The Blob Complex, part 2

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(joint work with Scott Morrison)

slides and prepreprint available at canyon23.net/math/
(or the URLs Scott gave)
Goals:
- n-category definition optimized for TQFTs (prove gluing theorem, blob complex product theorem)
- should be very easy to show that topological examples satisfy the axioms
- as simple as possible (but not simpler)
- both plain and infinity type categories
- also define modules, coends, tensor products, etc.
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Main ideas:
• don’t skeletonize (don’t try to minimize generators, don’t try to minimize relations)
• build in “strong” duality from the start
• non-recursive (don’t need to know what an (n-1)-category is)
Ingredients for an n-category:
1. morphisms in dimensions 0 through n
2. domain/range/boundary
3. composition
4. identity morphisms
5. special behavior in dimension n
Morphisms

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- We will allow morphisms to be of any shape, so long as it is homeomorphic to a ball
Morphisms (preliminary version): For any $k$-manifold $X$ homeomorphic to the standard $k$-ball, we have a set of $k$-morphisms $C_k(X)$. 
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Balls could be PL, topological, or smooth. Also unoriented, oriented, Spin, Pin\(_\pm\). We will concentrate on the case of PL unoriented balls.
Examples

Let $T$ be a topological space.

$C_k(X^k) = \text{Maps}(X \to T)$, for $k < n$, $X$ a $k$-ball.

$C_n(X^n) = \text{Maps}(X \to T)$ modulo homotopy rel boundary
(fundamental $n$-groupoid of $T$)

$C_k(X^k) = \text{Maps}(X \to T)$, for $k < n$, $X$ a $k$-ball.

$C_n(X^n) = C_\ast(\text{Maps}(X \to T))$ (singular chains)
($\infty$ version of fundamental groupoid of $T$)
\[ C_k(X^k) = \{ \text{embedded decorated cell complexes in } X \}, \text{ for } k < n. \]
\[ C_n(X^n) = \{ \text{embedded decorated cell complexes in } X \} \mod \text{iso and other local relations} \]

\[
\begin{align*}
\bigcirc & = q^5 + q^4 + q + 1 + q^{-1} + q^{-4} + q^{-5} \\
\bigcirc & = q^3 + q^2 + q + q^{-1} + q^{-2} + q^{-3} \\
& = 0 \\
\bigcirc & = -(q^3 + q^2 + q + q^{-1} + q^{-2} + q^{-3}) \\
\bigcirc & = (q^2 + 1 + q^{-2}) \\
\bigcirc & = -(q + q^{-1}) \left( \bigcirc + \bigcirc \right) + (q + 1 + q^{-1}) \left( \bigcirc + \bigcirc \right) \\
\bigcirc & = -\left( \bigcirc + \bigcirc + \bigcirc + \bigcirc + \bigcirc \right) + \left( \bigcirc + \bigcirc + \bigcirc + \bigcirc + \bigcirc \right) \\
\bigcirc & = -\left( \bigcirc - \bigcirc - \frac{1}{q^2 - 1 + q^{-2}} \right) \left( \bigcirc + \frac{1}{q + 1 + q^{-1}} \right) \\
\end{align*}
\]

(Kuperberg)

More examples
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Let $A$ be a traditional linear $n$-category with strong duality (e.g. pivotal 2-category).

$C_k(X^k) = \{A\text{-string diagrams in } X\}$, for $k < n$.

$C_n(X^n) = \{\text{finite linear combinations of } A\text{-string diagrams in } X\}$ modulo diagrams which evaluate to zero.

