Disoriented and confused: fixing the functoriality of Khovanov homology.

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1 What's wrong with Khovanov homology?

- It's almost functorial
- ... but not quite
- ... and it ought to be!

2 How do we fix it?

- Disorientations
- Movie moves
- Calculations
- Confusions

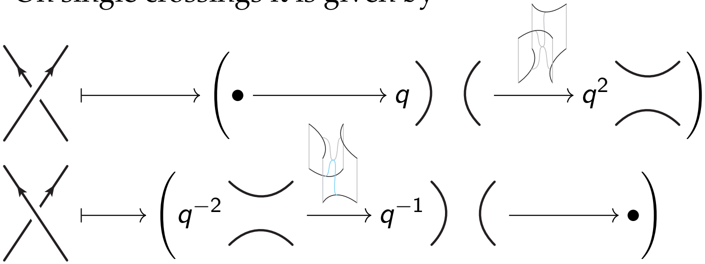
3 Odds and ends.

- Recovering the original theory
- Decategorifying
- Conclusions

What is Khovanov homology?

Khovanov homology is a map from tangles to up-to-homotopy complexes of (matrices of) cobordisms.

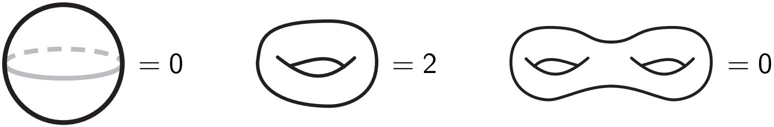
- On single crossings it is given by



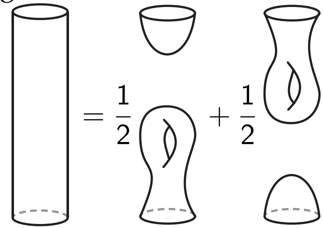
- It is a map of planar algebras: to compose two tangles in a planar way, take the tensor product of the corresponding complexes, combining objects and morphisms using the specified planar operation. details...

We need to impose some relations on cobordisms in order to make this a tangle invariant.

- Closed surface relations:

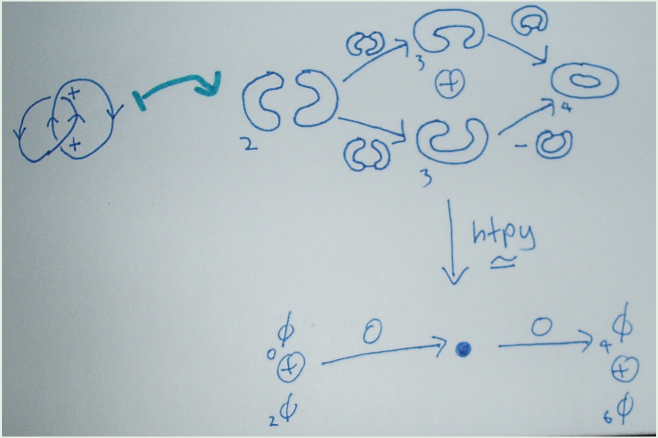


- The "neck cutting" relation:



Example

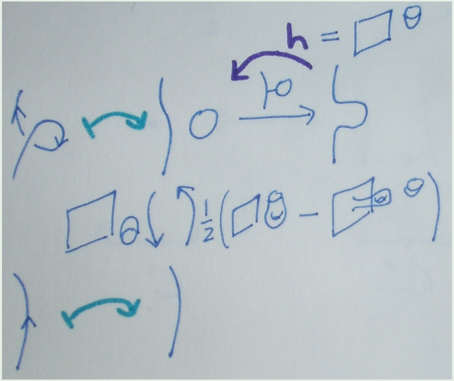
The hopf link.



Why is it actually an invariant of tangles?

We need to construct homotopy equivalences between the complexes on either side of each Reidemeister move

Example



Khovanov homology is almost functorial

So far, I've described an invariant associated to tangles.
We can try to make Khovanov homology functorial, associating to a cobordism between two tangles some chain map between the associated complexes.
Link cobordisms can be given presentations as 'movies'. Each frame of a movie is a tangle diagram. Between each pair of frames, one of the 'elementary movies' takes place:

- a Reidemeister move, in either direction
- the birth of death of a circle
- a morse move between two parallel arcs

We need to assign chain maps to each of the elementary movies.

- All the morse moves are easy; there are obvious cobordisms implementing them.
- To each Reidemeister move, we assign the chain map we constructed when showing that the two sides of the Reidemeister move were homotopically equivalent complexes.

