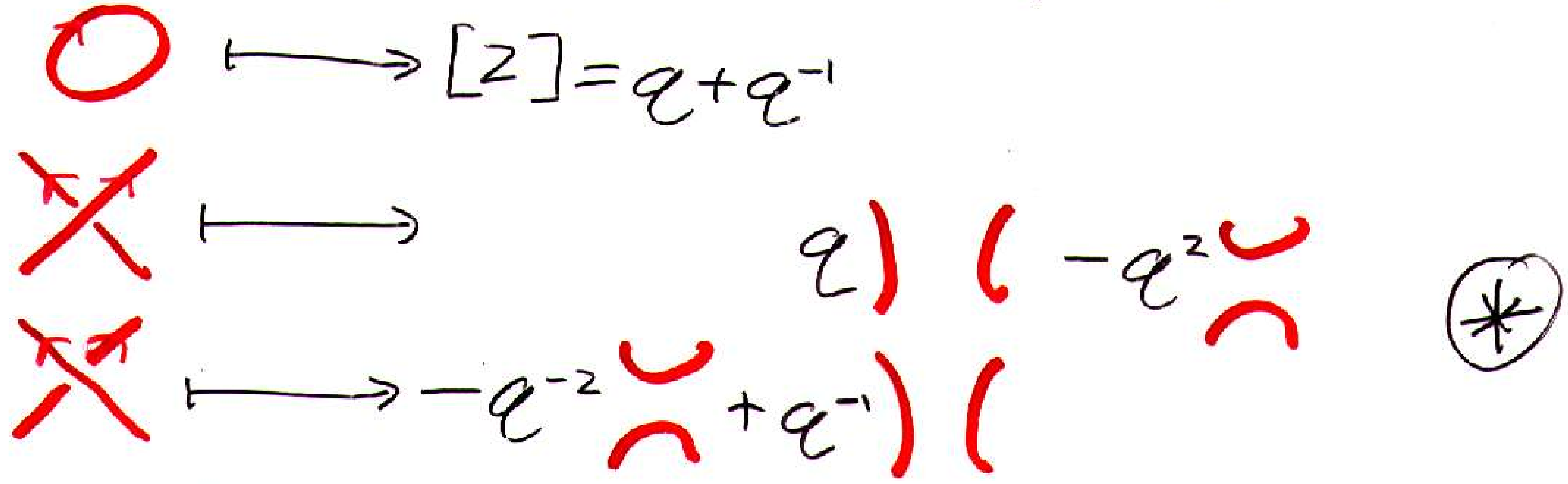
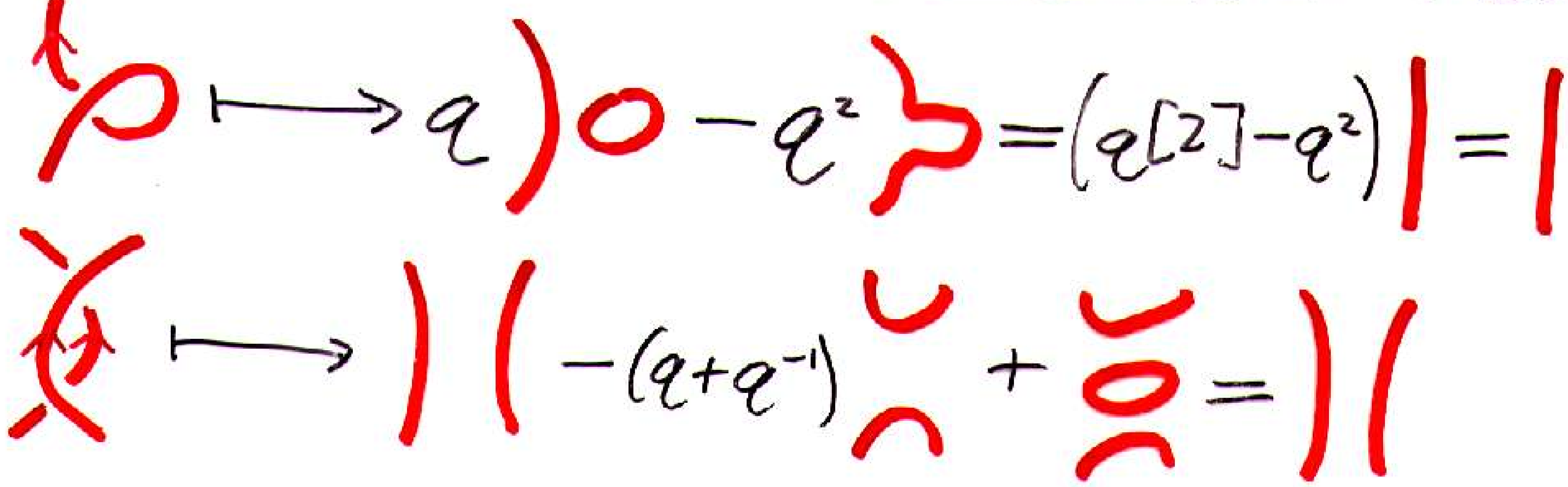
The Jones Polynomial

(a map from the planar algebra of tangles to the $\mathbb{Z}[q,q^{-1}]$ – Temperley-Lieb planar algebra.)

