}_2 \cong$ the Temperley-Lieb category

(= the free pivotal category with one object, no extra generating morphisms, modulo $\bigcirc = [2]$)

and (Kuperberg, '96)

$\mathbf{FundRep}\, U_q \mathfrak{sl}_3 \cong$



$$\begin{tikzcd} FS_n \arrow[d, "\mathcal{R}"] \\ \mathrm{FundRep}\, U_q\mathfrak{sl}_n \end{tikzcd}$$

$\mathcal{R}$ is a full functor

① Kuperberg's argument for the $\mathfrak{sl}_3$ case still works.

(recognise the image of FS_n under $\mathcal{R}$ as a category of representations by a Tannaka-Krein theorem)

② There's also a more direct proof.

(using the fact that

$$\mathcal{B}_n \twoheadrightarrow \mathrm{End}(V_1^{\otimes n})$$

via R-matrices is surjective, and the centre of $SL(n)$.)

$$
\begin{tikzcd}
FS_n \arrow[d, two heads, "R"] & \\
\mathrm{FundRep}\,U_q\mathfrak{sl}_n \arrow[d] & \\
\mathrm{Rep}\,U_q\mathfrak{sl}_n \arrow[r, "GT", "\text{`forget from } U_q\mathfrak{sl}_n \text{ down to } U_q\mathfrak{sl}_{n-1}\text{'}"'] & \mathrm{Rep}\,U_q\mathfrak{sl}_{n-1}
\end{tikzcd}
$$

$$V_a \longmapsto V_{a-1} \oplus V_a$$

$$
\begin{tikzcd}
FS_n \arrow[d, two heads, "R"] & \\
\mathrm{FundRep}\,U_q\mathfrak{sl}_n \arrow[r, "GT"] \arrow[d] & \mathrm{Mat}(\mathrm{FundRep}\,U_q\mathfrak{sl}_{n-1}) \arrow[d] \\
\mathrm{Rep}\,U_q\mathfrak{sl}_n \arrow[r, "GT", "\text{`forget from } U_q\mathfrak{sl}_n \text{ down to } U_q\mathfrak{sl}_{n-1}\text{'}"'] & \mathrm{Rep}\,U_q\mathfrak{sl}_{n-1}
\end{tikzcd}
$$

$$V_a \longmapsto V_{a-1} \oplus V_a$$

Example

A map $\varphi : V_a \longrightarrow V_b \otimes V_c$ becomes a map

$$GT(\varphi) : V_{a-1} \oplus V_a \longrightarrow V_{b-1} \otimes V_{c-1} \oplus V_{b-1} \otimes V_c \oplus V_b \otimes V_{c-1} \oplus V_b \otimes V_c$$

which we can write as a matrix of maps between the direct summands

$$GT(\varphi) = \begin{array}{c} \\ V_{b-1}\otimes V_{c-1} \\ V_{b-1}\otimes V_c \\ V_b \otimes V_{c-1} \\ V_b \otimes V_c \end{array} \begin{array}{c} \begin{array}{cc} V_{a-1} & \quad V_a \end{array} \\ \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \\ \varphi_{31} & \varphi_{32} \\ \varphi_{41} & \varphi_{42} \end{pmatrix} \end{array}$$

$$\begin{tikzcd}
FS_n \arrow[d, two heads, "R_n"] & \\
\mathrm{FundRep}\,U_q\underline{\underline{\mathfrak{sl}}}_n \arrow[r, hook, "GT"] & \mathrm{Mat}(\mathrm{FundRep}\,U_q\underline{\underline{\mathfrak{sl}}}_n)
\end{tikzcd}$$

- Notice that GT is <u>faithful</u>.

$$\begin{tikzcd}
FS_n \arrow[d, "R_n", two heads] & & \mathrm{Mat}(FS_{n-1}) \arrow[d, "\mathrm{Mat}(R_{n-1})", two heads] \\
\mathrm{FundRep}\,U_q\underline{\underline{sl}}_n \arrow[rr, hook, "GT"] & & \mathrm{Mat}(\mathrm{FundRep}\,U_q\underline{\underline{sl}}_{n-1})
\end{tikzcd}$$

- Notice that GT is <u>faithful</u>.

- We can extend R_{n-1} to work with matrices of diagrams.

$$\begin{tikzcd}
FS_n \arrow[r, "?dGT"] \arrow[d, "R_n"] & Mat(FS_{n-1}) \arrow[d, "Mat(R_{n-1})"] \\
FundRepU_q\underline{\underline{sl}}_n \arrow[r, hook, "GT"] & Mat(FundRepU_q\underline{\underline{sl}}_{n-1})
\end{tikzcd}$$

- Notice that GT is <u>faithful</u>.

