Coincidences of tensor categories

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Algebraic structures in knot theory
UC Riverside, November 7 2009

slides: http://tqft.net/UCR-identities
article: http://tqft.net/identities
Outline

1. Quantum knot invariants
2. Mysterious identities
3. Modular $\otimes$-categories
   - De-equivariantisation
   - Level-rank duality
   - Kirby-Melvin symmetry
4. Conclusion
   - Putting it all together
   - Thank you!
Quantum knot invariants

Reshetikhin-Turaev define a polynomial knot invariant for every
- quantum group $U_q(g)$, with $g$ a complex simple Lie algebra,
- and irreducible representation $V$ of $U_q(g)$:

$$\mathcal{J}_{U_q(g), V(K)}(q).$$

Example

$$\mathcal{J}_{U_q(sl_4), \bigwedge^2 \mathbb{C}^4}(q) = q^{16} + q^{12} + q^{10} + q^{-10} + q^{-12} + q^{-16}.$$ 

These invariants generalise the Jones polynomial ($SU(2)$, $\mathbb{C}^2$), the coloured Jones polynomials ($\text{Sym}^n \mathbb{C}^2$), HOMFLYPT ($SU(n)$, $\mathbb{C}^n$) and the 2-variable Kauffman polynomial ($SO(n)$ or $Sp(2n)$, $V^q$).
We can compute these invariants!

A computer can calculate the universal $\mathcal{R}$-matrix acting on any irreducible representation. A braid presentation of the knot tells us a sequence of matrices with entries in $\mathbb{Z}[q, q^{-1}]$ to multiply, and then take trace.

Really!

See my QuantumGroups' package, available as part of the KnotTheory' package from http://katlas.org/.

Example

```
<<KnotTheory'
QuantumKnotInvariant[A_3][Irrep[A_3][0,1,0]][Knot[4,1]]
== q^{16} + q^{12} + q^{10} + q^{-10} + q^{-12} + q^{-16}
```
Let’s search for identities between these polynomials, specialising $q$ to roots of unity.

We find lots of examples!

\[
\begin{align*}
\mathcal{J}_{SU(2),(2)}(K)(\exp(\frac{2\pi i}{12})) &= 2 \\
\mathcal{J}_{SU(2),(4)}(K)(\exp(\frac{2\pi i}{20})) &= 2 \mathcal{J}_{SU(2),(1)}(K)(\exp(\frac{-2\pi i}{10})) \\
\mathcal{J}_{SU(2),(6)}(K)(\exp(\frac{2\pi i}{28})) &= 2 \mathcal{J}_{SU(4),(1,0,0)}(K)(\exp(\frac{-2\pi i}{14})) \\
\mathcal{J}_{SU(2),(8)}(K)(\exp(\frac{2\pi i}{36})) &= 2 \mathcal{J}_{SO(8),(1,0,0,0)}(K)(-\exp(\frac{-2\pi i}{18})) \\
\mathcal{J}_{SU(2),(12)}(K)(\exp(\frac{2\pi i}{52})) &= 2 \mathcal{J}_{G_2,V_{(1,0)}}(K)(\exp(\frac{2\pi i\cdot 23}{26}))
\end{align*}
\]

Question

What’s going on? Is there some algebraic structure underlying these strange identities between knot polynomials?
Some mysterious identities

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\end{align*}

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What’s going on? Is there some algebraic structure underlying these strange identities between knot polynomials?
At a root of unity, the representation theory of a quantum group truncates to a **modular** $\otimes$-category with finitely many objects.

Example ($SU(3)$ at ‘level 3’, $q = \exp(\frac{2\pi i}{12})$)
Braided tensor categories are like finite groups

- Not all automorphisms come from ‘group-like’ sub-categories.
- Not all quotients are ‘modular’, or even $\otimes$.

These algebraic operations explain identities between the corresponding knot invariants.
We’d like to prove

\[ \mathcal{J}_{SU(2),(6)}(K)(\exp(\frac{2\pi i}{28})) = 2\mathcal{J}_{SU(4),(1,0,0)}(K)(\exp(-\frac{2\pi i}{14})). \]

On right hand side, we look at the modular tensor category \( SU(2) \)

at \( q = \exp(\frac{2\pi i}{28}) \). This has 12 objects, so we call it \( SU(2)_{11} \) (‘\( SU(2) \) at level 11’).
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On right hand side, we look at the modular tensor category \( SU(2) \) at \( q = \exp(\frac{2\pi i}{28}) \). This has 12 objects, so we call it \( SU(2)_{11} \) (‘\( SU(2) \) at level 11’).