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$C_n(X^n) = \text{blob complex of } X \text{ based on } A\text{-string diagrams}$
Boundaries (domain and range), part 1: For each $0 \leq k \leq n - 1$, we have a functor $C_k$ from the category of $k$-spheres and homeomorphisms to the category of sets and bijections.
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Domain + range → boundary: Let $S = B_1 \cup_E B_2$, where $S$ is a $k$-sphere ($0 \leq k \leq n - 1$), $B_i$ is a $k$-ball, and $E = B_1 \cap B_2$ is a $k-1$-sphere. Let $C(B_1) \times_{C(E)} C(B_2)$ denote the fibered product of the two maps $\partial : C(B_i) \to C(E)$. Then (axiom) we have an injective map

$$g_{1E} : C(B_1) \times_{C(E)} C(B_2) \to C(S)$$

which is natural with respect to the actions of homeomorphisms.
• Let $\mathcal{C}(S)_E \subset \mathcal{C}(S)$ denote the image of $\text{gl}_E$. 
\begin{itemize}
  \item Let $\mathcal{C}(S)_E \subset \mathcal{C}(S)$ denote the image of $\text{gl}_E$
  \item Given $c \in \mathcal{C}(\partial(X))$, let $\mathcal{C}(X; c) \overset{\text{def}}{=} \partial^{-1}(c)$
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• Given $E \subset \partial X$, let $\mathcal{C}(X)_E \overset{\text{def}}{=} \partial^{-1}(\mathcal{C}(\partial X)_E)$
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• In most examples, we require that the sets $\mathcal{C}(X; c)$ (for all $n$-balls $X$ and all boundary conditions $c$) have extra structure, e.g. vector space or chain complex
**Composition:** Let $B = B_1 \cup_Y B_2$, where $B$, $B_1$ and $B_2$ are $k$-balls ($0 \leq k \leq n$) and $Y = B_1 \cap B_2$ is a $k-1$-ball. Let $E = \partial Y$, which is a $k-2$-sphere. Note that each of $B$, $B_1$ and $B_2$ has its boundary split into two $k-1$-balls by $E$. We have restriction (domain or range) maps $C(B_i)_E \to C(Y)$. Let $C(B_1)_E \times_{C(Y)} C(B_2)_E$ denote the fibered product of these two maps. Then (axiom) we have a map

$$gl_Y : C(B_1)_E \times_{C(Y)} C(B_2)_E \to C(B)_E$$

which is natural with respect to the actions of homeomorphisms, and also compatible with restrictions to the intersection of the boundaries of $B$ and $B_i$. If $k < n$ we require that $gl_Y$ is injective. (For $k = n$, see below.)
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**Strict associativity:** The composition (gluing) maps above are strictly associative.
Multi-composition: Given any decomposition $B = B_1 \cup \cdots \cup B_m$ of a $k$-ball into small $k$-balls, there is a map from an appropriate subset (like a fibered product) of $C(B_1) \times \cdots \times C(B_m)$ to $C(B)$, and these various $m$-fold composition maps satisfy an operad-type strict associativity condition.
**Product (identity) morphisms:** Let $X$ be a $k$-ball and $D$ be an $m$-ball, with $k + m \leq n$. Then we have a map $\mathcal{C}(X) \to \mathcal{C}(X \times D)$, usually denoted $a \mapsto a \times D$ for $a \in \mathcal{C}(X)$. If $f : X \to X'$ and $\tilde{f} : X \times D \to X' \times D'$ are maps such that the diagram

$$\begin{align*}
X \times D & \xrightarrow{\tilde{f}} X' \times D' \\
\pi & \downarrow \quad \downarrow \pi \\
\tilde{X} & \xrightarrow{f} \tilde{X}'
\end{align*}$$

commutes, then we have

$$\tilde{f}(a \times D) = f(a) \times D'.$$

Product morphisms are compatible with gluing (composition) in both factors:

$$(a' \times D) \circ (a'' \times D) = (a' \circ a'') \times D$$

and

$$(a \times D') \circ (a \times D'') = a \times (D' \circ D'').$$

Product morphisms are associative:

$$(a \times D) \times D' = a \times (D \times D').$$

(Here we are implicitly using functoriality and the obvious homeomorphism $(X \times D) \times D' \to X \times (D \times D')$.) Product morphisms are compatible with restriction:

$$\text{res}_{X \times E}(a \times D) = a \times E$$

for $E \subset \partial D$ and $a \in \mathcal{C}(X)$. 