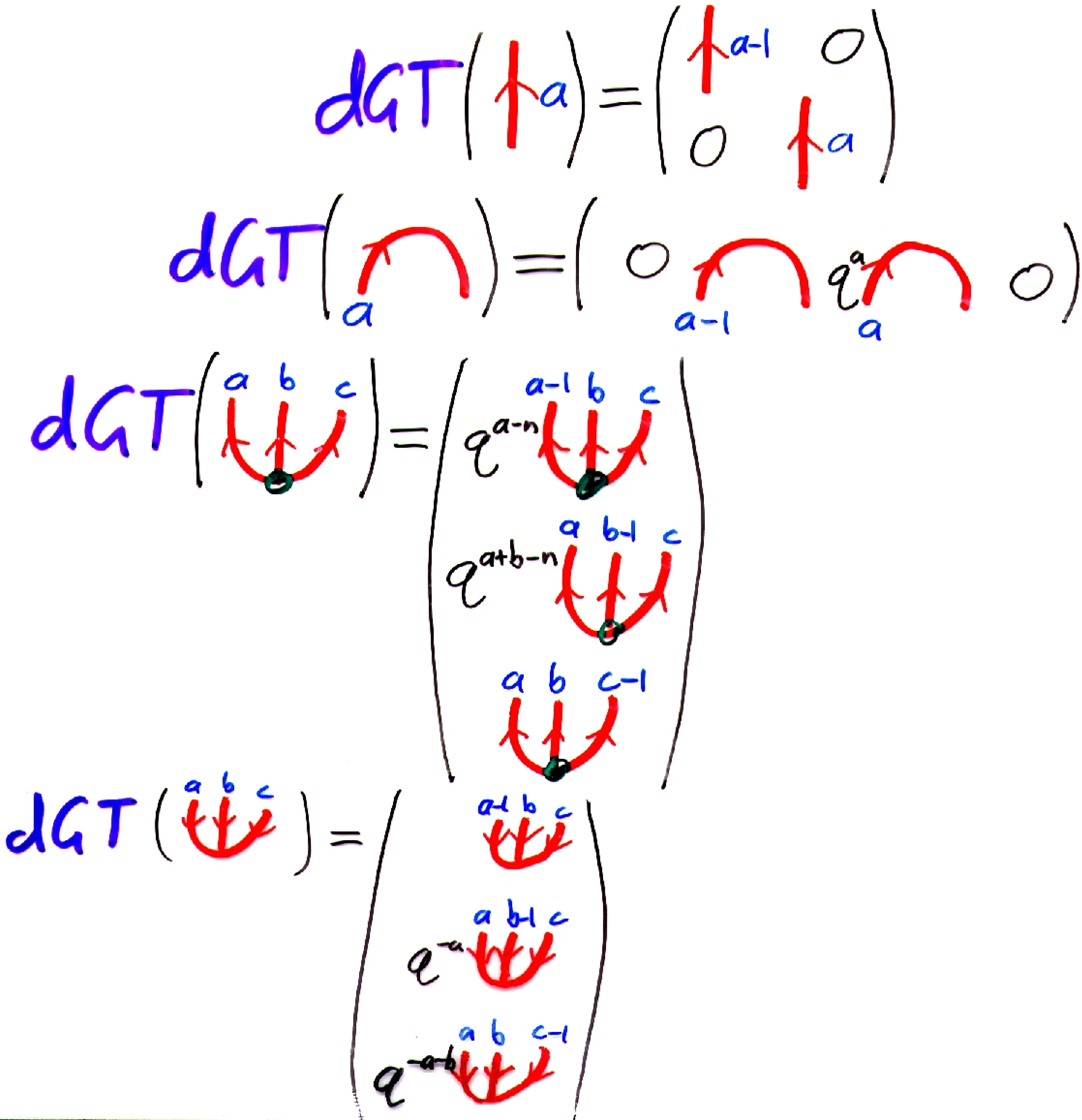
- We can extend R_{n-1} to work with matrices of diagrams.

<u>Question</u> Can we 'lift' the Gelfand-Tsetlin functor to a functor defined (combinatorially) on diagrams?

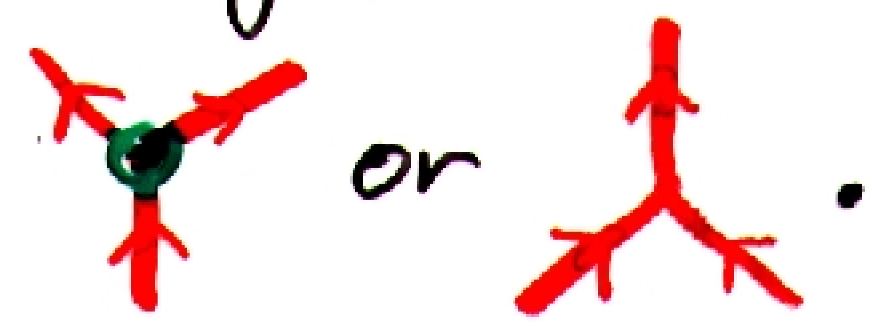
With such a lift, we can hope to recursively determine the relations:

