Disoriented and confused: fixing the functoriality of Khovanov homology.

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May 5, 2006

What's wrong with Khovanov homology?
- It's almost functorial
- ... but not quite
- ... and it ought to be!

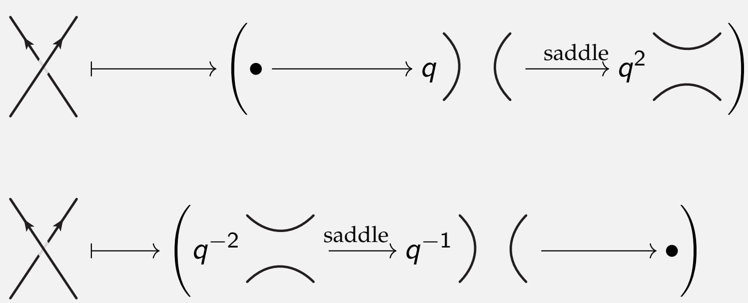
How do we fix it?
- Disorientations
- Movie moves
- Calculations
- Confusions

Odds and ends.
- Recovering the original theory
- Decategorifying

What is Khovanov homology?

Khovanov homology is a map from tangles to up-to-homotopy complexes of (matrices of) cobordisms.

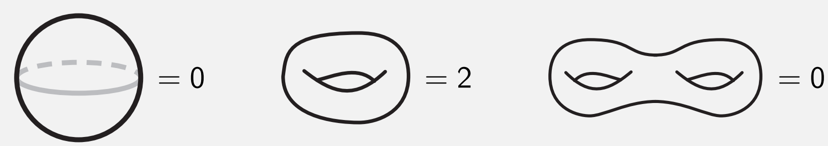
- On single crossings it is given by



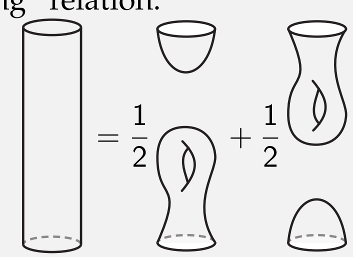
- It is a map of planar algebras: to compose two tangles in a planar way, take the tensor product of the corresponding complexes, combining objects and morphisms using the specified planar operation.

We need to impose some relations on cobordisms in order to make this a tangle invariant.

- Closed surface relations:

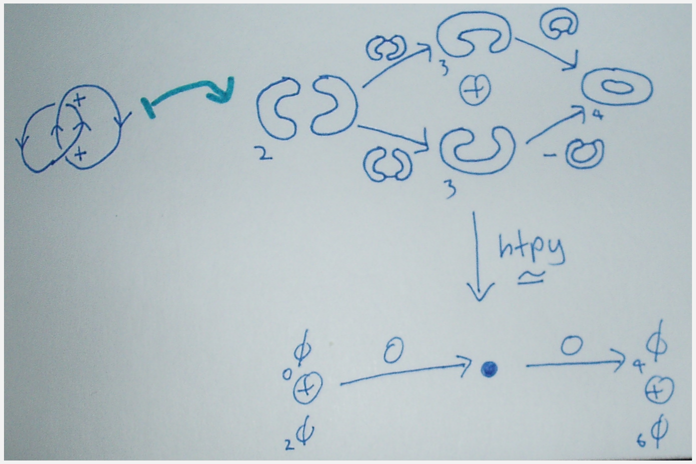


- The "neck cutting" relation:



Example

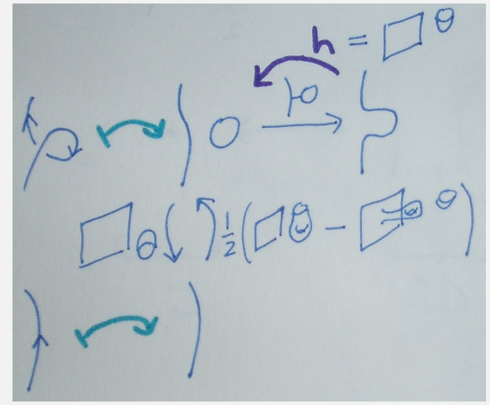
The hopf link.



Why is it actually an invariant of tangles?