To assign a chain map to an arbitrary link cobordism, we choose a movie presentation, and compose the chain maps associated to each elementary piece. Is this well defined?

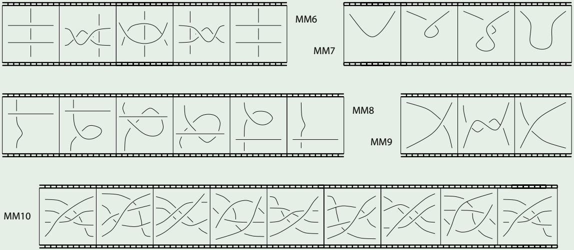
... but not quite

Theorem (Carter and Saito)

Two movies are presentations of the same link cobordism exactly if they are related by a sequence of 'movie moves'.

Example (Movie moves 6-10)

Each movie here is equivalent to the 'do nothing' movie.



Thus to check our proposed invariant of link cobordisms is well defined, we 'only' need to check that we assign the same chain map (up to homotopy equivalence!) to either side of each movie move.

Theorem (Bar-Natan, 2004)

The two sides of a movie move agree up to sign!

Theorem (Jacobsson, 2002)

The signs don't come out right. You can shuffle them around, but not make them go away.

... and it ought to be!

It would be nice if Khovanov homology really were functorial.

- Functors are good!
- You could identify generators in the Khovanov homologies calculated from two different presentations of a knot.
- Khovanov's construction of a categorification of the coloured Jones polynomial would be easier.
- You could build a doubly monoidal 4-category out of Khovanov homology.
- It may help Bar-Natan's 're-embeddability' argument for mutation invariance work.

How do we fix it?

To fix the sign problems in Khovanov homology, we'll make two modifications to the 'target category' of cobordisms.

disorientations Objects and cobordisms will be 'piecewise oriented', with 'disorientation lines' where the orientations disagree.

confusions Extra morphisms called 'confusions' fix some defects in the category, and make proofs manageable. They are 'spinorial' objects.

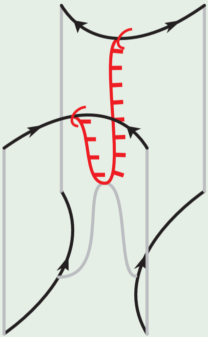
Disorientations

We'll replace the unoriented cobordism category previously used with a category of 'disoriented cobordisms'.

Objects Non-crossing arcs embedded in a disc, each piecewise oriented. Each 'disorientation mark' separating oppositely oriented intervals also has a preferred direction.

Morphisms Surfaces are piecewise oriented, with 'disorientation' lines marking the boundaries between regions with opposite orientations. Each disorientation line has a 'fringe', indicating a preferred side.

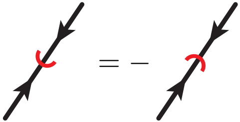
Example



In the oriented regions, we impose the usual cobordism relations. We also need some rules for removing closed disorientation lines, and reconnecting parallel disorientation lines.

Motivation

- Our original motivation was to find a suitable modification of Bar-Natan's cobordism category which 'decategorified' to the disoriented $\mathfrak{su}_2$ skein theory. (See, for example, Kirby and Melvin.)
- There we have the relation



reflecting the fact that the standard representation of $\mathfrak{su}_2$ is *anti-symmetrically* self-dual.

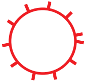
Disorientation relations

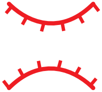
Fix a parameter ω, such that $\omega^4 = 1$.

- At $\omega = 1$, we recover the old theory by forgetting all orientation data. (We also recover the sign problems!)
- At $\omega = i$, we'll have functoriality!

Introduce some relations on disorientations:


$= \omega$

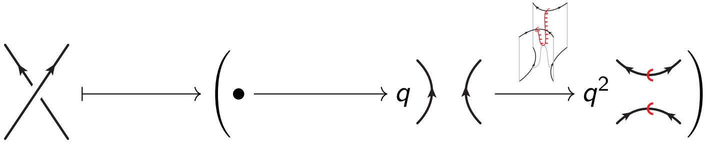

$= \omega^{-1}$

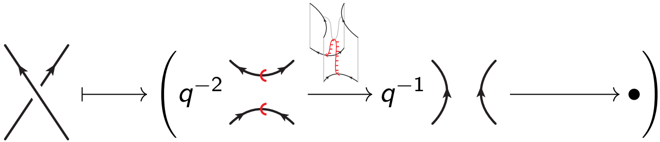

$= \omega^{-1}$


These are consistent!