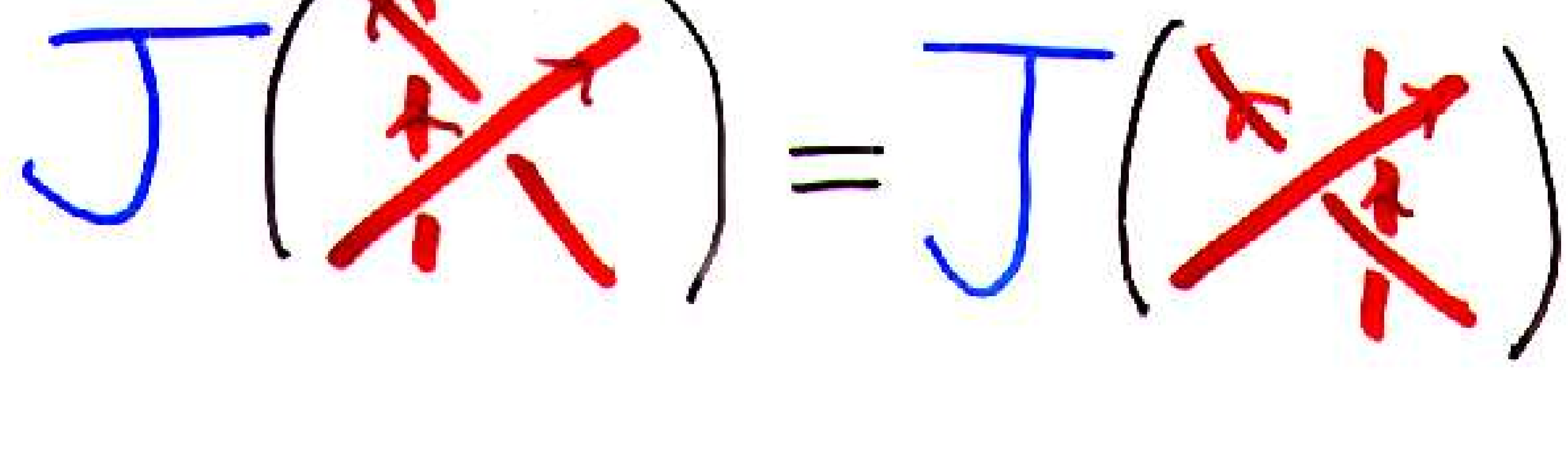


...riance under the first two Reidemeister moves—

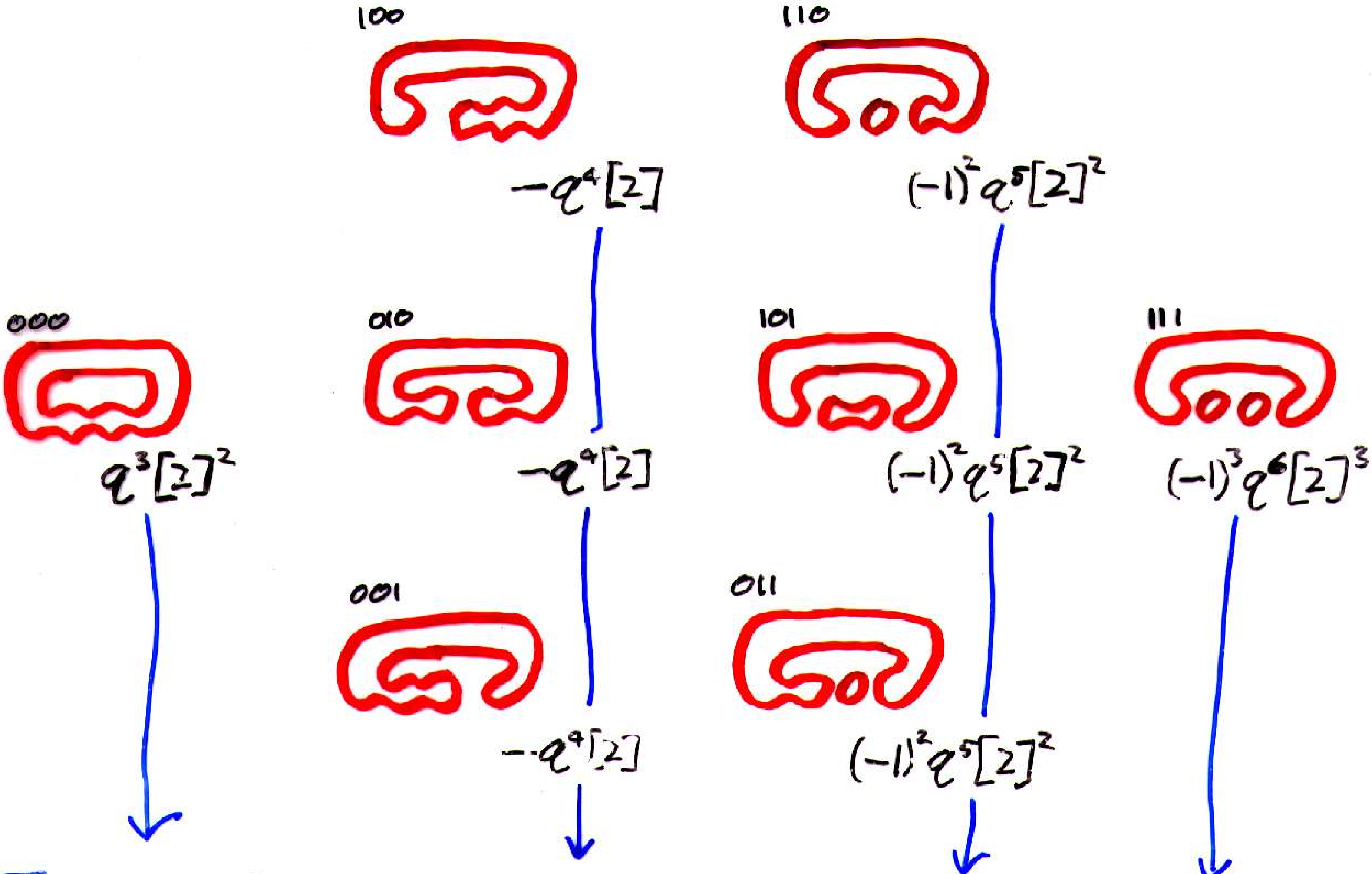


And a miracle occurs—

as long as we choose the coefficients in (*) so we have invariance under R1 and R2, invariance under R3 comes for free!