We need to construct homotopy equivalences between the complexes on either side of each Reidemeister move

Example



Khovanov homology is almost functorial

So far, I've described an invariant associated to tangles.
We can try to make Khovanov homology functorial, associating to a cobordism between two tangles some chain map between the associated complexes.
Link cobordisms can be given presentations as 'movies'. Each frame of a movie is a tangle diagram. Between each pair of frames, one of the 'elementary movies' takes place:

- a Reidemeister move, in either direction
- the birth of death of a circle
- a morse move between two parallel arcs

We need to assign chain maps to each of the elementary movies.

- All the morse moves are easy; there are obvious cobordisms implementing them.
- To each Reidemeister move, we assign the chain map we constructed when showing that the two sides of the Reidemeister move were homotopically equivalent complexes.

To assign a chain map to an arbitrary link cobordism, we choose a movie presentation, and compose the chain maps associated to each elementary piece. Is this well defined?

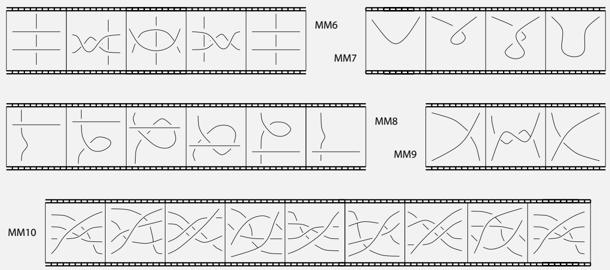
... but not quite

Theorem (Carter and Saito)

Two movies are presentations of the same link cobordism exactly if they are related by a sequence of 'movie moves'.

Example (Movie moves 6-10)

Each movie here is equivalent to the 'do nothing' movie.



Thus to check our proposed invariant of link cobordisms is well defined, we 'only' need to check that we assign the same chain map (up to homotopy equivalence!) to either side of each movie move.

Theorem (Bar-Natan, 2004)

The two sides of a movie move agree up to sign!

Theorem (Jacobsson, 2002)

The signs don't come out right. You can shuffle them around, but not make them go away.

... and it ought to be!

It would be nice if Khovanov homology really were functorial.

- Functors are good!
- You could identify generators in the Khovanov homologies calculated from two different presentations of a knot.
- Khovanov's construction of a categorification of the coloured Jones polynomial would be easier.
- You could build a doubly monoidal 4-category out of Khovanov homology.

How do we fix it?

To fix the sign problems in Khovanov homology, we'll make two modifications to the 'target category' of cobordisms.

disorientations Objects and cobordisms will be 'piecewise oriented', with 'disorientation lines' where the orientations disagree.

confusions Extra morphisms called 'confusions' fix some defects in the category, and make proofs manageable. They are 'spinorial' objects.

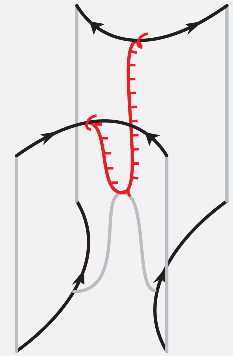
Disorientations

We'll replace the unoriented cobordism category previously used with a category of 'disoriented cobordisms'.

Objects Non-crossing arcs embedded in a disc, each piecewise oriented. Each 'disorientation mark' separating oppositely oriented intervals also has a preferred direction.

Morphisms Surfaces are piecewise oriented, with 'disorientation' lines marking the boundaries between regions with opposite orientations. Each disorientation line has a 'fringe', indicating a preferred side.

Example



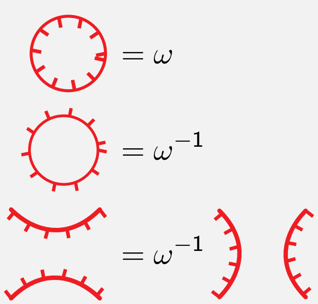
In the oriented regions, we impose the usual cobordism relations. We also need some rules for removing closed disorientation lines, and reconnecting parallel disorientation lines.

Disorientation relations

Fix a parameter ω, such that $\omega^4 = 1$.

- At $\omega = 1$, we recover the old theory by forgetting all orientation data. (We also recover the sign problems!)
- At $\omega = i$, we'll have functoriality!