Modifying the tangle invariant

Now tangles are mapped to (up-to-homotopy) complexes of disoriented cobordisms. It's obvious where to put the seams in, if we want to preserve orientation data away from crossings.





Disorientation marks face to the right, relative to the direction of the crossing.

Theorem (M&W)

This is still an invariant of tangles. We'll see all the homotopy equivalences for Reidemeister moves soon!

Movie moves

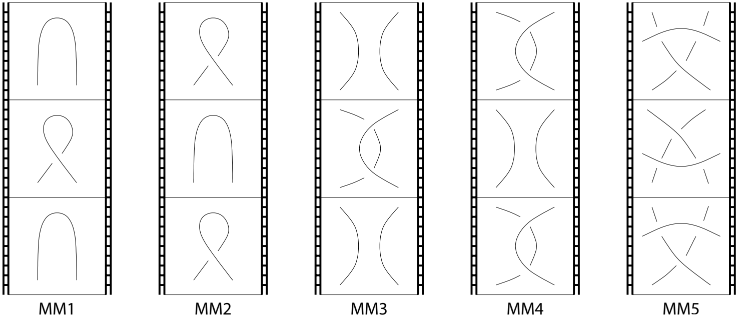
Now we need to check 15 movie moves. These come in several types.

Inverses These almost trivial moves insist that the time reverse of a Reidemeister move is also its inverse.

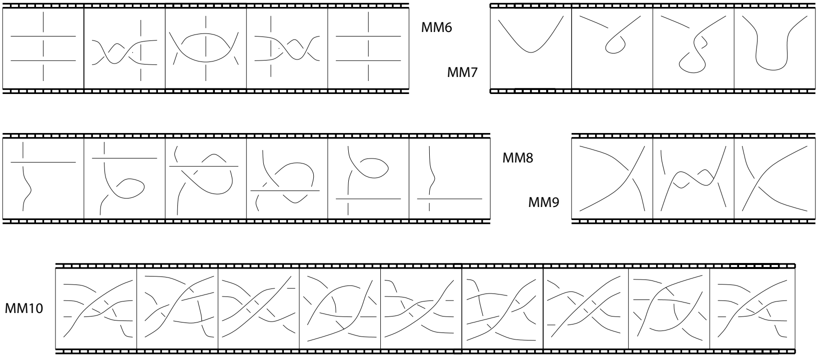
Circular clips These 'circular' clips should be equivalent to the identity. These include the 3 'hard' clips that involve a type III Reidemeister move.

Non-reversible clips These pairs of clips should be equivalent, when read either up or down.

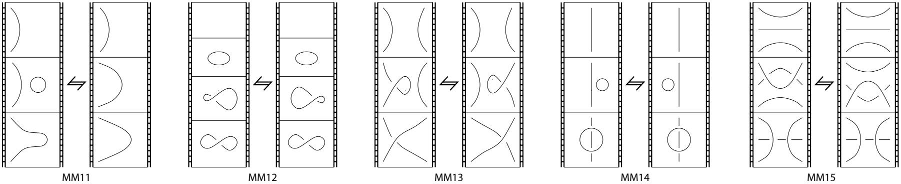
Inverse moves



These are boring; we know these are identities, because the two successive steps are a homotopy equivalence and its inverse.



These are 'hard'; moves 6, 8 and 10 involve the third Reidemeister move.



- Each pair of clips should give the same map, whether read up or down.
- These ones don't seem so bad, but there are lots of sign problems lurking here!
- Often there's a sign problem one way but not the other.

Jacobsson's sign tables

Jacobsson reported sign problems in almost every move!

MM	J♯	$\pm$
6	15	-
7	13	+
7 (mirror)	13	-
8	6	-
8 (mirror)	6	+
9	14	-
9 (mirror)	14	+
10	7	+

MM	J♯	$\downarrow$	$\uparrow$
11	9	+	+
12	11	-	+
12 (mirror)	11	+	+
13	12	-	+
13 (mirror)	12	+	-
14	8	+	-
15	10	-	+

We can calculate the corresponding table for the disoriented theory, as a function of ω.

- At $\omega = 1$, we recover the tables above.
- At $\omega = i$, all the signs agree.

What about all the orientations!?

- At this point it appears we need to check many orientations of each of these movie moves; up to 16 in the worst case.
- For now, we'll ignore this, and just check the signs for one oriented representative of each movie move.
- Later, the introduction of 'confusions' will deal with the rest.