$$\ker(R_n) = dGT^{-1}(\ker(Mat(R_{n-1})))$$

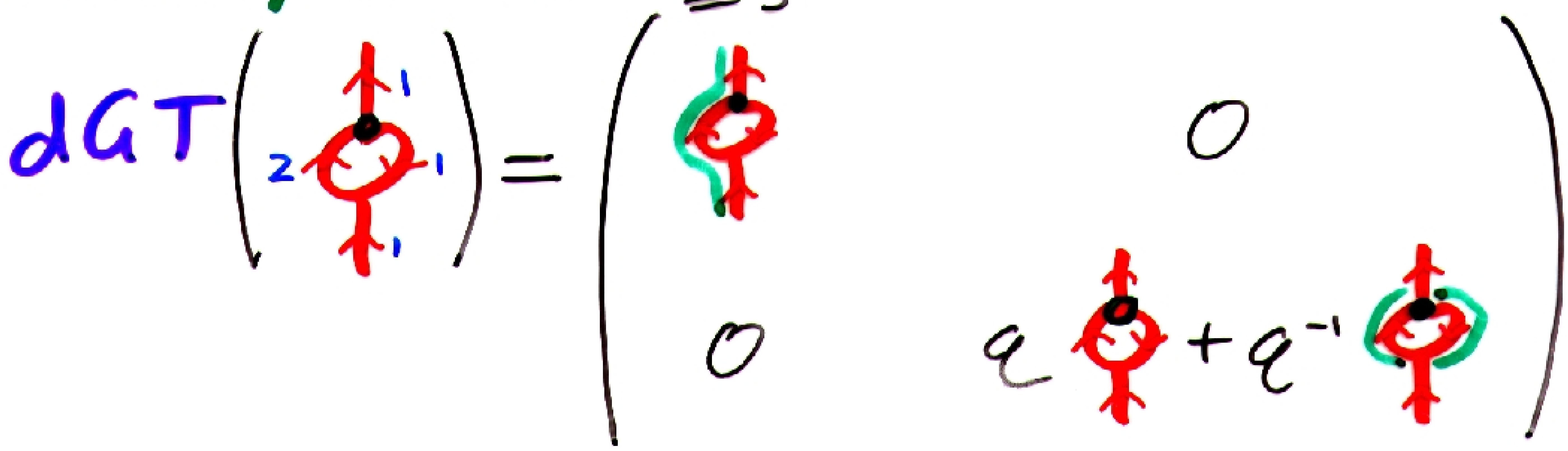
The 'diagrammatic Gelfand-Tsetlin' functor is determined by its values on generating morphisms

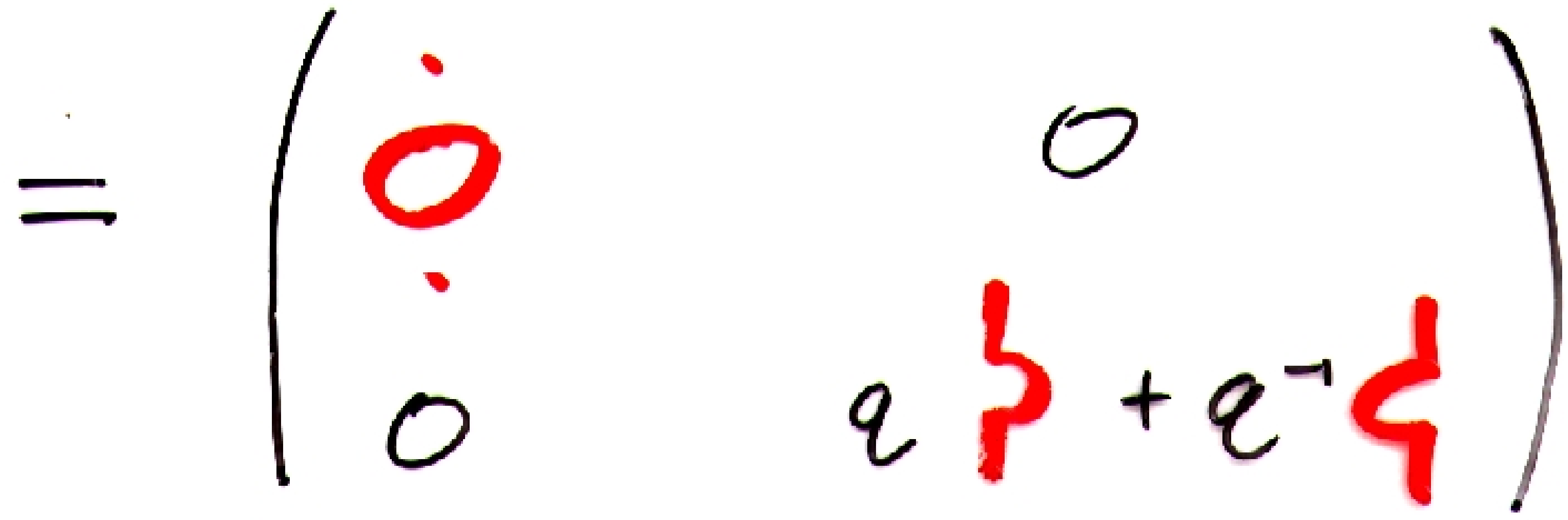


It's easy to calculate dGT.