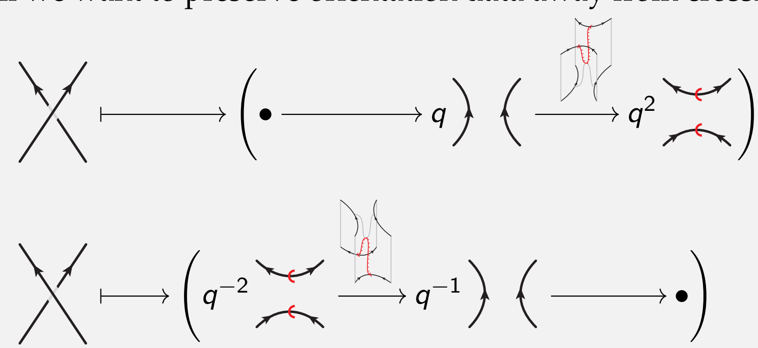
Introduce some relations on disorientations:



▸ These are consistent!

Modifying the tangle invariant

Now tangles are mapped to (up-to-homotopy) complexes of disoriented cobordisms. It's obvious where to put the seams in, if we want to preserve orientation data away from crossings.



Disorientation marks near a crossing face to the right.

Theorem (M&W)

This is still an invariant of tangles. We'll see all the homotopy equivalences for Reidemeister moves soon!

It not obvious at this point what the relationship is with the old theory. We expect that it will be equivalent for knots and links, but different for tangles. This is only based on some small examples, however!

Movie moves

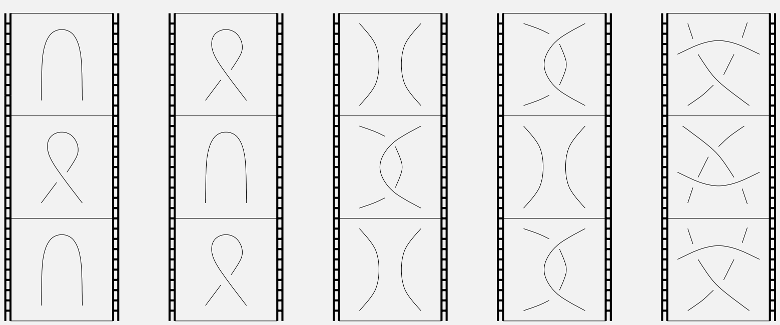
Now we need to check 15 movie moves. These come in several types.

Inverses These almost trivial moves insist that the time reverse of a Reidemeister move is also its inverse.

Circular clips These 'circular' clips should be equivalent to the identity. These include the 3 'hard' clips that involve a type III Reidemeister move.

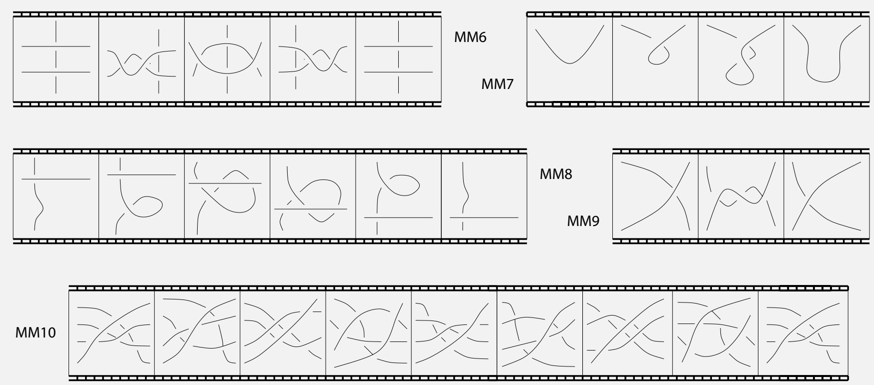
Non-reversible clips These pairs of clips should be equivalent, when read either up or down.

Inverse moves

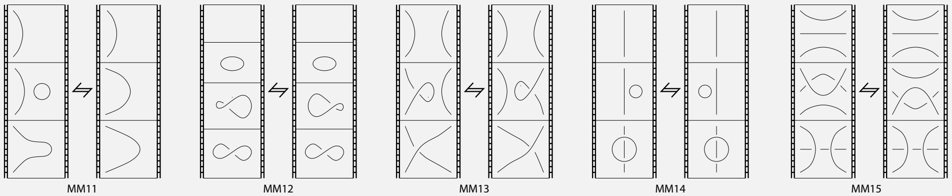


MM1 MM2 MM3 MM4 MM5

These are boring; we know these are identities, because the two successive steps are a homotopy equivalence and its inverse.