$J\Big($  $\Big)$



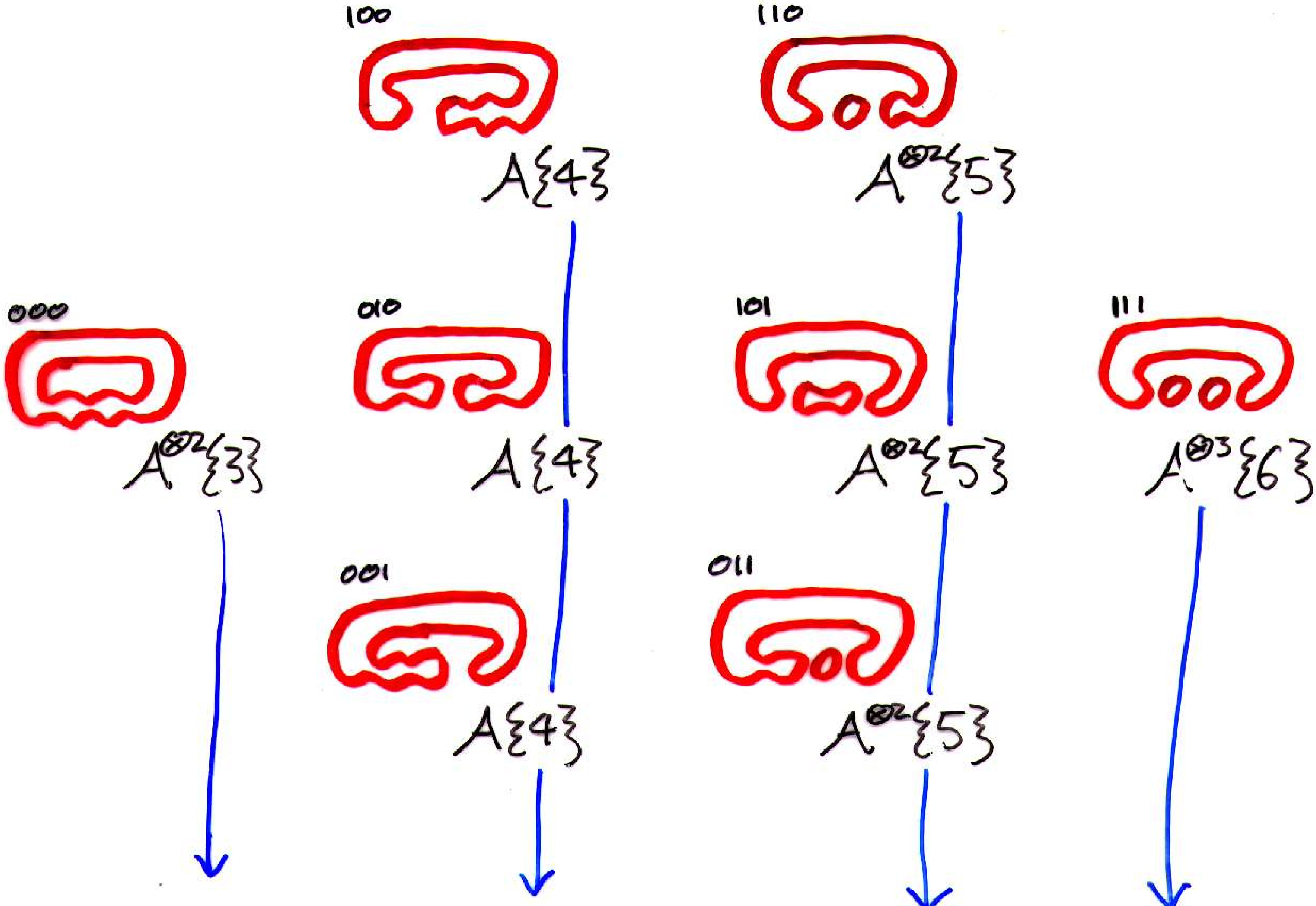
$$J = q^3[2]^2 - (q^4[2]+q^4[2]+q^4[2]) + (3q^5[2]^2) - q^6[2]^3$$
$$= q + q^3 + q^5 - q^9$$

Next, introduce a <u>graded</u> <u>module</u> $A = R\{1\} \oplus R\{-1\}$
(where $\{n\}$ denotes a grading shift)

Then $q\dim A = q + q^{-1} = [2]$,

and, for example, $q\dim A^{\otimes 2}\{3\} = q^3[2]^2$

$J\Bigg($  $\Bigg)$



$C^0 = A^{\otimes 2}\{3\}$ $\quad C^1 = \bigoplus_3 A\{4\}$ $\quad C^2 = \bigoplus_3 A^{\otimes 2}\{5\}$ $\quad C^3 = A^{\otimes 3}\{6$