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“extended isotopy”
Extended isotopy invariance in dimension $n$: Let $X$ be an $n$-ball and $f : X \to X$ be a homeomorphism which restricts to the identity on $\partial X$ and is extended isotopic (rel boundary) to the identity. Then $f$ acts trivially on $C(X)$. 
Plain n-cat:

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Infinity n-cat:

**Families of homeomorphisms act in dimension n.** For each $n$-ball $X$ and each $c \in \mathcal{C}(\partial X)$ we have a map of chain complexes

$$C_*(\text{Homeo}_\partial(X)) \otimes \mathcal{C}(X; c) \to \mathcal{C}(X; c).$$

Here $C_*$ means singular chains and $\text{Homeo}_\partial(X)$ is the space of homeomorphisms of $X$ which fix $\partial X$. These action maps are required to be associative up to homotopy, and also compatible with composition (gluing).
Equivalences between this n-cat definition and more traditional ones (at least for n=1 or 2)

A-string diagrams with canonical relations

"topological" n-cat \( C \)

restrict \( C \) to standard \( \eta \)-ball, \( \omega \geq \eta \)

traditional n-cat \( A \)
Colimit construction

- Let $\mathcal{C}$ be in $n$-category.
- We want to extend $\mathcal{C}$ to arbitrary $k$-manifolds $Y$, $0 \leq k \leq n$. 
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- There is a functor which assigns to a decomposition $Y = \bigcup_i X_i$ the set (or vector space or chain complex) $\bigotimes_i \mathcal{C}(X_i)$. 
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- There is a functor which assigns to a decomposition $Y = \bigcup_{i} X_{i}$ the set (or vector space or chain complex) $\bigotimes_{i} \mathcal{C}(X_{i})$.

- Define $\mathcal{C}(Y)$ to be the colimit (or homotopy colimit) of this functor.
Newfangled blob complex

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- $\mathcal{D}$ is in some sense the free resolution of $C$ as an $A_\infty$ $n$-category.

  - Let $M^n = F^{n-k} \times Y^k$. Let $C$ be a plain $n$-category. Let $\mathcal{F}$ be the $A_\infty$ $k$-category which assigns to a $k$-ball $X$ the old-fashioned blob complex $B^C_\ast(X \times F)$. 
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- Given a plain $n$-category $C$, we can construct an $A_\infty$ $n$-category $D$ by defining $D(X) = B_*^C(X)$ for each $n$-ball $X$.

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- Theorem: $\mathcal{F}(Y) \simeq B_*^C(F \times Y)$. 
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- Given a plain $n$-category $C$, we can construct an $A_\infty$ n-category $D$ by defining $D(X) = B^C_*(X)$ for each $n$-ball $X$.

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  - Let $M^n = F^{n-k} \times Y^k$. Let $C$ be a plain $n$-category. Let $F$ be the $A_\infty$ $k$-category which assigns to a $k$-ball $X$ the old-fashioned blob complex $B^C_*(X \times F)$.

  - Theorem: $F(Y) \simeq B^C_*(F \times Y)$.

- Corollary: $D(M) \simeq B^C_*(M)$ for any $n$-manifold $M$. (Proof: Let $F$ above be a point.) So the old-fashioned and newfangled blob complexes are homotopy equivalent.
Modules

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- Modules for $C$ are defined in a similar style.
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- A marked $k$-ball is a pair $(B, M)$ which is homeomorphic to the standard pair $(B_k, B^{k-1})$. 

\[ B \backslash \{ m \} \]
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- A $\mathcal{C}$-module $\mathcal{M}$ is a collection of functors $\mathcal{M}_k$ from the category of marked $k$-balls to the category of sets, $0 \leq k \leq n$. 
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  ![Diagram of a marked $k$-ball]

  - A $\mathcal{C}$-module $\mathcal{M}$ is a collection of functors $\mathcal{M}_k$ from the category of marked $k$-balls to the category of sets, $0 \leq k \leq n$.