These are 'hard'; moves 6, 8 and 10 involve the third Reidemeister move.



- Each pair of clips should give the same map, whether read up or down.
- These ones don't seem so bad, but there are lots of sign problems lurking here!
- Often there's a sign problem one way but not the other.

Jacobsson's sign tables

Jacobsson reported sign problems in almost every move! (Unfortunately he used a different numbering.)

MM	J♯	$\pm$
6	15	-
7	13	+
7 (mirror)	13	-
8	6	-
8 (mirror)	6	+
9	14	-
9 (mirror)	14	+
10	7	+

MM	J♯	$\downarrow$	$\uparrow$
11	9	+	+
12	11	-	+
12 (mirror)	11	+	+
13	12	-	+
13 (mirror)	12	+	-
14	8	+	-
15	10	-	+

We can calculate the corresponding table for the disoriented theory, as a function of ω.

- At $\omega = 1$, we recover the tables above.
- At $\omega = i$, all the signs agree.

What about all the orientations!?

- At this point is appears we need to check many orientations of each of these movie moves; up to 16 in the worst case.
- For now, we'll ignore this, and just check the signs for one oriented representative of each movie move.
- Later, the introduction of 'confusions' will deal with the rest.

Bar-Natan's proof

Bar-Natan gave a simple proof that Khovanov homology is well-defined 'up-to-sign'.

- Certain tangles are *simple*, in that the automorphism group of the associated complex consists only of multiples of the identity.
- Each of movie moves 1-10 starts and ends with a 'simple tangle', and so must be a multiple of the identity.
- (Movie moves 11-15 can be done easily by hand.)

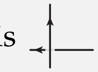
In our situation, many small tangles are still 'simple' in this sense, although now there are more units in our coefficient ring: $\pm 1, \pm i$. We'll make use of this often.

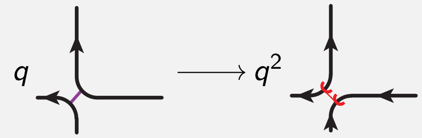
Detecting the sign

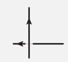
Bar-Natan's result ensures that movie moves are well-defined up to sign. We can relatively easily detect this sign.

- Cobordisms between loopless diagrams are all in non-positive degree.
- Because of the grading shifts in the definition of Khovanov homology, homotopies must be in strictly positive degree.
- Not many homotopies are possible. We call a direct summand of an object in a complex *homotopically isolated* if there are no possible homotopies in or out.

Example

The initial (and final) frame of MM8 is , whose associated complex is



Neither of the objects have loops, so both objects are isolated. If f :  $\to$  is homotopic to the identity, it must be the identity on the nose; $f - I = dh + hd = 0$.

We can often detect the sign associated to a movie move by choosing an isolated summand in the complex, and observing its image under the movie move.

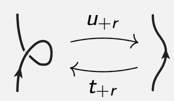
Calculations

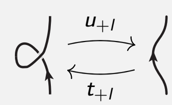
It's now time to do the real work! We need to

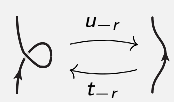
- calculate explicit chain maps corresponding to Reidemeister moves.
  - These are unique up to a unit, by Bar-Natan's result.
  - We can easily write these down for the R1 and R2, but R3 will take some work; we'll use Bar-Natan's cone construction to organise this.
- detect the signs for each movie move, in at least one orientation,
- and explain away all the other orientations!

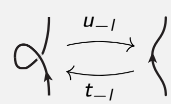
Twist maps

The twist maps implement the Reidemeister I moves. There are four variations.

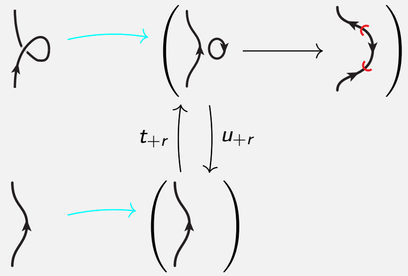
Positive right twist 

Positive left twist 

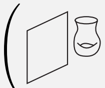
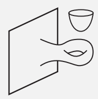
Negative right twist 