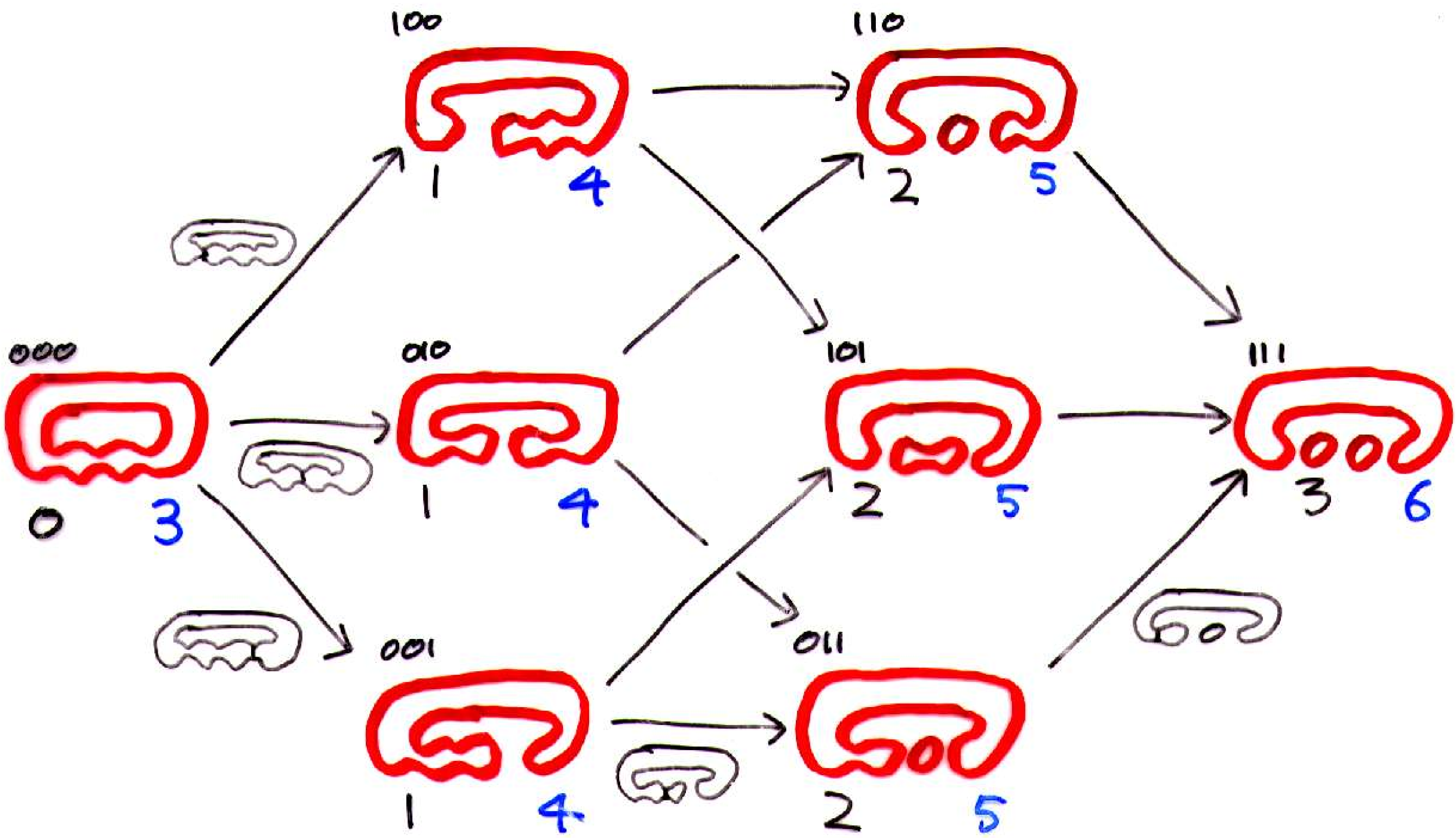
Then $J = \chi_q(C^*)$

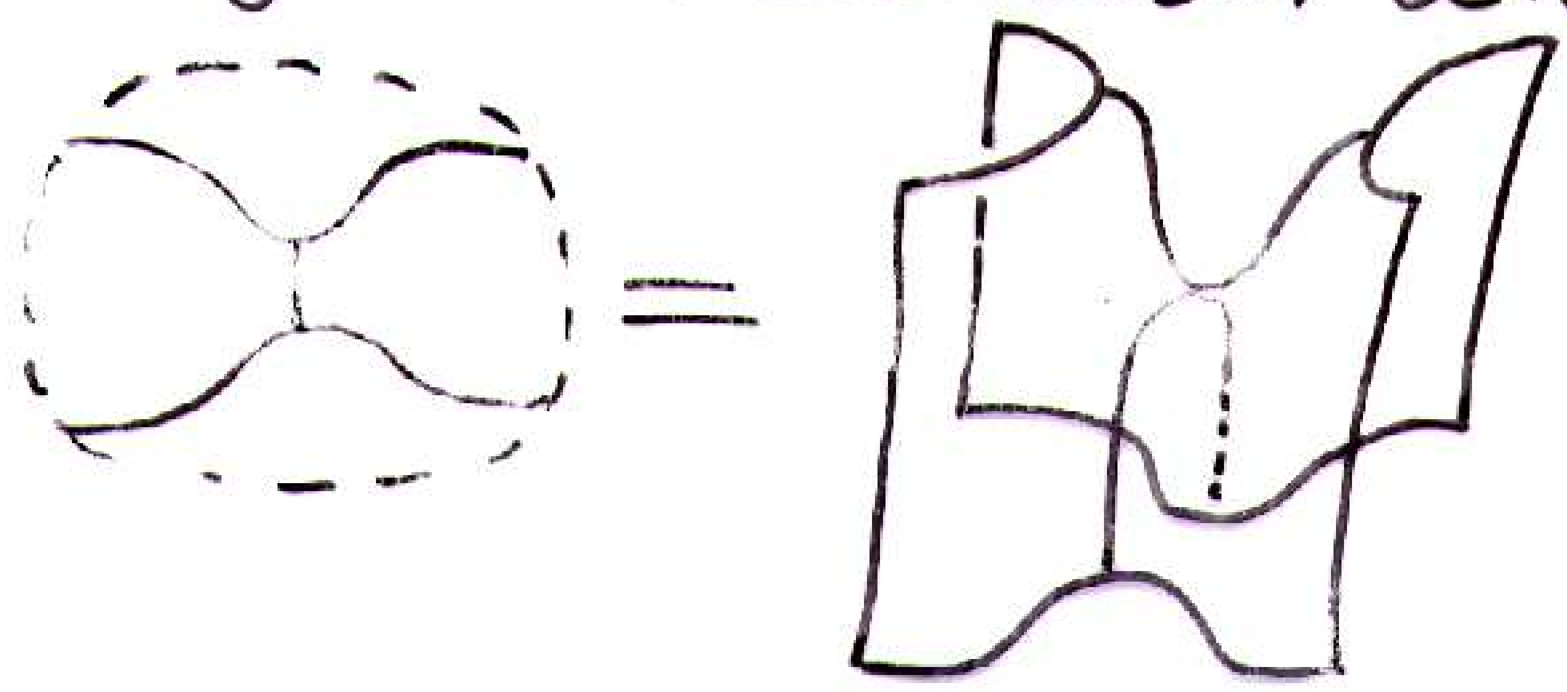
$$= q\dim C^0 - q\dim C^1 + q\dim C^2 - q\dim C^3$$

Writing J as an Euler characteristic gave the original motivation for Khovanov homology.

Kh()

Step 1: build the commutative cube



- This picture is a <u>commutative cube</u> in the category of 1+1 cobordisms
  - each vertex is a 1-manifold 
    - decorated with a degree and a grading
  - each edge is a cobordism between 1-manifolds
  - 

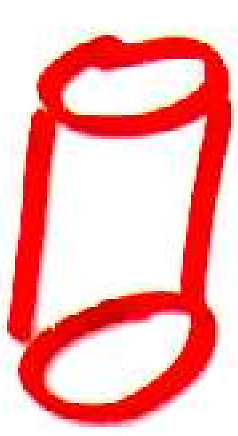
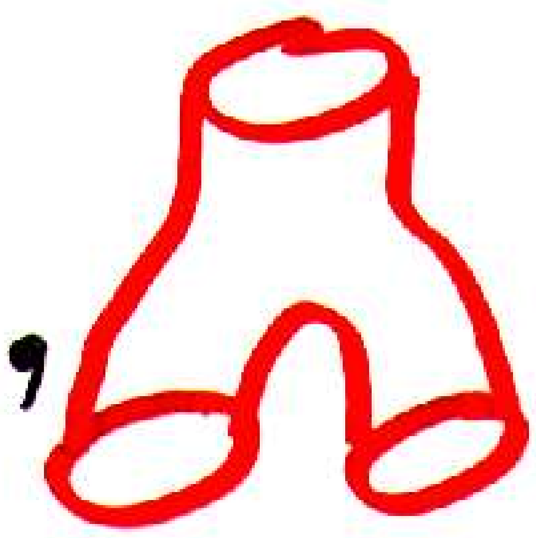
Kh()   Step 2: Apply a TQFT

$$\begin{tikzcd}
 & C_{100}\{4\} \arrow[r] \arrow[dr] & C_{110}\{5\} \arrow[dr] & \\
C_{000}\{3\} \arrow[ur] \arrow[r] \arrow[dr] & C_{010}\{4\} \arrow[ur] \arrow[dr] & C_{101}\{5\} \arrow[r] & C_{111}\{6\} \\
 & C_{001}\{4\} \arrow[ur] \arrow[r] & C_{011}\{5\} \arrow[ur] &
\end{tikzcd}$$