- In the top dimension $n$ we have the same extra structure as $\mathcal{C}$ (vector space, chain complex, ...).
Motivating example: Let $W$ be an $m+1$-manifold with non-empty boundary. Let $\mathcal{E}$ be an $m+n$-category.

Let $\mathcal{C}$ be the $n$-category with $\mathcal{C}(X) \overset{\text{def}}{=} \mathcal{E}(X \times \partial W)$. 
• Motivating example: Let $W$ be an $m+1$-manifold with non-empty boundary. Let $\mathcal{E}$ be an $m+n$-category.

• Let $\mathcal{C}$ be the $n$-category with $\mathcal{C}(X) \overset{\text{def}}{=} \mathcal{E}(X \times \partial W)$.

• Define the $\mathcal{C}$-module $\mathcal{M}$ by

$$\mathcal{M}(M, B) \overset{\text{def}}{=} \mathcal{E} \left( (B \times \partial W) \bigcup_{M \times \partial W} (M \times W) \right).$$

![Diagram of $\mathcal{M}(M, B)$]
- Two different ways of cutting a marked $k$-ball into two pieces, so two different kinds of composition. (One is composition within $\mathcal{M}$, the other is the action of $\mathcal{C}$ on $\mathcal{M}$.)

\[ \text{action} \]

\[ \mathcal{M}\text{-composition} \]
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- $\mathcal{M}$ can be thought of as a collection of $n-1$-categories with some extra structure.
- For $n = 1, 2$ this is equivalent to the usual notion of module.
Decorated colimit construction

- Let $W$ be a $k$-manifold. Let $Y_i$ be a collection of disjoint codimension 0 submanifolds of $\partial W$.

- Let $\mathcal{C}$ be an $n$-category and $\mathcal{N} = \{\mathcal{N}_i\}$ be a collection of $\mathcal{C}$-modules, thought of as labels of $\{Y_i\}$. 
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- We can use a variation on the above colimit construction to define a set (or vector space or chain complex if $k = n$) $\mathcal{C}(W, \mathcal{N})$. 
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We can use a variation on the above colimit construction to define a set (or vector space or chain complex if $k = n$) $\mathcal{C}(W, \mathcal{N})$.

The object of the colimit are decompositions of $W$ into (plain) balls $X_j$ and marked balls $(B_i, M_i)$, with $M_i = B_i \cap \{Y_i\}$. 

$$\bigotimes_j \mathcal{C}(X_j) \bigotimes \mathcal{N}(B_i, M_i)$$
Decorated colimit construction

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- We can use a variation on the above colimit construction to define a set (or vector space or chain complex if $k=n$) $\mathcal{C}(W, \mathcal{N})$.

- The object of the colimit are decompositions of $W$ into (plain) balls $X_j$ and marked balls $(B_l, M_l)$, with $M_l = B_l \cap \{Y_i\}$.

- This defines an $n-k$-category which assigns $\mathcal{C}(D \times W, \mathcal{N})$ to a ball $D$. (Here $N_i$ labels $D \times Y_i$.)
Tensor products and gluing

- As a simple special case of this construction, given $\mathcal{C}$-modules $\mathcal{N}_1$ and $\mathcal{N}_2$, define the tensor product $\mathcal{N}_1 \otimes \mathcal{N}_2$ (an $n-1$-category) to be the result of taking $W$ to be an interval and letting $\mathcal{N}_1$ and $\mathcal{N}_2$ label the endpoints of the interval.
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- Gluing theorem: Let $M^{n-k} = M_1 \cup_Y M_2$. Let $\mathcal{C}$ be an $n$-category. The above constructions give a $k$-category $\mathcal{C}(M)$, a $k-1$-category $\mathcal{C}(Y)$, and two $\mathcal{C}(Y)$-modules $\mathcal{C}(M_i)$. Then

$$\mathcal{C}(M) \simeq \mathcal{C}(M_1) \otimes_{\mathcal{C}(Y)} \mathcal{C}(M_2).$$