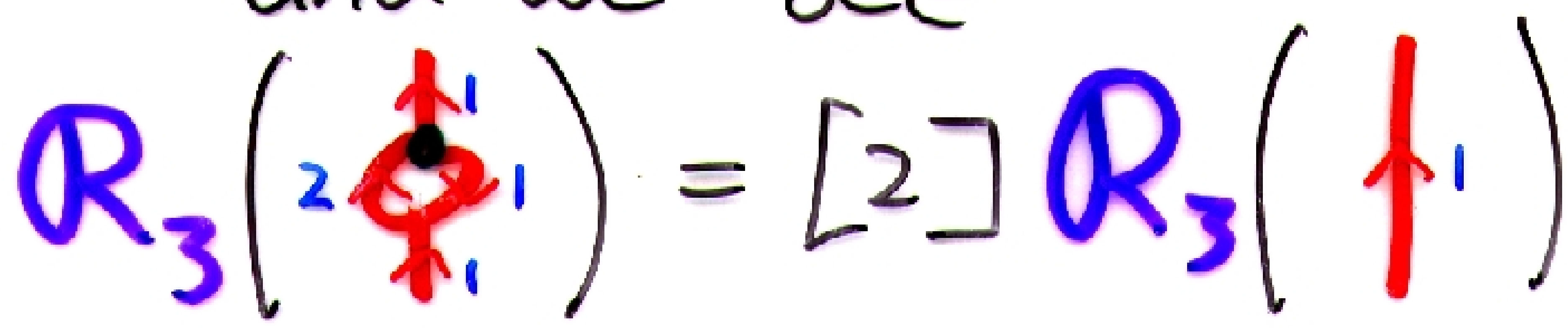
- Arrange the orientations in a diagram so every vertex looks like  .
- For every subset of the boundary, consider all collections of nonoverlapping paths with that boundary.
- Each matrix entry is a sum of 'reductions along the paths', with coefficients.

Example for $\mathfrak{sl}_3$.



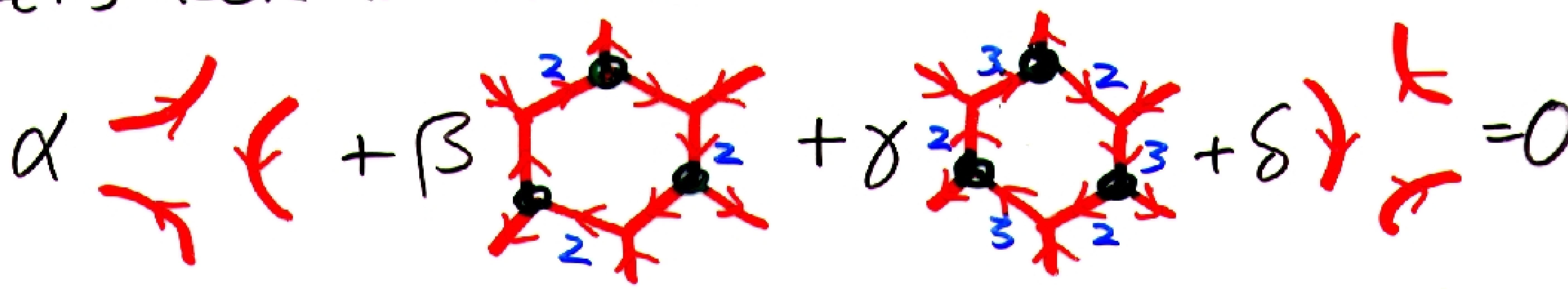


and we see

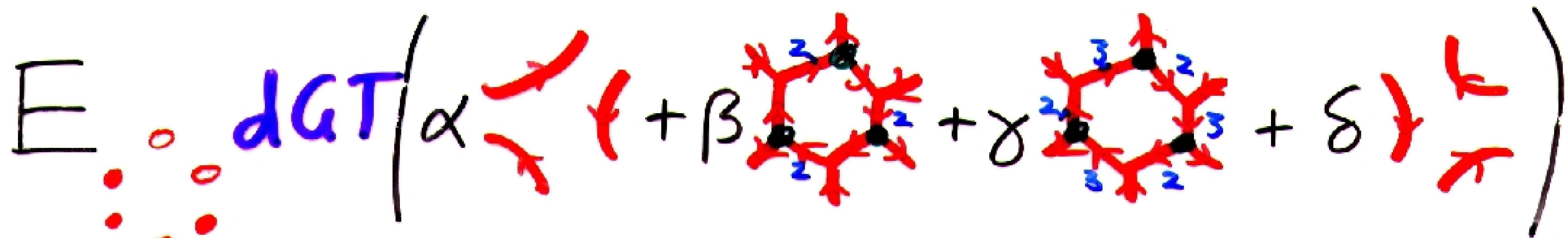


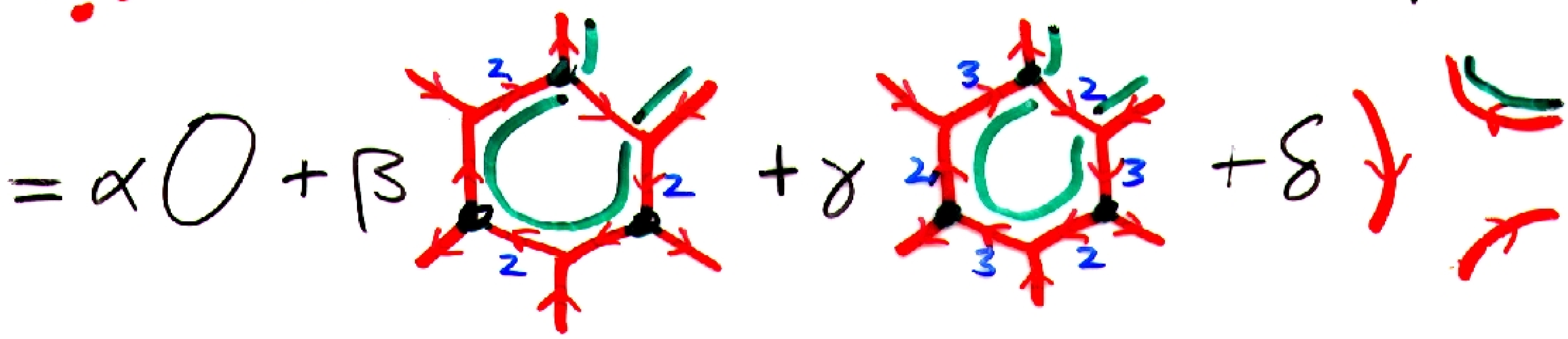
An $\mathfrak{sl}_4$ Example (at $q=1$, for simplicity)