- This is a commutative cube in the category of graded R-modules.
  - A TQFT takes 1-manifold to R-modules

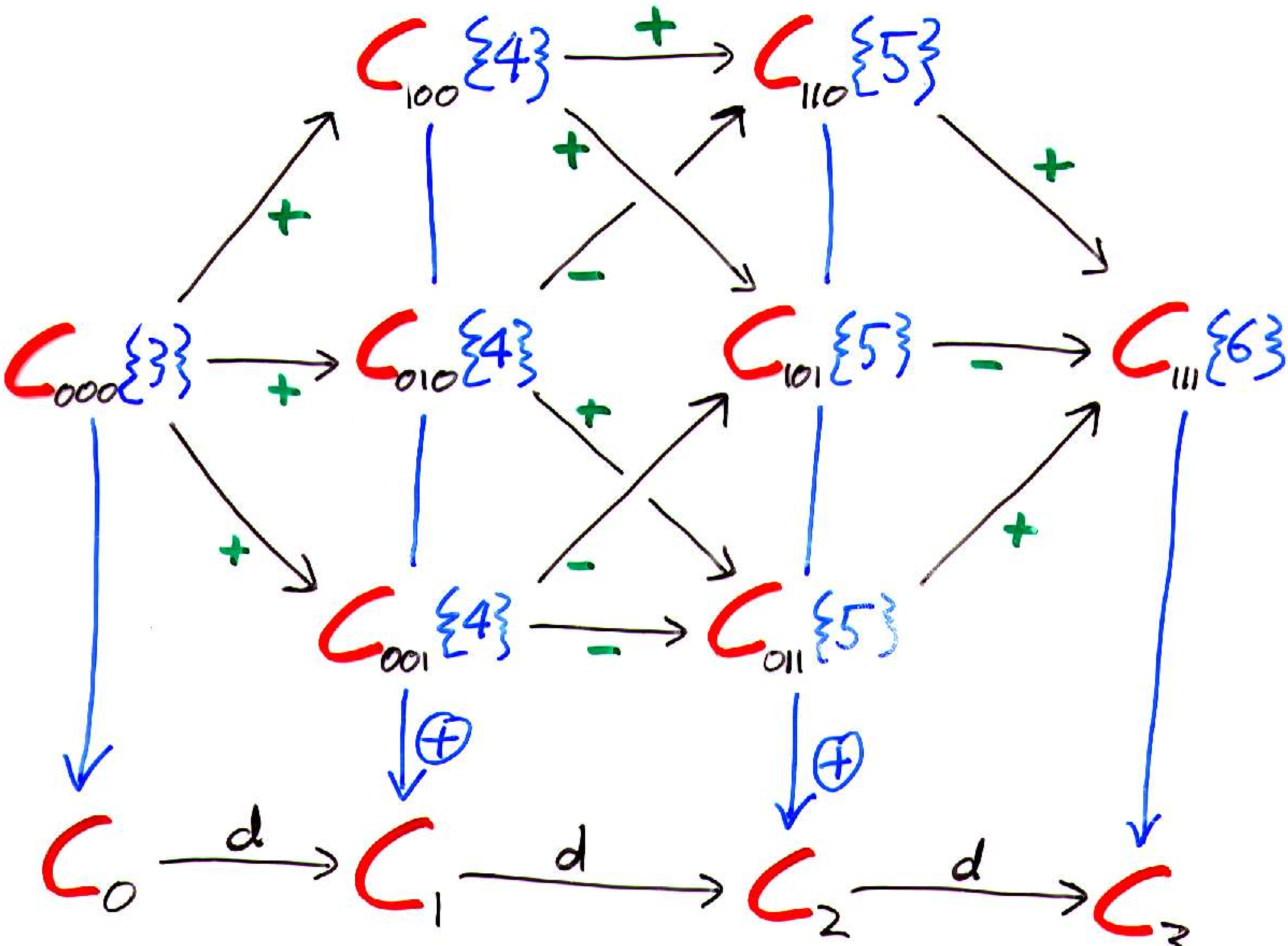
$$\underbrace{\bigcirc\ \bigcirc \cdots \bigcirc}_{k} \longmapsto A^{\otimes k}$$

  - and cobordisms to maps between them

 $\longmapsto id_A$,  $\longmapsto (A \otimes A \xrightarrow{m} A)$, etc

Kh()

Step 3: Sprinkle signs and collapse to a complex.

