KHOVANOV HOMOLOGY FOR 4-MANIFOLDS

SCOTT MORRISON

1. Introduction

Khovanov homology is a *categorical knot invariant.*

$$
\begin{tikzcd}
\boxed{\text{links embedded in } S^3} \arrow[r, "\mathrm{Kh}"] & \boxed{\text{chain complexes in graded vector spaces}} \arrow[r, "H_*"] & \boxed{\text{doubly graded vector spaces}} \\
\boxed{\text{isotopies}} \arrow[r, "\mathrm{Kh}"] & \boxed{\text{chain equivalences}} \arrow[r, "H_*"] & \boxed{\text{isomorphisms}} \\
\boxed{\text{2-isotopies / 3-isotopies}} \arrow[r, "\mathrm{Kh}"] & \boxed{\text{homotopies}} &
\end{tikzcd}
$$

(Actually, this description is over-optimistic; the honest statements are for links in B^3 rather than S^3. We'll return to this later.)

The definition is combinatorial. We choose presentation of links (in terms of crossings), isotopies (in terms of Reidemeister moves) and 2-isotopies (in terms of Roseman/Carter-Saito movie moves), and associate appropriate algebraic data to the pieces, along with instructions for assembling the pieces.

In fact, Khovanov homology is somewhat stronger than the diagram above indicates. In the second row, we can instead write

$$\text{cobordisms} \xrightarrow{Kh} \text{chain maps} \xrightarrow{H_*} \text{linear maps}$$

References

- Khovanov, "A categorification of the Jones polynomial", `arXiv:math.QA/9908171`.
- Bar-Natan, "Khovanov's homology for tangles and cobordisms", `arXiv:math.GT/0410495`.
- Clark, Morrison & Walker, "Fixing the functoriality of Khovanov homology", `arXiv:math.GT/0701339`.

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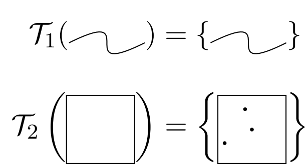
How do we build a 4-manifold invariant out of these pieces? With the right language in place, there's an idiomatic construction. The above data is sufficient to build a "disklike" 4-category, and then we can follow a standard recipe to produce an invariant of 4-manifolds. I'm not going to attempt to describe in general what a disklike 4-category is, or give the recipe in full generality. (Although you can find this in our "Blob complex" paper, `arXiv:1009.5025`, or by coming to Teichner's course this semester.) Instead, I'll just describe directly the 4-category associated to Khovanov homology, and implement the recipe for this particular example.

2. 4-categories

It's actually pretty exciting that Khovanov homology lets us build a 4-category. In fact, this is one of the very few constructions of a higher (≥ 4) category that is 'dimension-specific'. (That is, you couldn't just substitute 7 for 4 everywhere and obtain a 7-category!)

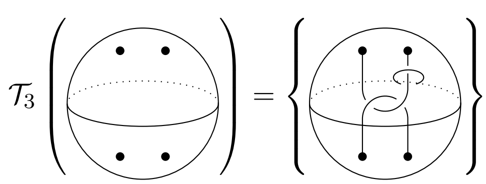
2.1. **Tangles and cobordisms.** To begin, let's first describe the closely related 4-category $\mathcal{T}$ of tangles and cobordisms. A disklike 4-category should associate a set to each k-ball with an appropriately decorated boundary, for each $0 \leq k \leq 4$. We'll describe the first few levels before actually explaining what 'appropriately decorated' means.

$$\mathcal{T}_0(\bullet) = \{\bullet\}$$



That is, $\mathcal{T}$ associates to each 0-ball the singleton set containing that 0-ball, and similarly for each 1-ball. To any arbitrary 2-ball (i.e. any manifold which happens to be homeomorphic to the standard 2-ball), $\mathcal{T}$ associates the collection of finite subsets of the interior.

So far, we haven't had any decorations on the boundaries. Once we reach 3-balls, $\mathcal{T}$ not only associates a set to each 3-ball, but also to each 3-ball decorated with a finite subset in its boundary. In particular, it associated the set of all tangles with the given boundary points.



Note that these tangles are actual particular embedded tangles, and are not considered up to isotopy.

At this point we can stop and explain what the boundary conditions are in general. For each k-ball, consider some way of splitting the boundary into two or more $(k-1)$-balls, and labelling each of these with an element of the corresponding set generated by $\mathcal{T}_{k-1}$. The labels must agree on the boundaries between the $(k-1)$ balls. This constitutes a possible boundary condition for the k-ball, although later we'll tweak this slightly to remove the dependence on the actual splitting.