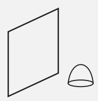
Negative left twist 

The positive right twist map is



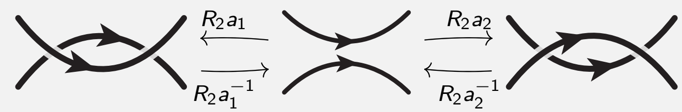
where t_{+r} and u_{+r} are given by

$t_{+r} = \frac{1}{2}\Big($  $- \omega^{-2}$  $\Big)$

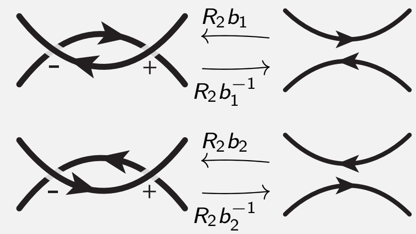
$u_{+r} =$ 

Tuck maps

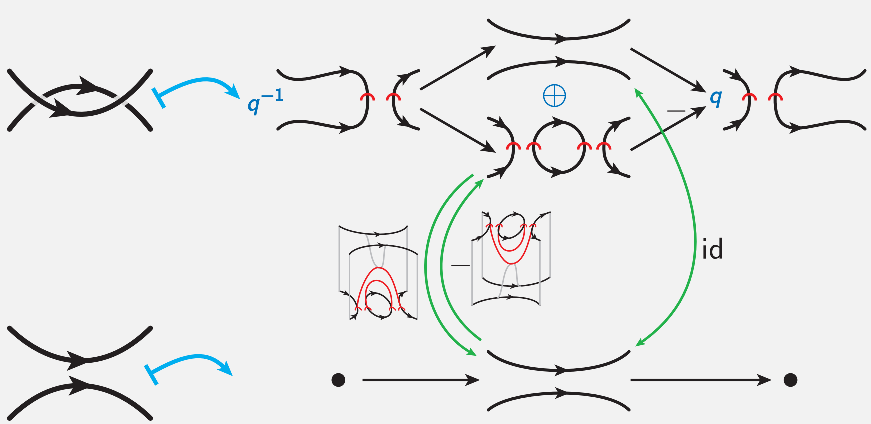
RIIa The upper strand can be on the left or the right.



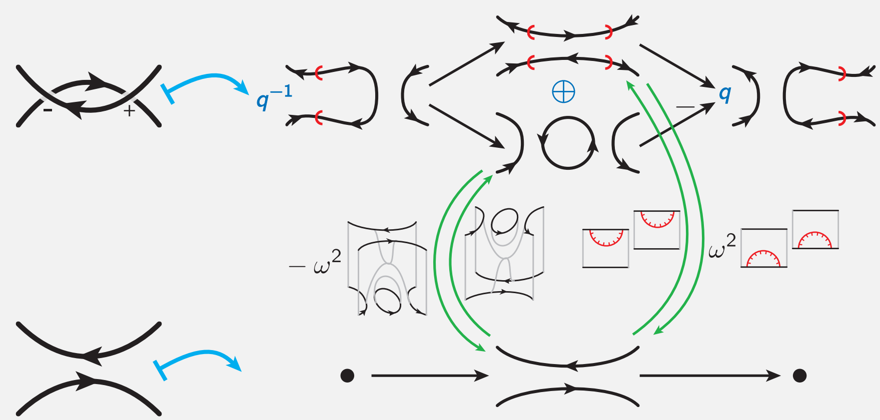
RIIb The upper strand can go from the positive crossing to the negative crossing, or vice versa.



RIIa maps



RIIb maps



The cone construction, and RIII

Obtaining the RIII map takes some work! We follow through Bar-Natan's proof of RIII invariance, keeping track of the explicit homotopy equivalence being constructed.

Lemma

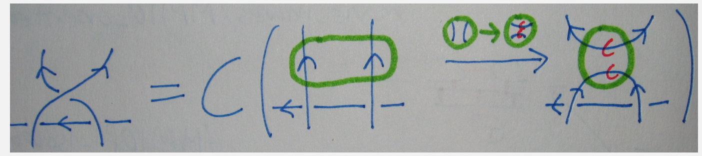
The RII moves are strong deformation retracts.

Lemma

If $f : A^\bullet \to B^\bullet$ is a chain map, and $r : B^\bullet \to C^\bullet$ is a strong deformation retract, $C(rf) \simeq C(f)$.