Bar-Natan's argument

Bar-Natan gave a simple proof that Khovanov homology is well-defined up-to-sign:

- Certain tangles are *simple*, in that the automorphism group of the associated complex consists only of multiples of the identity.
- Each of movie moves 1-10 starts and ends with a 'simple tangle', and so must be a multiple of the identity.
- (Movie moves 11-15 can be done easily by hand.)

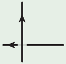
In our situation, many small tangles are still 'simple' in this sense, although now there are more units in our coefficient ring: $\pm 1, \pm i$. We'll make use of this often.

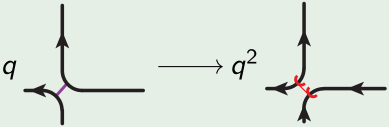
Detecting the sign

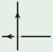
We can now easily detect the sign associated to a movie move.

- Cobordisms between loopless diagrams are all in non-positive grading.
- Because of the grading shifts in the complex associated to a tangle diagram, homotopies must be in strictly positive grading.
- This means not many homotopies are possible. We call a direct summand of an object in a complex *homotopically isolated* if there are no possible homotopies in or out.

Example

The initial (and final) frame of MM8 is , whose associated complex is



Neither of the objects have loops, so both objects are *isolated*. If f :  $\rightarrow$  is homotopic to the identity, it must be the identity on the nose; $f - I = dh + hd = 0$.

We can often detect the sign associated to a movie move by choosing an isolated summand in the complex, and observing its image under the movie move.

Calculations

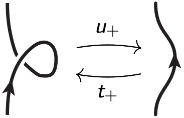
It's now time to do the real work! We need to

- calculate explicit chain maps corresponding to Reidemeister moves.
  - These are unique up to a unit.
  - We can easily write these down for the R1 and R2, but R3 will take some work; we'll use Bar-Natan's cone construction to organise this.
- detect the signs for each movie move, in at least one orientation,
- and explain away all the other orientations!

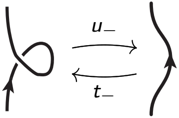
Twist maps

The twist maps implement the Reidemeister I moves. There are two variations.

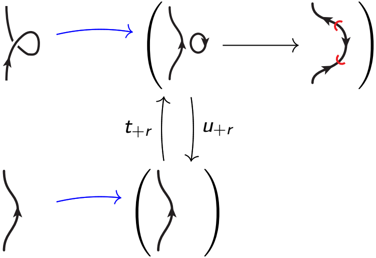
Positive twist



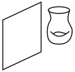
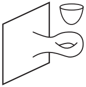
Negative twist

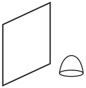


The positive twist map is

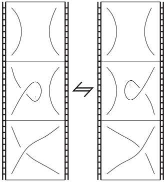


where t_{+r} and u_{+r} are given by

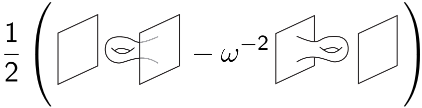
$t_{+r} = \frac{1}{2}\Big($  $- \omega^{-2}$  $\Big)$

$u_{+r} =$ 

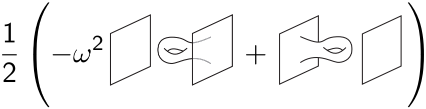
MM13



Each side of MM13 consists
of a twist move followed by a morse move.
Reading down the left side, we get



and on the right



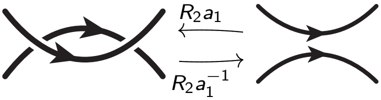
Thus we see the two sides of MM13 differ by a sign of $-\omega^2$!

Skip to MM8...
Skip other movie moves...

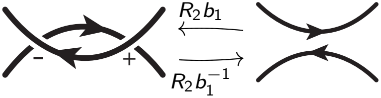
Tuck maps

We have the distinguish between the braid-like and non-braid-like R2 moves.