Thus $\mathcal{T}_4$ should associate some set to every 4-ball with a link in its boundary (because if we split the boundary into 3-balls, each must be labelled by some tangle, and at the boundaries these tangles must match up). For the 4-category of tangles and cobordisms we associate the set of all surfaces bounding the given link, *modulo isotopy* fixing the boundary. Recall that we didn't allow isotopies at lower dimensions; this is an instance of the common theme that 'top level morphisms' satisfy different axioms than the lower ones.

Note in this example that the $\mathcal{T}_k$ are actually *functorial*: given a homeomorphism of a k-ball, there is an obvious isomorphism between the associated sets, given by carrying everything along. For a general n-category we'd insist on this as part of the data.

So far we haven't actually defined a particularly interesting example — you might quickly realise that for any $0 \leq k \leq n$ there is an n-category of codimension k submanifolds, which generalises this $n = 4, k = 2$ example.

We've now essentially specified the *data* of our 4-category. What structure do these sets $\mathcal{T}_k(X)$ have? Given

- two k-balls X and Y, with
$$\partial X = S \cup U \qquad\qquad \partial Y = S^{op} \cup V$$
where S, U and V are $k-1$ balls, and
- fields $x \in \mathcal{T}_k(X)$, $y \in \mathcal{T}_k(Y)$ such that
$$\partial x = s \bullet u \qquad\qquad \partial y = s \bullet v$$
where $s \in \mathcal{T}_{k-1}(S)$, $u \in \mathcal{T}_{k-1}(U)$, $v \in \mathcal{T}_{k-1}(V)$,

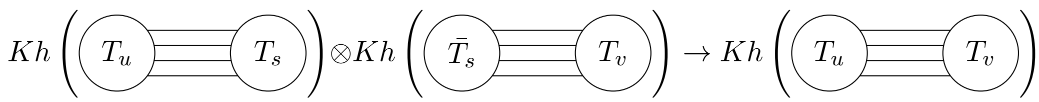
we notice that there's a natural way (just gluing codimension 2 submanifolds together!) to glue the fields x and y together, to obtain a field $x \bullet y$ in $\mathcal{T}_k(X \cup_S Y)$. Note that this gluing operation is strictly associative as we glue more balls together (unlike in many definitions of higher categories, which keep track of more and more associators).

2.2. **The Khovanov 4-category.**

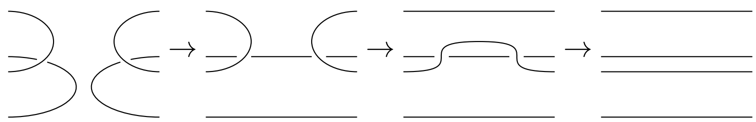
We define the lower levels of the Khovanov 4-category exactly as we did for the category of tangles and cobordisms:
$$Kh_k = \mathcal{T}_k \qquad \text{for } 0 \leq k \leq 3.$$
All we change is the definition of Kh_4 — to a 4-ball with a link in its boundary, we simply associate the Khovanov homology of the link! (Note

that values of Kh_4 are vectors spaces rather than sets as before; this is a 4-category enriched in vector spaces, in fact doubly graded vector spaces.)

We now have more work to do, however, to specify the gluing operation on 4-balls. Now that the definition is not just topological, the gluing operation has to reflect some of the algebraic structure from Khovanov homology. What we need are linear maps



for arbitrary tangles T_u, T_s and T_v. These linear maps can be induced by the cobordism that cancels T_s with $\bar{T}_s$. Here's an example:



One can verify, using the functoriality of Kh, that this defines an associative gluing rule.

In fact there's a functor $\mathcal{T} \to Kh$, which is just the identity up to dimension 3, and sends

$$\underbrace{\Sigma : \emptyset \to L}_{\text{a surface bounding a link}} \quad \mapsto \quad \underbrace{Kh(\Sigma) : \mathbb{C} \to Kh(L)}_{\text{essentially an element of } Kh(L)}$$

The existence of this functor tells us that we can think of the Khovanov 4-category as a braided monoidal 2-category.

Now, what do we do with a 4-category?

3. 4-manifold invariants

Given a 4-manifold W, we define the *poset of ball decompositions* $\mathcal{D}(W)$. The elements of the poset are ways to write W as a union of balls, $W = \bigcup_{i \in I} B_i$. A technical detail: there should exist a sequence of maps, $\bigsqcup B_i \to W_1 \to W_2 \to \cdots \to W_k = W$., so that each map glues together a pair of opposite codimension 0 submanifolds of the boundary. The arrows in the poset are ways to glue some of the balls together into a larger ball.