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\[ \mathbb{Z}/2\mathbb{Z} \subset SO(3)_6 \subset SU(2)_{11} \]

Here’s the subcategory \( SO(3)_6 \).

\[ SO(3)_6 = \begin{array}{ccccccccc}
    f^{(0)} & f^{(2)} & f^{(4)} & f^{(6)} & f^{(8)} & f^{(10)} & f^{(12)}
\end{array} \]
Take the quotient $SO(3)_6/2$; it’s a new modular tensor category.

$$SO(3)_6/2 = \begin{array}{c}
\text{P} \\
\text{f}^{(0)} \\
\text{f}^{(2)} \\
\text{f}^{(4)} \\
\text{Q}
\end{array}$$

Quotients of braided $\otimes$-categories are usually called ‘de-equivariantisations’.

To match conventions between $SU$ and $SO$, replace $q$ with $q^2$. The object (6) splits into two pieces, $P$ and $Q$, with the same knot invariants.

Corollary

$$\mathcal{J}_{SU(2)_{11},(6)}(K)(\exp(\frac{2\pi i}{28})) = 2\mathcal{J}_{SO(3)_6/2,P}(K)(\exp(\frac{2\pi i}{14})).$$
Level-rank duality is tricky! The correct statement is

**Theorem**

*With n odd, q a* 4(n + m − 2)-th root of unity,*

\[ SO(n)|_{q/2} \cong SO(m)|_{-q^{-1}/2}. \]

Translating to levels, this is \( \text{SO}(n)_{m/2} \cong \text{SO}(m)_{n/2} \), but not at the obvious root of unity!

The quotients are by \( V_{me_1} \) and \( V_{ne_1} \), the highest multiples of the standard representation.
Example \(((\text{SO}(3)_6)/2 \cong (\text{SO}(6)_3)/2)\)

Here we show \(\text{SO}(6)_3\) as the ‘vector’ subset of \(\text{Spin}(6)_4 \cong \text{SU}(4)_3\).

Corollary

\[
\mathcal{J}_{\text{SO}(3)/2, P}(K)(\exp(\frac{2\pi i}{14})) = \mathcal{J}_{\text{SO}(6)/2, (200)}(K)(-\exp(\frac{-2\pi i}{14}))
\]
Kirby-Melvin symmetry

‘Include up’ to all of $SU(4)$. There’s a “Kirby-Melvin symmetry” given by $- \otimes (300)$, interchanging $(200)$ and $(100)$.

Kirby-Melvin symmetries aren’t quite ‘quotients’ unless we change the pivotal structure. Knot invariants may change by a sign.

**Corollary**

$$\mathcal{J}_{SU(4),(200)}(K)(- \exp(-\frac{2\pi i}{14})) = -\mathcal{J}_{SU(4),(100)}(K)(- \exp(-\frac{2\pi i}{14})).$$
Theorem

\[ \mathcal{J}_{SU(2),(6)}(K)|_{q=\exp(\frac{2\pi i}{28})} = 2 \mathcal{J}_{SU(4),(1,0,0)}(K)|_{q=\exp(-\frac{2\pi i}{14})} \]

Proof.

\[ \mathcal{J}_{SU(2),(6)}(K)(e^{\frac{2\pi i}{28}}) = \mathcal{J}_{SO(3)_6,(6)}(K)(e^{\frac{2\pi i}{14}}) \]  
(sub-category)

\[ = 2 \mathcal{J}_{SO(3)_6/2,P}(K)(e^{\frac{2\pi i}{14}}) \]  
(quotient)

\[ = 2 \mathcal{J}_{SO(6)_3/2,2e_3}(K)(-e^{-\frac{2\pi i}{14}}) \]  
(level-rank)

\[ = 2 \mathcal{J}_{SU(4),2e_1}(K)(-e^{-\frac{2\pi i}{14}}) \]  
\(D_3 = A_3\)

\[ = -2 \mathcal{J}_{SU(4),e_1}(K)(-e^{-\frac{2\pi i}{14}}) \]  
(Kirby-Melvin)

\[ = 2 \mathcal{J}_{SU(4),e_1}(K)(e^{-\frac{2\pi i}{14}}) \]  
(parity)
Conclusion

It’s fun to explain strange identities between knot polynomials by understanding algebraic relationships between the underlying modular tensor categories.

Read our paper http://tqft.net/identities for

- all the coincidences and automorphisms related to $SO(3)_{m/2}$,
- a nice summary of level-rank duality, especially for $SO(3)$,
- the best description of Kirby-Melvin symmetry in the literature,
- many more pretty pictures!

Thank you!