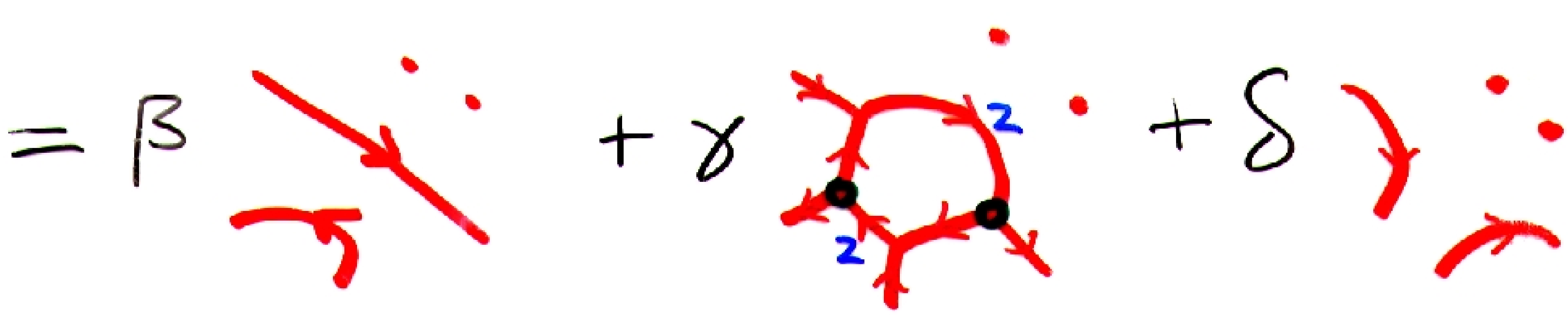
Let's look for a relation of the form



We'll apply **dGT**, then pick out just one matrix entry





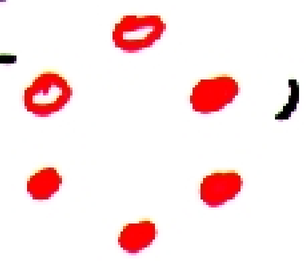


For this to be zero, we must have

$$\beta = -\gamma = \delta,$$

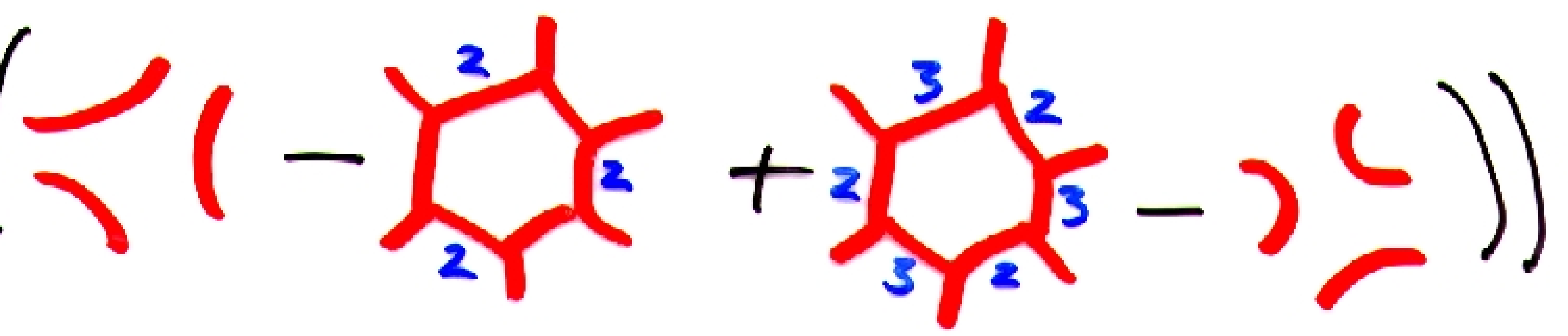
by Kuperberg's 'square' relation for $\mathfrak{sl}_3$