Any 4-category then defines a functor from the poset of ball decompositions. In the case of Khovanov homology, enriched in vector spaces, we get

$$Kh : \mathcal{D}(W) \to \mathbf{Vec}$$

Each ball decomposition is sent to the tensor product of the corresponding vector spaces for the individual balls. In the presence of boundary conditions, we also need to take an (enormous!) direct sum over all consistent boundary conditions. For Khovanov homology, this means that for each ball decomposition, we look at the boundaries of the 4-balls and see that these boundaries can be divided up into 3-ball along which the 4-balls are being glued. In each of these 3-balls we draw some tangle, so that each 3-sphere now contains a link. To this picture we associate the tensor product of the Khovanov homologies of all the links, and to the ball decomposition we associate the direct sum of the tensor products, over all ways of drawing tangles in the boundaries. Corresponding to arrows in the poset we have maps of the corresponding vector spaces, simply given by the gluing maps of the 4-category.

Finally we're ready to define the actual invariant $\underrightarrow{Kh}(W)$ as the colimit of this functor. Thus we assemble a huge vector space, which is a direct sum indexed by $\mathcal{D}(W)$ of the values of Kh, modulo the relation that following any arrow of the poset gives an identification. That is,

$$\underrightarrow{Kh}(W) = \left(\bigoplus_{b \in \mathcal{D}(W)} HKh(b) \right) / \{x - Kh(g)(x) | x \in Kh(b), g : b \mapsto b'\}$$

You might rightly worry about this definition: we start with such a huge and flabby infinite dimensional space, is there any hope that the quotient collapses down to something small enough (finite dimensional even?) to be computable, but not just trivial? There's some evidence we should be optimistic. It's not too hard to establish the following to two facts

Lemma 3.1. $\underrightarrow{Kh}(S^4) = \mathbb{C}$

Lemma 3.2. $\underrightarrow{Kh}(B^4; L) = HKh(L)$

(That is, the TQFT invariant of the 4-ball with a link in its boundary is just the Khovanov homology of the link. It's easy to see that it must be some quotient of the Khovanov homology, and slightly harder to see that it's no smaller.)

The other reason is hope that this definition is a reasonable one is that it's just a special case of a uniform recipe that encompasses (the codimension 1 part of) Turaev-Viro invariants, Reshetikhin-Turaev invariants, Dijkgraaf-Witten invariants and more.

For example, given a 2-category and a surface, this construction produces a vector space, which is the usual Turaev-Viro space; obtaining the 3-dimensional part of the theory requires more work, and more conditions on the 2 category. We don't expect that construction to have an analogue for Khovanov homology (i.e. giving numerical invariants of 5 manifolds), although c.f. Witten's recent paper `arXiv:1101.3216`.

Any invariant of manifolds built in this way from an n-category $\mathcal{C}$ automatically satisfies some nice gluing formulas. Briefly, for an $(n-1)$-manifold Y, we can look at the invariant $\underrightarrow{\mathcal{C}}(Y \times I)$, and realise that this vector space actually has the structure of a category: objects are given by the boundary conditions on $Y \times \{0\}$ and $Y \times \{1\}$, and composition is via gluing two copies of I end to end. Moreover, if an n-manifold W has a copy of Y in its boundary, then $\underrightarrow{(\ \mathcal{C})}(W)$ is naturally a module over $\underrightarrow{\mathcal{C}}(Y \times I)$: the module action corresponds to gluing a collar on to Y. If W has two copies of Y in its boundary, then we have the following gluing formula

$$\underset{\to}{\mathcal{C}}(W \bigcup_{Y} \circlearrowleft) \cong \underset{\to}{\mathcal{C}}(W) \bigotimes_{\underset{\to}{\mathcal{C}}(Y)} \circlearrowleft.$$

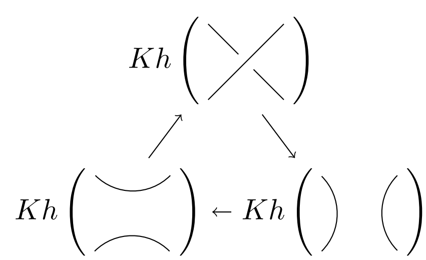
We've tried doing some small calculations based on this formula, for $B^3 \times S^1$, with some simple links in the boundary.

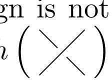
I'll just mention briefly that there's an integer-valued invariant (Rasmussen's s-invariant) that can be extracted from the usual Khovanov homology of a link. This invariant gives a lower bound for the genus of a surface $\Sigma \subset B^4$ bounding the link. It appears that this result will generalize fairly directly to the general case of a link in the boundary of an arbitrary 4-manifold.