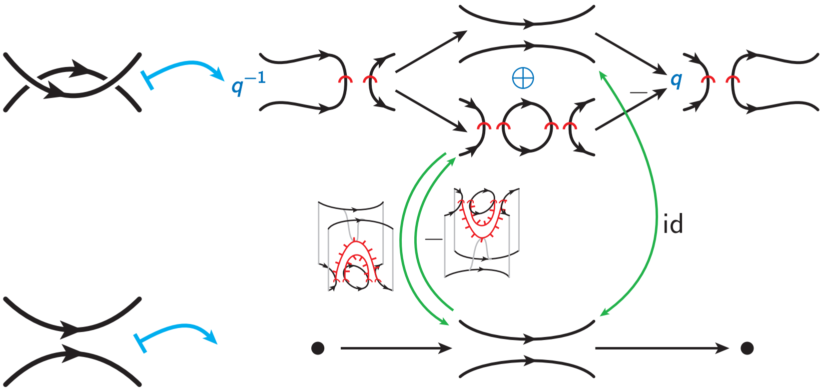
R2a



R2b

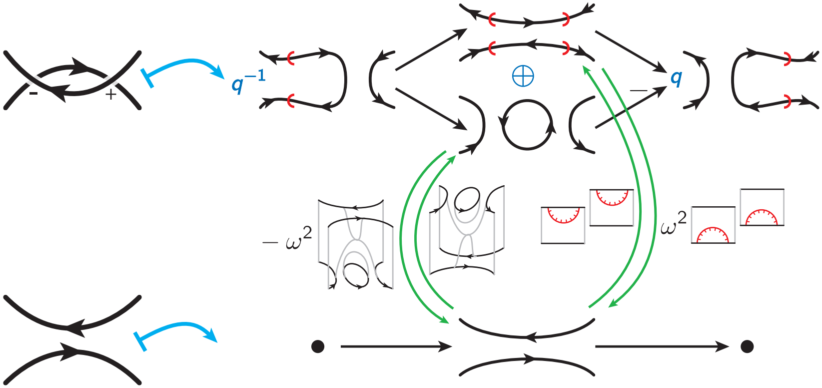


R2a maps



Skip other moves...

R2b maps



Skip R3 moves...

The cone construction, and R3

Obtaining the R3 map takes some work! We follow through Bar-Natan's proof of R3 invariance, keeping track of the explicit homotopy equivalence being constructed.

Lemma

The R2 untuck moves are strong deformation retracts.

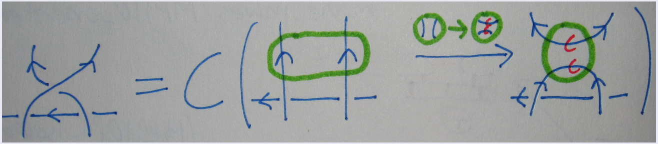
Lemma

If $f : A^\bullet \to B^\bullet$ is a chain map, and $r : B^\bullet \to C^\bullet$ is a strong deformation retract, then $C(rf) \simeq C(f)$.

details

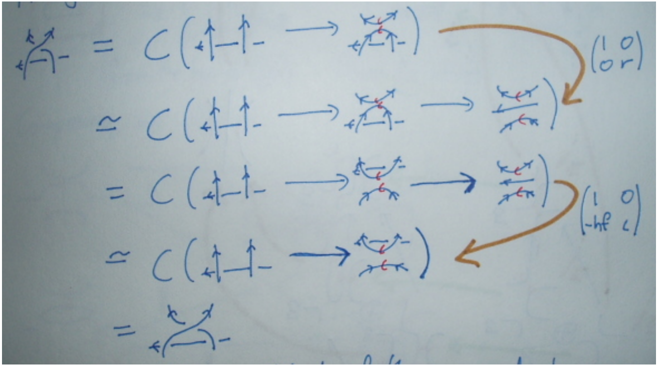
Lemma ('Categorified Kauffman trick')

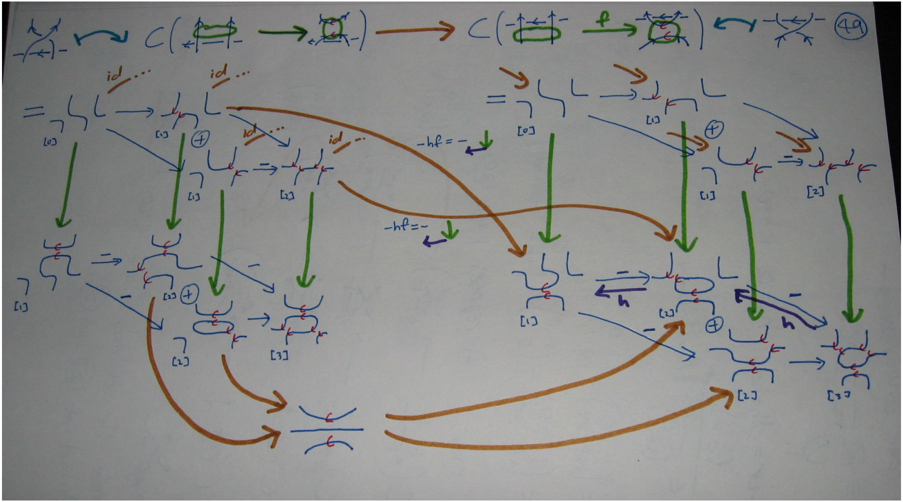
Each side of the R3 move can be realised as a cone over the morphism switching between two smoothings of the 'central' crossing.