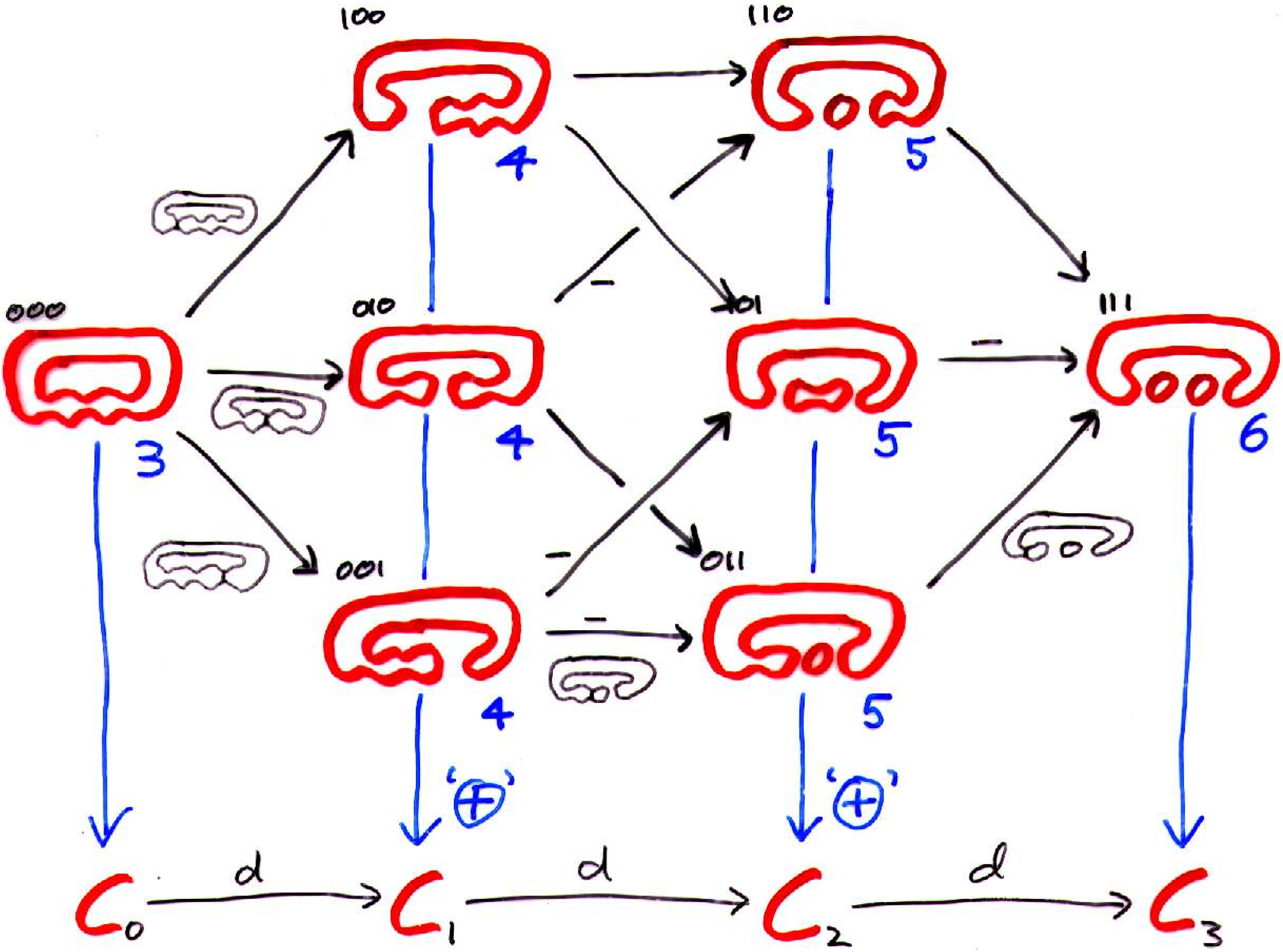


- This is a <u>complex</u> in the category of R-modules
  - $d^2 = 0$ since every square <u>anticommutes</u>

<u>Theorem</u> (Khovanov) The homology of this complex, Kh, is a knot invariant.

BN 

Forget the TQFT, sprinkle signs, and collapse to a complex using 'formal' direct sums.

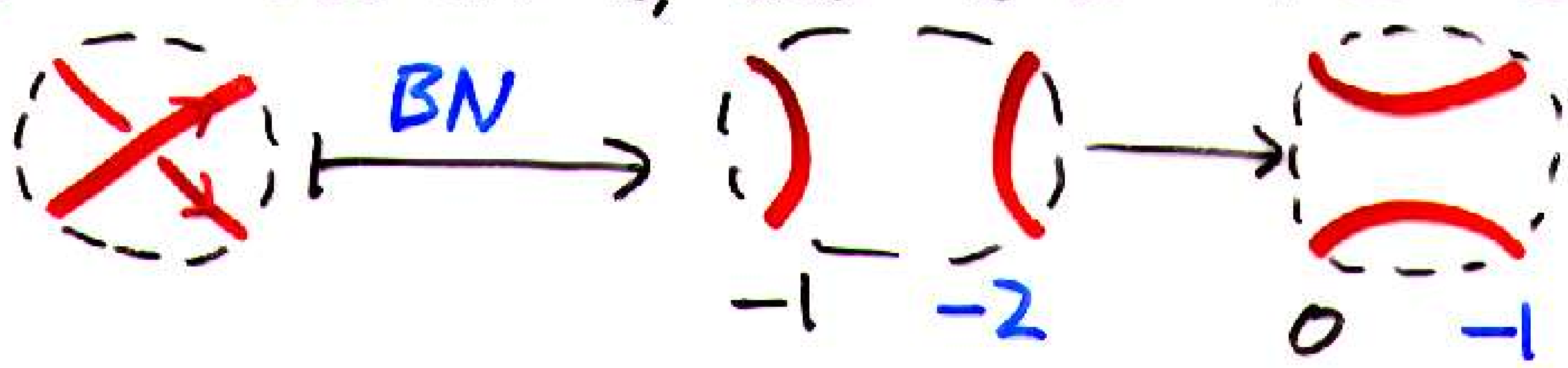


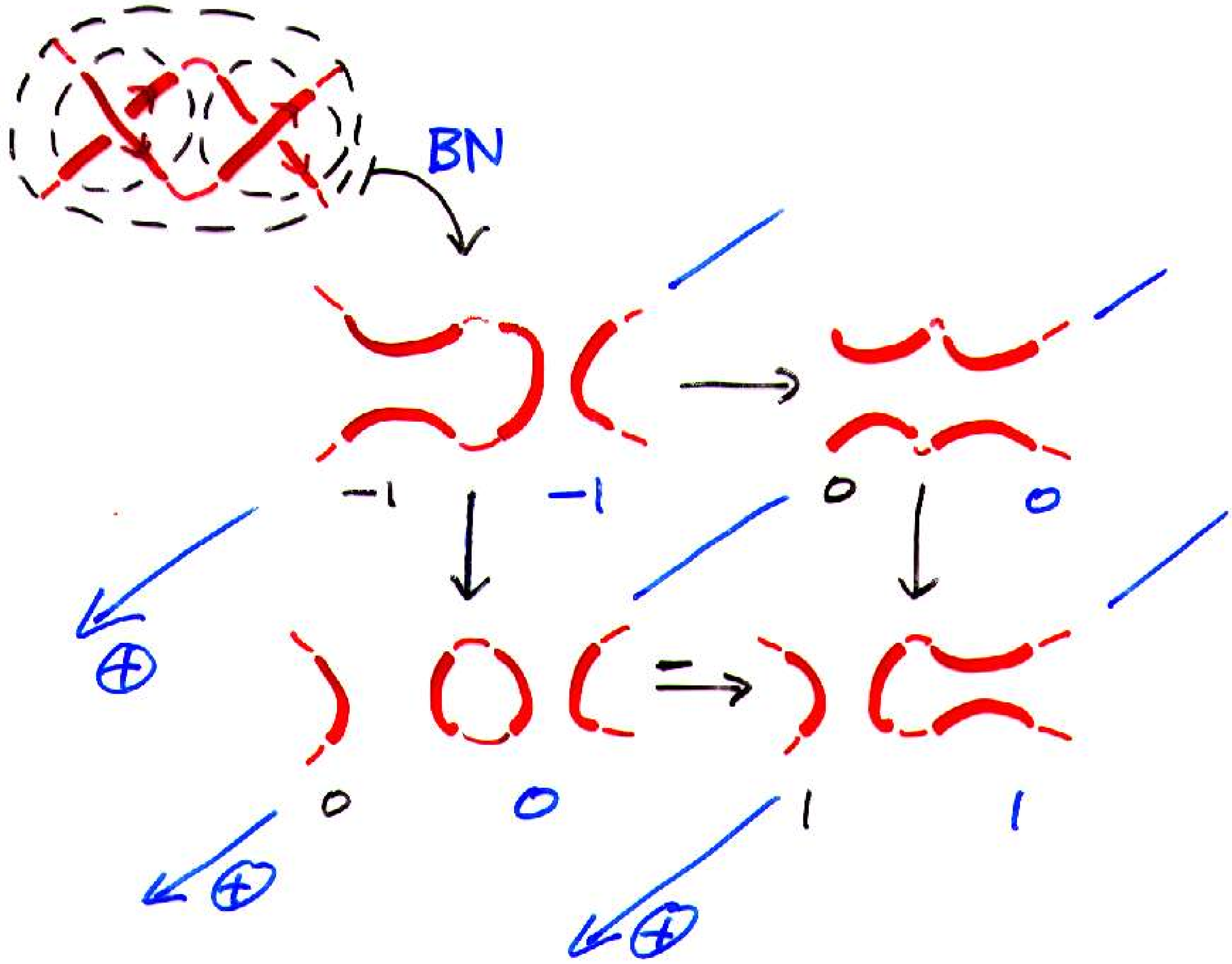
- This is a complex in the category of matrices of R-linear combinations of cobordisms.