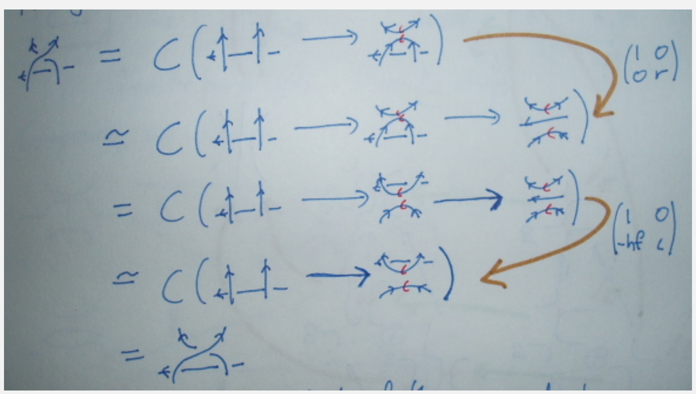
Lemma

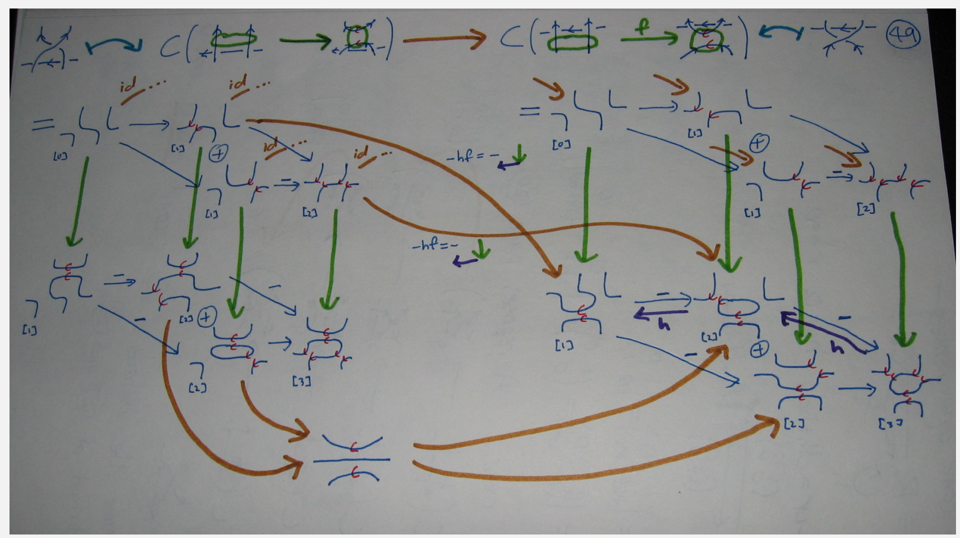
Each side of the RIII move can be realised as a cone over the morphism switching between two smoothings of the 'central' crossing.



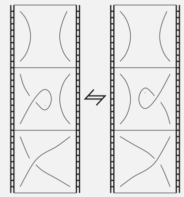
We can then compose this morphism with the 'untuck' move, a strong deformation retract. Doing this to either side of the RIII move, we obtain the same cone!

Putting this together, we have

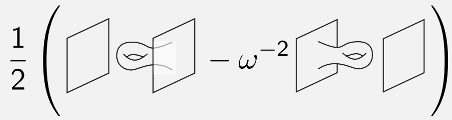




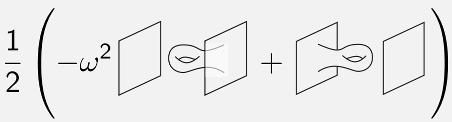
MM13



Each side of MM13 consists of a twist move (t_{+r} and t_{+l} respectively) followed by a morse move. Reading down the left side, we get

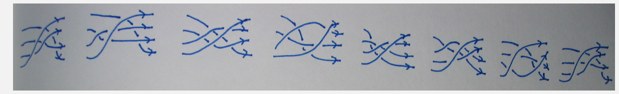


and on the right

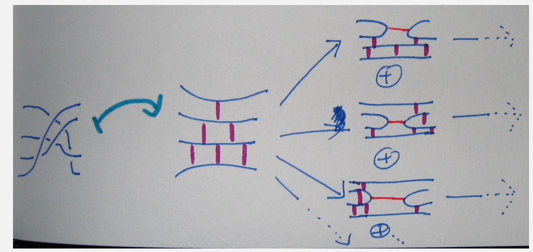


Thus we see the two sides of MM13 differ by a sign of $-\omega^2$!