We can then compose this morphism with the 'untuck' move, a strong deformation retract. Doing this to either side of the R3 move, we obtain the same cone!

Putting this together, we have




49
id ...
id ...
−hf = −
−hf = −
f
h
h

We will only need 2 facts about the R3 maps to calculate the movie move signs.

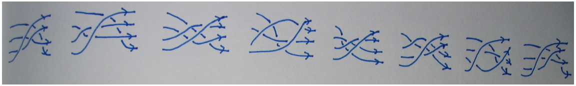
Lemma (for MM6, 8 and 10)

The top layer of the initial cone is mapped identically to the top layer of the final cone.

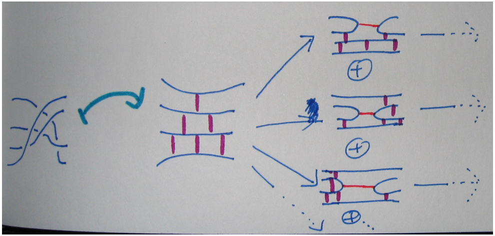
Lemma (for MM6 and 8)

The lowest and highest homological levels of the bottom layer are sent to zero.

MM10

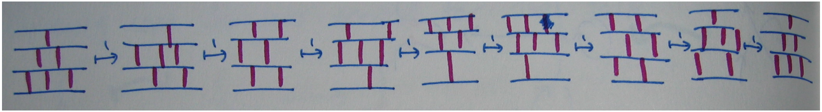


Look at the initial frame. The associated complex has one object in homological degree 0; the object we obtain from the 'positive smoothing' of each of the four crossings, and it's homotopically isolated:



We just need to calculate its image under the movie.

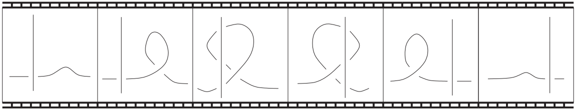
Happily, the cone construction tells us that the 'all positive smoothings' diagram on one side of a Reidemeister III move is taken, with coefficient one, to the 'all positive smoothings' diagram on the other side.



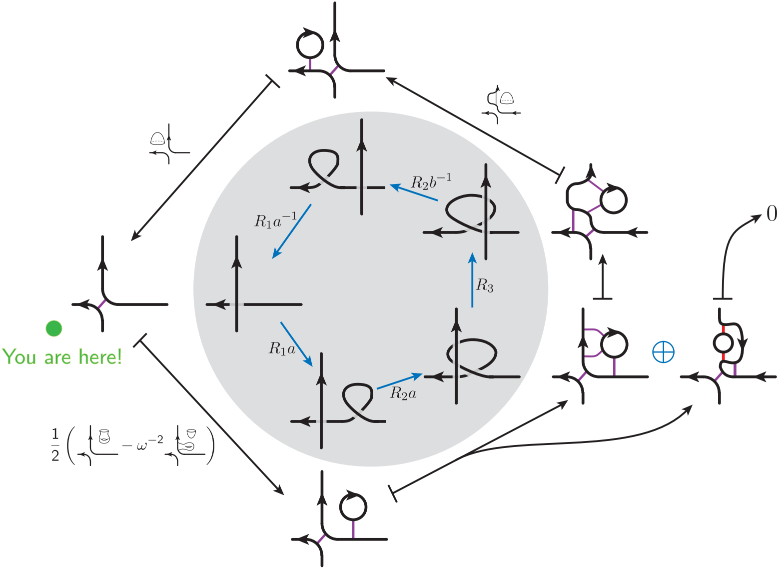
Thus the sign of MM10 is $1^8 = 1$.

Skip other movie moves...

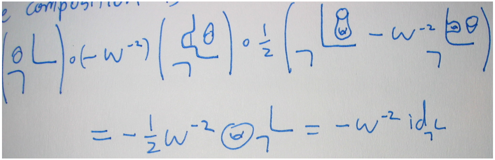
MM8



MM8 is the second hardest of the movie moves involving R3, but it turns out to barely depend on the details of the R3 map. We calculate the image of a homotopically isolated element of the initial complex.


R_2b^{-1}
R_1a^{-1}
R_3
R_1a
R_2a
0
You are here!
$\frac{1}{2}\left(\quad - \omega^{-2} \quad \right)$

Following the maps around the circular movie, starting at the left, we obtain the following composition:



Again, the disoriented theory gets the sign right!