Looking at another matrix entry, for example

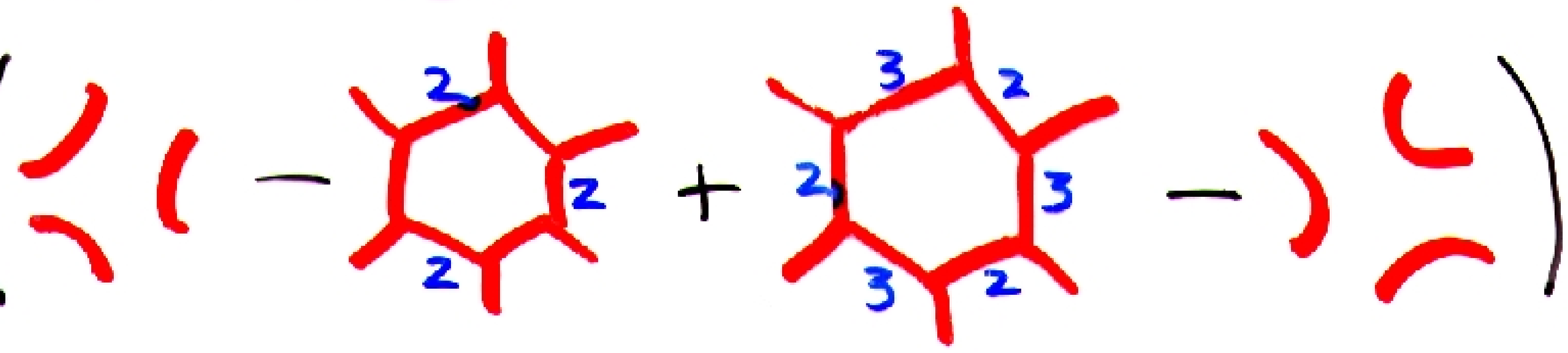
E  , we find $-\alpha = \beta = -\gamma$.

Thus, for there to be a relation amongst these diagrams, the coefficients must be alternating: $-\alpha = \beta = -\gamma = \delta$.

Checking <u>all</u> the matrix entries, we find

$\mathcal{R}_{n-1}\big(dGT\big($  $\big)\big) = 0$

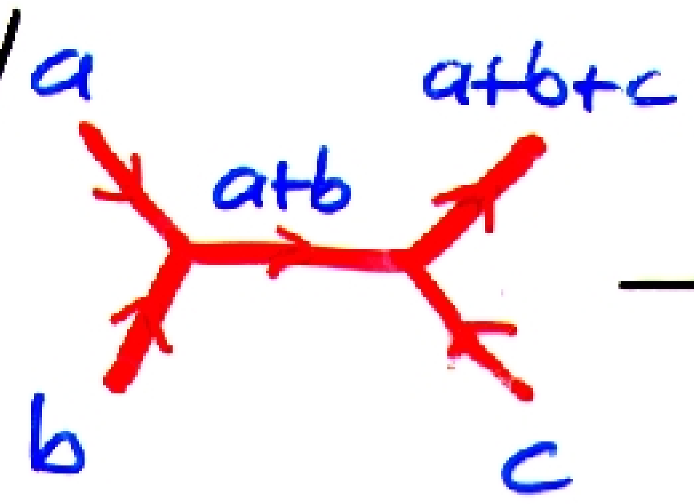
and therefore

$\mathcal{R}_n\big($  $\big) = 0$

providing us with a relation in the $\mathfrak{sl}_4$ representation theory.

$\operatorname{Ker} \mathcal{R}_n$

$\cup$



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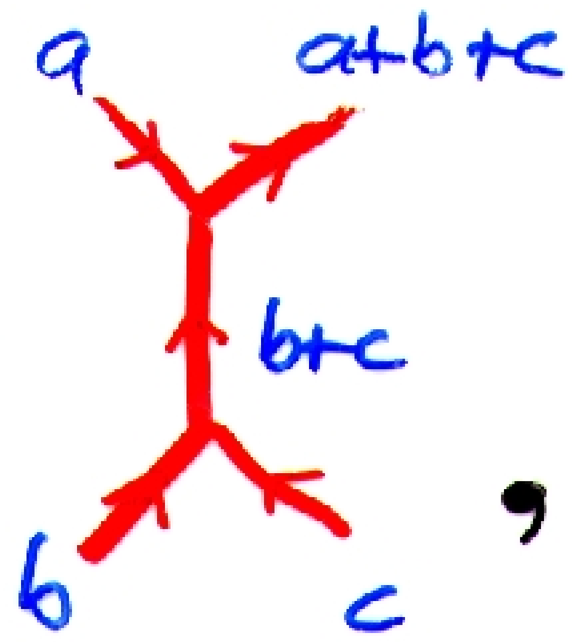

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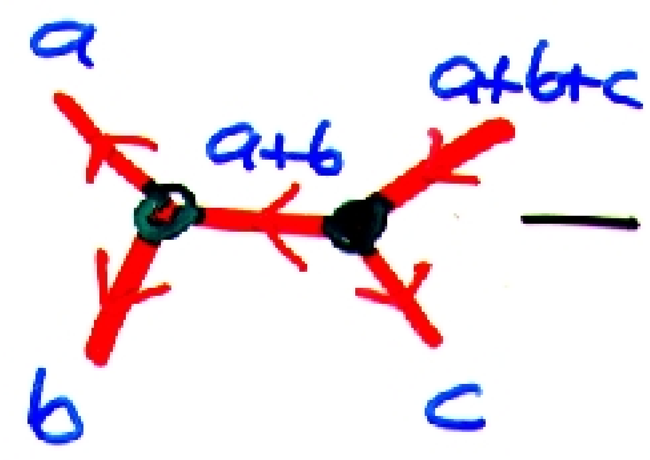