MM10

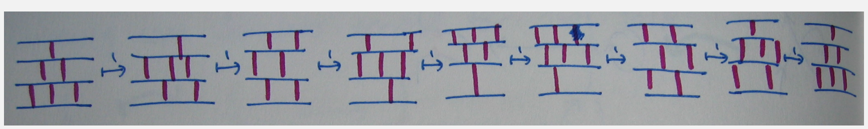


Look at the initial frame. The associated complex has one object in homological degree 0; the object we obtain from the 'positive smoothing' of each of the four crossings, and it's homotopically isolated:



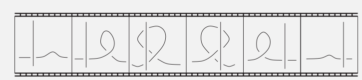
We just need to calculate its image under the movie.

Happily, the cone construction tells us that the 'all positive smoothings' diagram on one side of a Reidemeister III move is taken, with coefficient one, to the 'all positive smoothings' diagram on the other side.

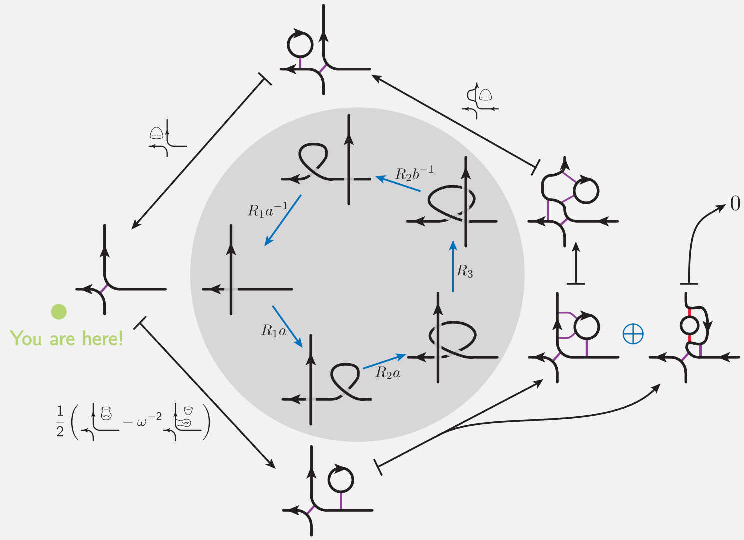


Thus the sign of MM10 is $1^8 = 1$.

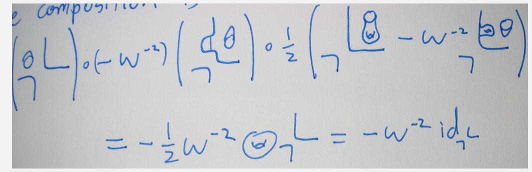
MM8



MM8 is the second hardest of the movie moves involving RIII, but it turns out to barely depend on the details of the RIII map. We calculate the image of a homotopically isolated element of the initial complex.



Following the maps around the circular movie, starting at the left, we obtain the following composition:



Again, the disoriented theory gets the sign right!

Theorem (M&W)

All the movie moves come out right in at least one orientation. At $\omega = 1$ we see the sign problem Jacobsson observed, but at $\omega = i$ movie moves really are equivalences.

What about all the other orientations?

Confusions

The disoriented cobordism category has some defects.

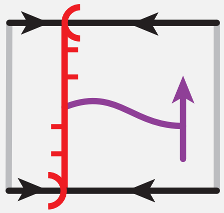
- There are no cobordisms from the empty diagram to a circle with two clockwise disorientation marks.
- If we extend the invariant to disoriented tangles, there's no nice equivalence allowing us to slide a disorientation mark past a crossing.

Introducing some new morphisms called 'confusions' solve both of these problems.

Definition

Confusions are points on a disorientation line at which the 'fringe' changes side. They have a spin framing, recorded with a (possibly twisted) ribbon connecting the confusion to a 'reference framing'.