Theorem (M&W)

All the movie moves come out right in at least one orientation. At $\omega = 1$ we see the sign problem Jacobsson observed, but at $\omega = i$ movie moves really are equivalences.

(Actually, at present I can't reproduce Jacobsson's signs for MM6; perhaps there's an orientation or mirror image issue?)

Question

What about all the other orientations?

Confusions

The disoriented cobordism category has some defects.

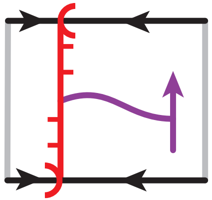
- There are no cobordisms from the empty diagram to a circle with two clockwise disorientation marks, for example.
- If we extend the invariant to disoriented tangles, there's no nice equivalence allowing us to slide a disorientation mark past a crossing.
- We have to deal with all the orientations of movie moves separately.

Introducing some new morphisms called 'confusions' solves all of these problems.

Definition

Confusions are points on a disorientation line at which the 'fringe' changes side. They have a spin framing, recorded with a (possibly twisted) ribbon connecting the confusion to a 'reference framing'.

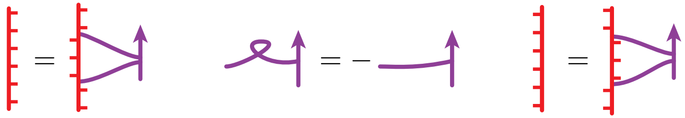
Thus the simplest appearance of a confusion is



This is a map between two disoriented strands, which changes the preferred direction of the disorientation mark.

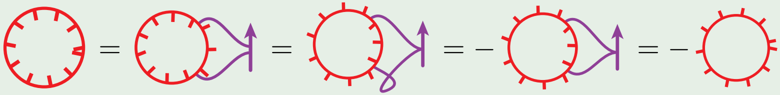
Confusion rules

We can create and annihilate confusion pairs, according to the following rules.

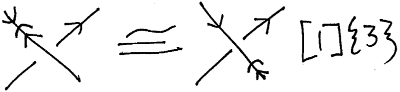


Example

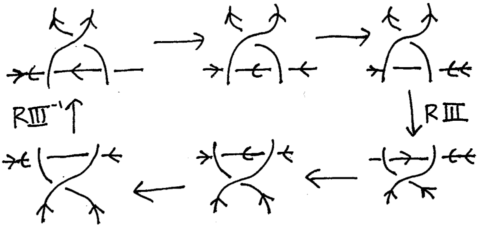
We can create a confusion pair, slide one around a circle, and then annihilate them.



There is now an isomorphism of complexes which allows us to slide a disorientation mark through a crossing. (At the expense of an overall grading and degree shift.)

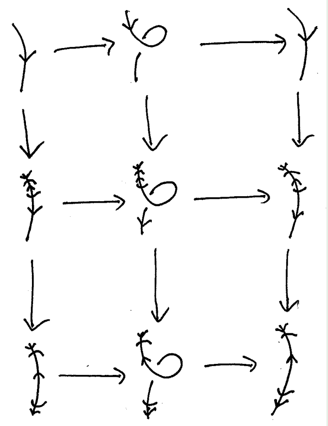


The disorientation slide isomorphisms commute with Reidemeister moves. (Warning – this hasn't been checked carefully!)



Now it's easy to show all the other orientations of movie moves are equivalent to the ones we've checked.

Example



Recovering the original theory

Theorem

The complex associated to a knot diagram is isomorphic to the complex constructed in the original theory.

Proof (sketch!?)

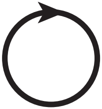
For each disoriented circle, fix an isomorphism with both the anticlockwise and clockwise oriented circles. (This involves up-to-sign choices). In the anticommutative cube associated to a knot, replace each diagram, using these isomorphisms, with a diagram in the 'standard' orientation; the orientation of a circle is determined by its nesting depth. This cube differs from the cube in the original theory simply by a sprinkling of units, and so gives an isomorphic complex. □

Decategorifying

- We can "decategorify" a category, (not quite the usual way) obtaining an abelian group:

$$\mathcal{C} \rightsquigarrow \left\langle \mathrm{Obj}(\mathcal{C}) \;\middle|\; \begin{array}{c} A = B + C \text{ whenever} \\ A \cong B \oplus C \end{array} \right\rangle$$

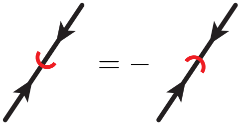
- What do we get? Just as with the unoriented cobordism category, we have the relation



$$= q + q^{-1}$$

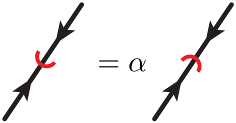
due to the neck cutting relation.