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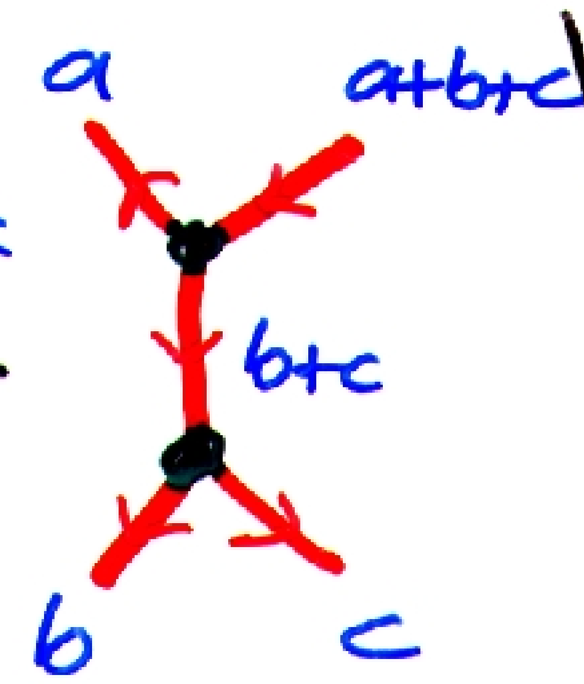
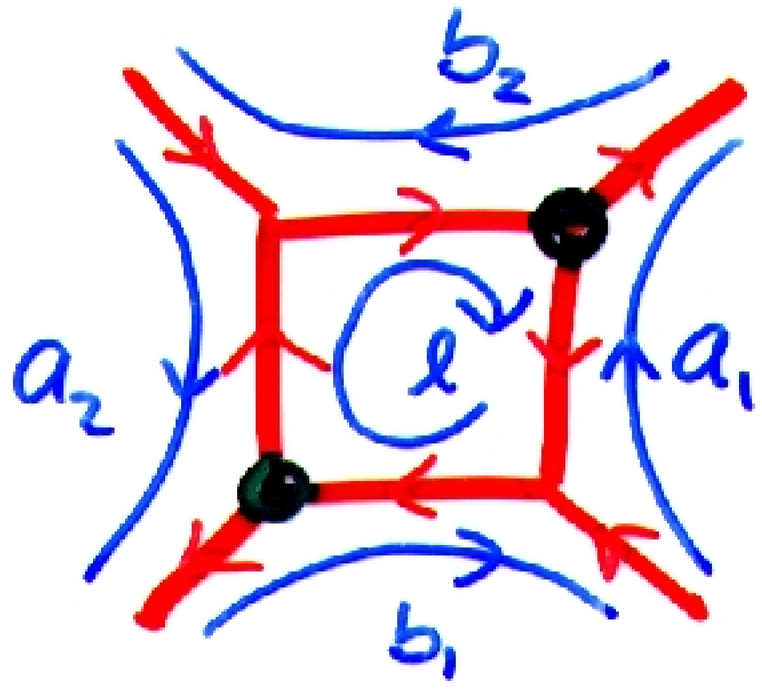
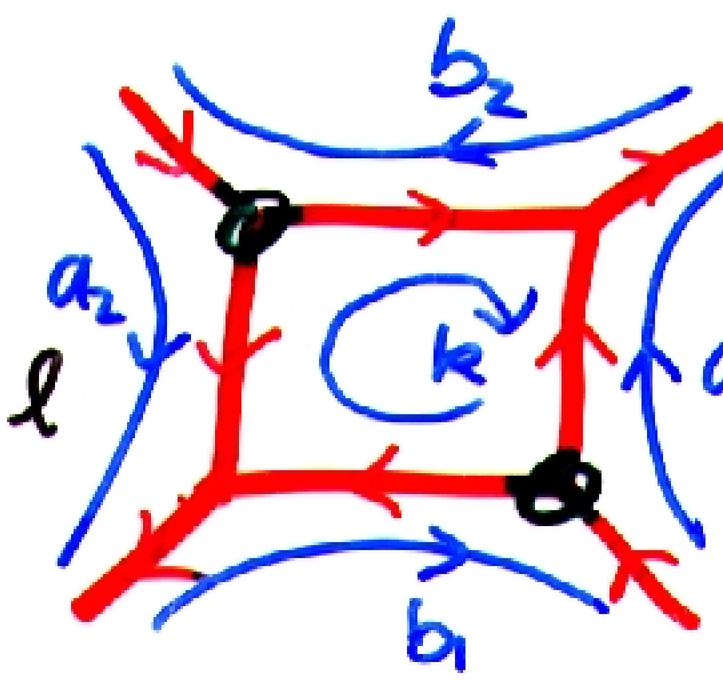
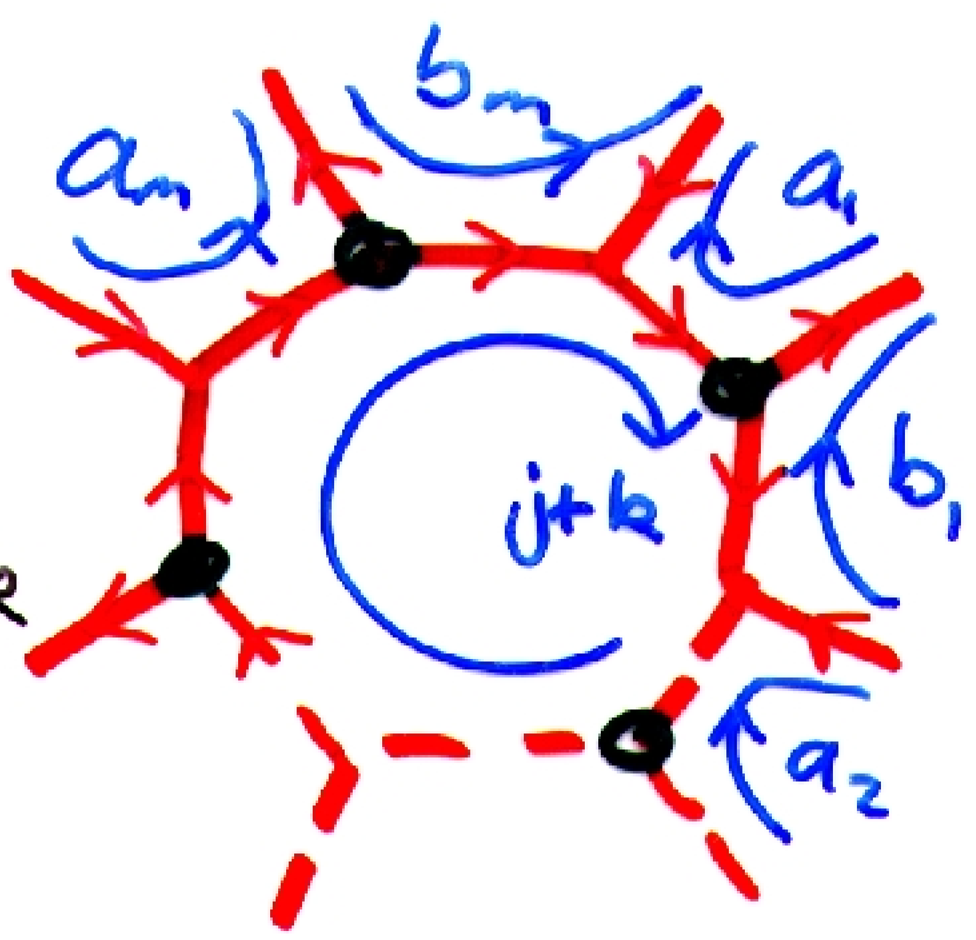