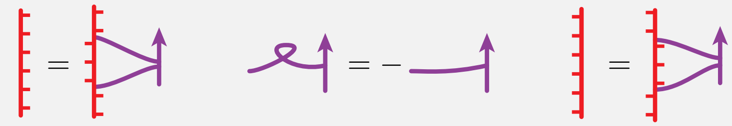
Thus the simplest appearance of a confusion is



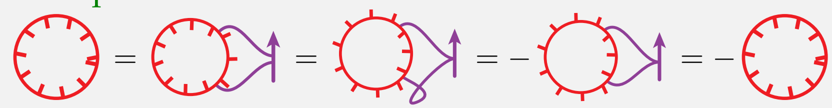
This is a map between two disoriented strands, which changes the preferred direction of the disorientation mark.

Confusion rules

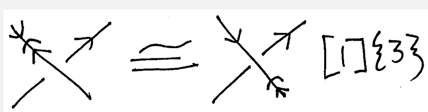
We can create and annihilate confusion pairs, according to the following rules.



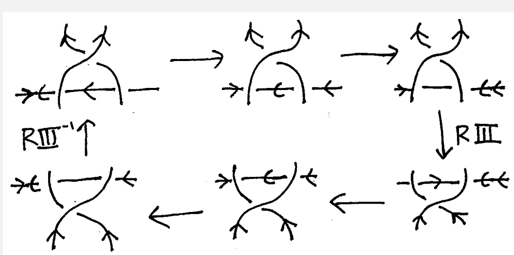
Example



We can now prove that there is a isomorphism of complexes which allows us to slide a disorientation mark through a crossing. (At the expense of an overall grading and degree shift.)

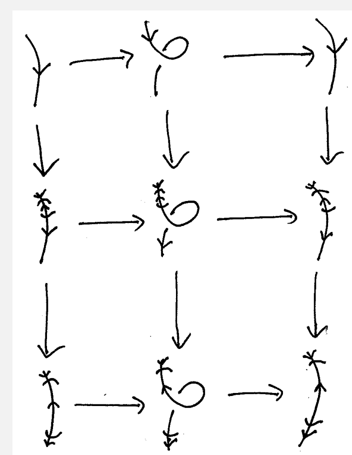


The disorientation slide isomorphisms commute with Reidemeister moves. (Warning – this hasn't been checked carefully!)



Now it's easy to show all the other orientations of movie moves are equivalent to the ones we've checked.

Example



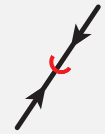
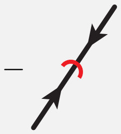
Recovering the original theory

Theorem
The complex associated to a knot or link is isomorphic to the complex constructed in the original theory.

Proof.
For each disoriented circle, fix an isomorphism with both the anticlockwise and clockwise oriented circles. (This involves up-to-sign choices). In the anticommutative cube associated to a knot, replace each diagram, using these isomorphisms, with a diagram in the 'standard' orientation; the orientation of a circle is determined by its nesting depth. This cube differs from the cube in the original theory simply by a sprinkling of units, and so gives an isomorphic complex. □

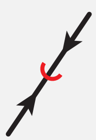
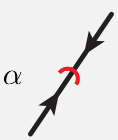
Decategorifying

- Our original motivation was to find a suitable modification of Bar-Natan's cobordism category which 'decategorified' to the disoriented $\mathfrak{su}_2$ skein theory. (See, for example, Kirby and Melvin.)
- There we have the relation

 $= -$ 

reflecting the fact that the standard representation of $\mathfrak{su}_2$ is *anti-symmetrically* self-dual.

- To recover this in our theory, we introduce an additional $\mathbb{Z}/2\mathbb{Z}$ grading on morphisms; the parity of the number of confusions. Now the decategorification consists of $\mathbb{Z}[q, q^{-1}, \alpha]$ modules, with $\alpha^2 = 1$.
- A confusion provides an isomorphism between the two diagrams above, but it is in the non-zero $\mathbb{Z}/2\mathbb{Z}$ grading, so in the decategorification we have the equation

 $= \alpha$ 

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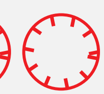
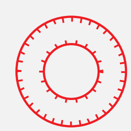
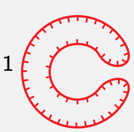
Disorientation relations are consistent

Example
We can create a disorienting seam, split it in two, then annihilate both parts:

$\omega =$  $= \omega^{-1}$   $= \omega^{-1}\omega^2$

Alternatively, we could create a pair, join them, and then annihilate:

$1 =$  $= \omega^{-1}$  $= \omega^{-1}\omega$

◂ return to disorientation relations