- To recover the relation



in our theory, we need to introduce an additional $\mathbb{Z}/2\mathbb{Z}$ grading on morphisms; the parity of the number of confusions. Now the decategorification consists of $\mathbb{Z}[q, q^{-1}, \alpha]$ modules, with $\alpha^2 = 1$.

- A confusion provides an isomorphism between the two diagrams above, but it is in the non-zero $\mathbb{Z}/2\mathbb{Z}$ grading, so in the decategorification we have the equation



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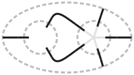
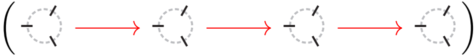
Conclusions

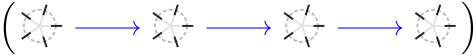
We've described a new setting for Khovanov homology, in a category of *disoriented cobordisms*.

- It is properly functorial with respect to link cobordisms.
- The complex associated to a knot diagram is isomorphic to the usual Khovanov complex, but not canonically so.
- Disoriented cobordisms decategorify to the disoriented $\mathfrak{su}_2$ skein theory.

Appendix: planar composition of complexes

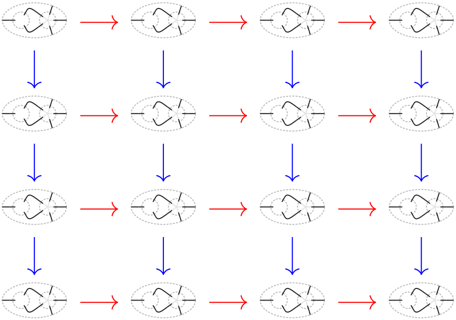
- Given a quadratic tangle,  and a pair of complexes associated to the inner discs,

$C_{\text{red}} =$ 

$C_{\text{blue}} =$ 

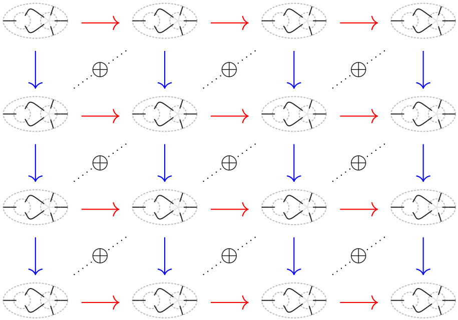
  we need to define a new complex associated to the outer disc.
- We'll imitate the usual tensor product operation on complexes, making use of the planar tangle to combine objects and morphisms.

- First construct a double complex.



- Each horizontal arrow is the planar composition of an original red arrow with the identity on the right disc.
- Each vertical arrow is the planar composition of an original blue arrow with the identity on the left disc.

- Then collapse the double complex to a complex, by taking direct sums along the anti-diagonals.



back to 'What is Khovanov homology?'

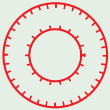
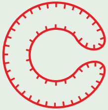
Appendix: Disorientation relations are consistent

Example

We can create a disorienting seam, split it in two, then annihilate both parts:

$\omega =$  $= \omega^{-1}$   $= \omega^{-1}\omega^{2}$

Alternatively, we could create a pair, join them, and then annihilate:

$1 =$  $= \omega^{-1}$  $= \omega^{-1}\omega$

◂ return to disorientation relations

Appendix: Deformation retracts

Definition

A chain map $r : B^\bullet \to C^\bullet$ is a *strong deformation retract* if there is a chain map $\imath : C^\bullet \to B^\bullet$ and a homotopy $h : B^\bullet \to B^{\bullet-1}$ such that $r\imath = \mathbf{1}_C$, $\imath r - \mathbf{1}_B = dh + hd$, and $rh = h\imath = 0$.

Lemma

If $f : A^\bullet \to B^\bullet$ is a chain map, and $r : B^\bullet \to C^\bullet$ is a strong deformation retract, then the cone $C(rf)$ is homotopic to the cone $C(f)$, via

$$C(f)^\bullet = A^{\bullet+1} \oplus B^\bullet \underset{\left(\begin{smallmatrix} 1 & 0 \\ -hf & \imath \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} 1 & 0 \\ 0 & r \end{smallmatrix}\right)}{\rightleftarrows}} A^{\bullet+1} \oplus C^\bullet = C(rf)^\bullet$$

Back to the cone construction.