$$- \sum_{k} c_{\bar a,\bar b;k,\ell}$$



$$\sum_{k=-\hat{\sum} b}^{-\hat{\sum} a+1} d_{\bar a,\bar b;j,k}$$



for each $j = n-1+\sum a, \dots, \sum b$

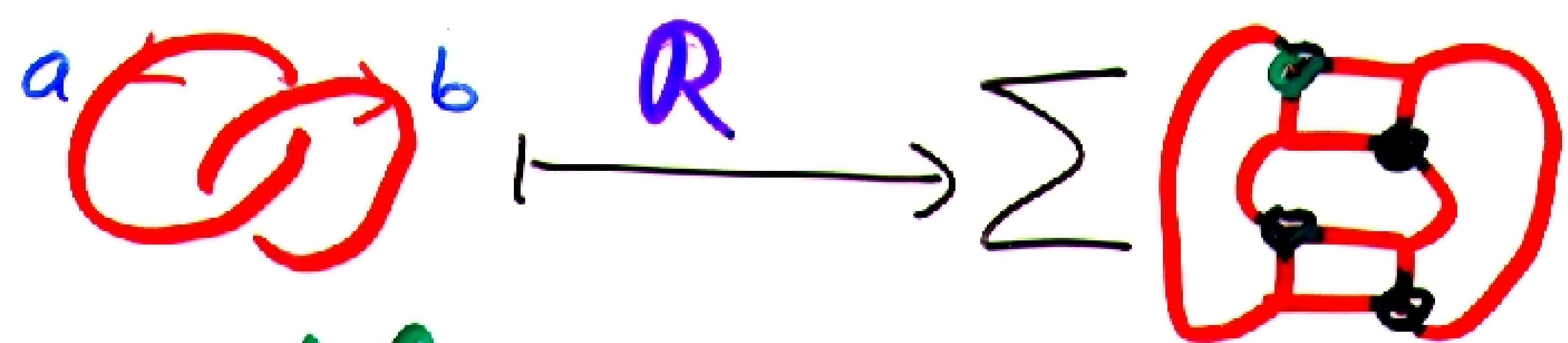
where $d_{\bar a,\bar b;j,k} = (-1)^{k+j} \begin{bmatrix} n-k-j-\max(a) \\ n-1-j-\sum a \end{bmatrix} \begin{bmatrix} k+j+\min(b) \\ j+\sum b \end{bmatrix}$

and $c_{\bar a,\bar b;k,\ell} = \begin{bmatrix} n-\sum a+\sum b \\ \sum b+k+\ell \end{bmatrix}$ if $n-\sum a+\sum b \geq 0$

or $(-1)^{k+\ell} \begin{bmatrix} \sum a-n-1+k+\ell \\ \sum b+k+\ell \end{bmatrix}$ otherwise.

Applications

- Efficient evaluation of coloured quantum knot invariants for $\underline{\underline{su}}(n)$.



- The subfactors associated to some fundamental representations at a certain root of unity have intermediate subfactors.
- (Optimistically) 'Foam' models for $\underline{\underline{sl}}_n$-Khovanov homology.

(Khovanov's $\underline{\underline{sl}}_3$ paper gave categorified versions of Kuperberg's $\underline{\underline{sl}}_3$ relations.)