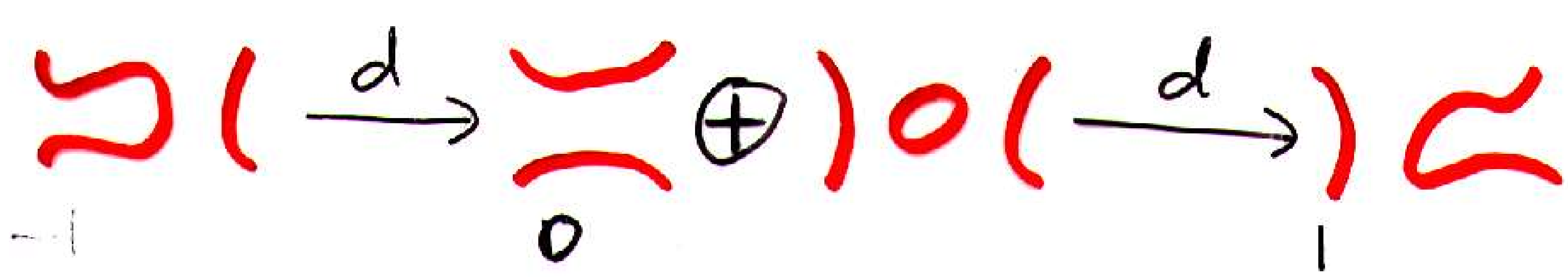
Q: Might the homotopy type of this complex be a knot invariant?

Bar-Natan's theory for tangles

- everything so far makes sense for tangles, as well a links.
- the theory is a map of planar algebras; to compose complexes of cobordisms, join up the cobordisms, and tensor the complexes.







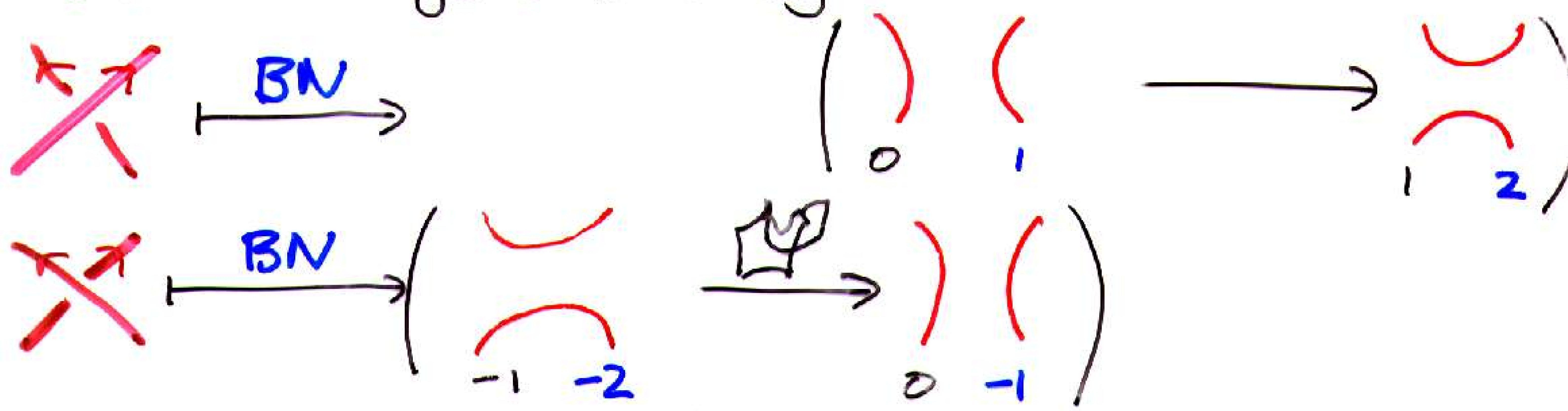
The story so far...

Bar-Natan's theory is a morphism of planar algebras

$$\text{Tangle-Diagrams} \xrightarrow{BN} \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}))$$

additive and graded!

defined on generators by



Kom(Mat(Cob)) is a planar algebra; take tensor products of complexes and matrices, and use the usual planar algebra operations on cobordisms.