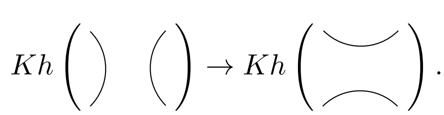
4. Functoriality in S^3

At this point we have to go back, and admit that the description of Khovanov homology given at the beginning was over-optimistic. We're going to have to get our hands dirty with some grungy details. In practice, the reason why Khovanov homology (for links in the 3-sphere) is computable is that it satisfies an exact triangle. Unfortunately, in our present understanding this exact triangle doesn't play well with functoriality.

Historically, the first version of Khovanov homology was for *unoriented* links, but it suffered from an unfortunate defect: it wasn't actually functorial, just "functorial up to sign". This means that the chain map associated to a cobordism of knots is not sufficiently well defined. If we modify that cobordism by an isotopy, the new chain map might not be homotopic to the old one, but instead could be homotopic to *minus* the old one. In this setting at least it's easy to describe the exact triangle.

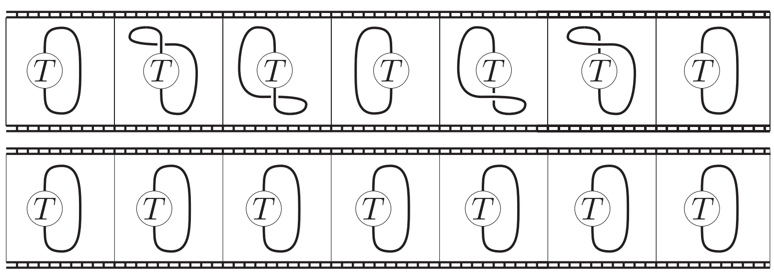


Each of the maps in the triangle is in fact induced by the obvious cobordism. Here the fact that these maps are only defined up to a sign is not so problematic; we can still calculate the isomorphism type of Kh 
from the (up to sign) map



Later, with David Clark and Kevin Walker, we modified the definition (now an invariant for oriented links) and fixed this problem. There's still one big problem, however. At the time, we only thought about links in B^3, rather than in S^3. If you stop and think about the TQFT constructions above, you'll see that functoriality in S^3 is essential. But what difference is there, actually? Well, up to an isotopy, every link avoids the north pole of S^3, so we can just extend the usual definition of the Khovanov complex at the level of links. Similarly every isotopy or cobordism or links generically avoids the north pole, so there's no problem associated chain maps to these. The difficulty arise because 2-isotopies do not avoid the north pole; in particular there are certain pairs of cobordisms that are *not* isotopic to each other in B^3, but become so in S^3. Thus, to work in S^3 we have to be sure that these cobordisms induce homotopic chain maps.

In fact, all we need to check is that the chain maps induced by the following movies are homotopic, for any tangle T with 2 boundary points.



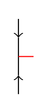
At this point, this seems hard!

Our hope is to extend the definition of Khovanov homology to so called 'disoriented links'; we expect that given such an extension, there will be an exact triangle that is nice and functorial. Using this exact triangle and the five lemma, we'll be able to proved that these movies give homotopic maps by inducting on the number of crossings in the tangle T.

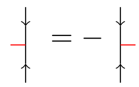
What are 'disoriented links', and why do we expect them to be helpful? We haven't actually mentioned this so far, but Khovanov homology is closely related to the Jones polynomial, via

$$J(L)(q) = \sum_{i,j} (-1)^i q^j \dim HKh_{i,j}(L),$$

and hence to the representation theory of $U_q(\mathfrak{su}_2)$. The standard representation of $SU(2)$ is self-dual, and in particular antisymmetrically self-dual. This means that in any diagrammatic version of the representation theory, we expect to see a box implementing this isomorphism, which might look something like

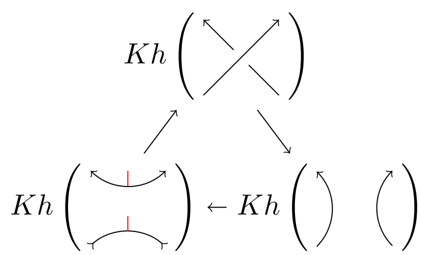


and which picks up a minus sign under 180 degree rotation:



A disoriented link is then a 'piecewise oriented' link, along with a choice at each break of a choice of side.

Our fix to Khovanov homology incorporated a certain categorification of these diagrammatics in the construction, but it remains unclear if we can actually extend the definition to allow disorientations on the input links. If so, we expect that there will be an exact triangle



and hope that this exact triangle will be natural with respect to cobordisms outside the indicated region.

5. Does $Kh(W^4)$ have an exact triangle?

[[I ran out of time to write about this in detail ...]]

No, it appears not; we got as far in our $B^3 \times S^1$ calculations to see this before giving up. This isn't unexpected, because our TQFT construction is not an exact functor. It must be time to learn about the blob complex, which is a 'derived analogue' of the TQFT construction, and where the exact triangle for Khovanov homology should survive as a spectral sequence.