We now need to show this descends to

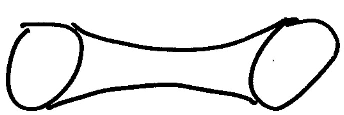
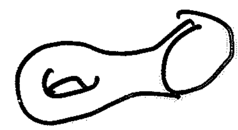
$$\text{Tangles} \cong \text{Tangle-Diagrams}/\text{Reidemeister moves} \xrightarrow{BN} \mathrm{Kom}/_{\mathrm{htpy}}(\mathrm{Mat}(\mathrm{Cob}/_{\sim}))$$

In fact, the maps in these complexes aren't honest cobordisms, but cobordisms modulo some relations:

"S":  $= 0$ "T":  $= 2$

"G":  $= 0$

and "Neck Cutting":

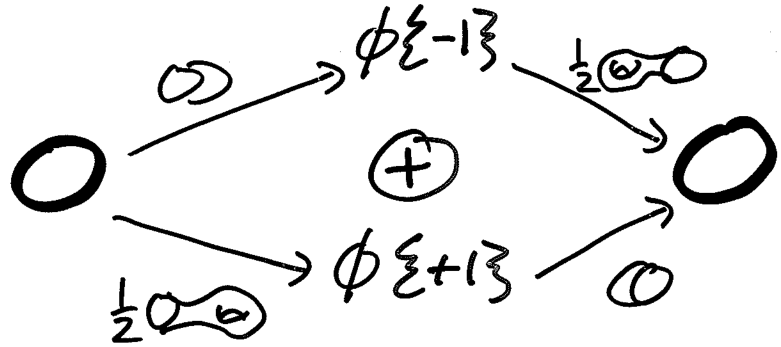
 $= \frac{1}{2}$   $+ \frac{1}{2}$  

(does it make any difference whether the cobordisms are 'abstract' or 'embedded'?)

The 'easy' proofs of isotopy invariance.

First, we need a consequence of the 'neck-cutting relation

namely that the object $\bigcirc$ is isomorphic to $\phi\{-1\} \oplus \phi\{+1\}$:



The composition $\bigcirc \longrightarrow \bigcirc$ is the identity by NC, while the composition $\phi\{-1\} \oplus \phi\{+1\} \longrightarrow \phi\{-1\} \oplus \phi\{+1\}$ is

using T, S and G.

Replacing $\bigcirc$ objects in a complex in this manner is called 'delooping'.

Second, we need the <u>reduction lemma</u>

If $\psi : b_1 \to b_2$ is an isomorphism, there's an isomorphism of complexes:

$$[C] \xrightarrow{\begin{pmatrix}\alpha \\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\psi & \delta \\ \gamma & \varepsilon\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{(\mu \ \nu)} [F]$$

$$\cong$$

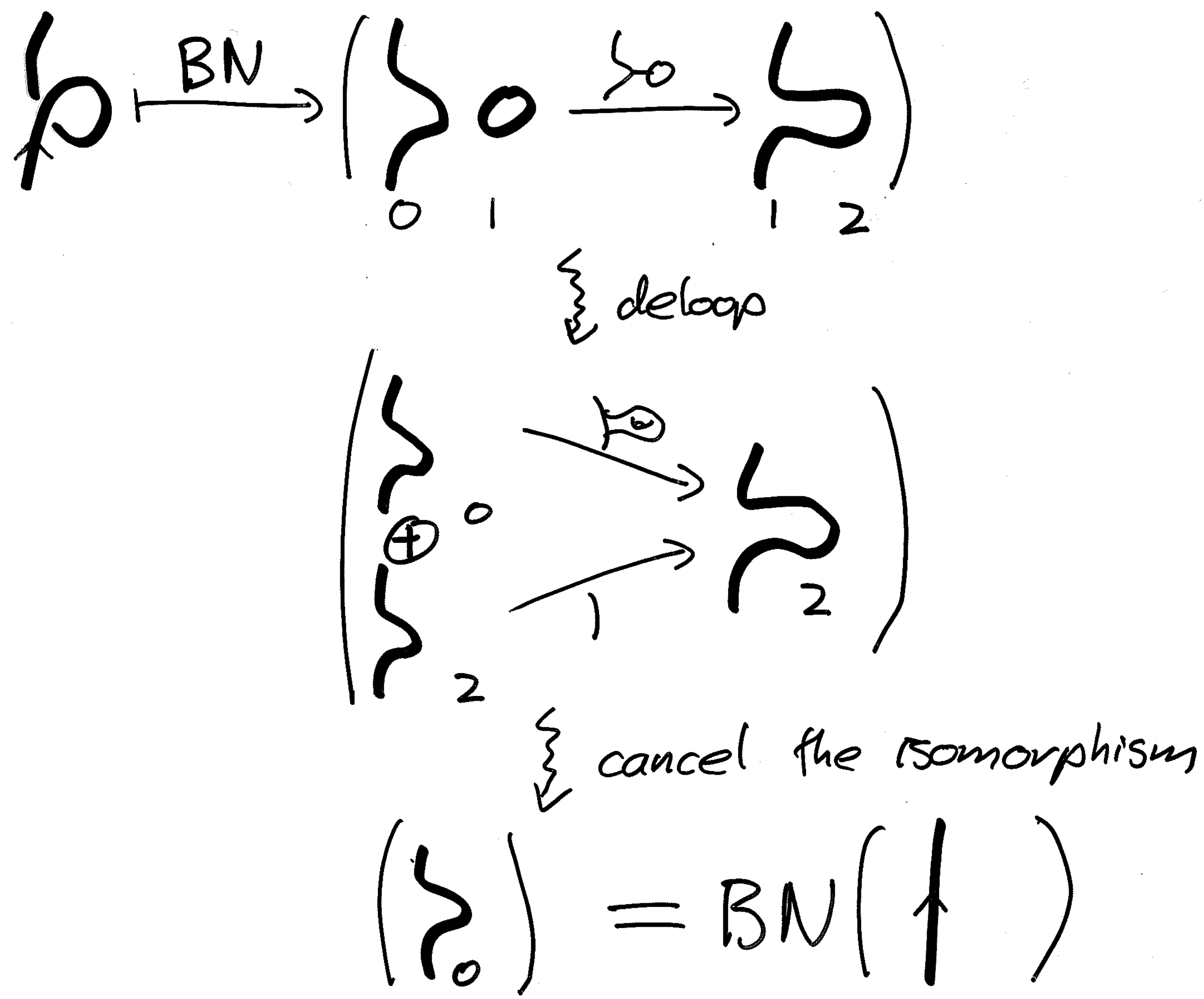
$$[C] \xrightarrow{\begin{pmatrix}0 \\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\psi & 0 \\ 0 & \varepsilon - \gamma\psi^{-1}\delta\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{(0 \ \nu)} [F]$$

This second complex has a contractible direct summand $[b_1] \xrightarrow{\psi} [b_2]$ and so up to homotopy it's just

$$[C] \xrightarrow{(\beta)} [D] \xrightarrow{(\varepsilon - \gamma\psi^{-1}\delta)} [E] \xrightarrow{(\nu)} [F]$$

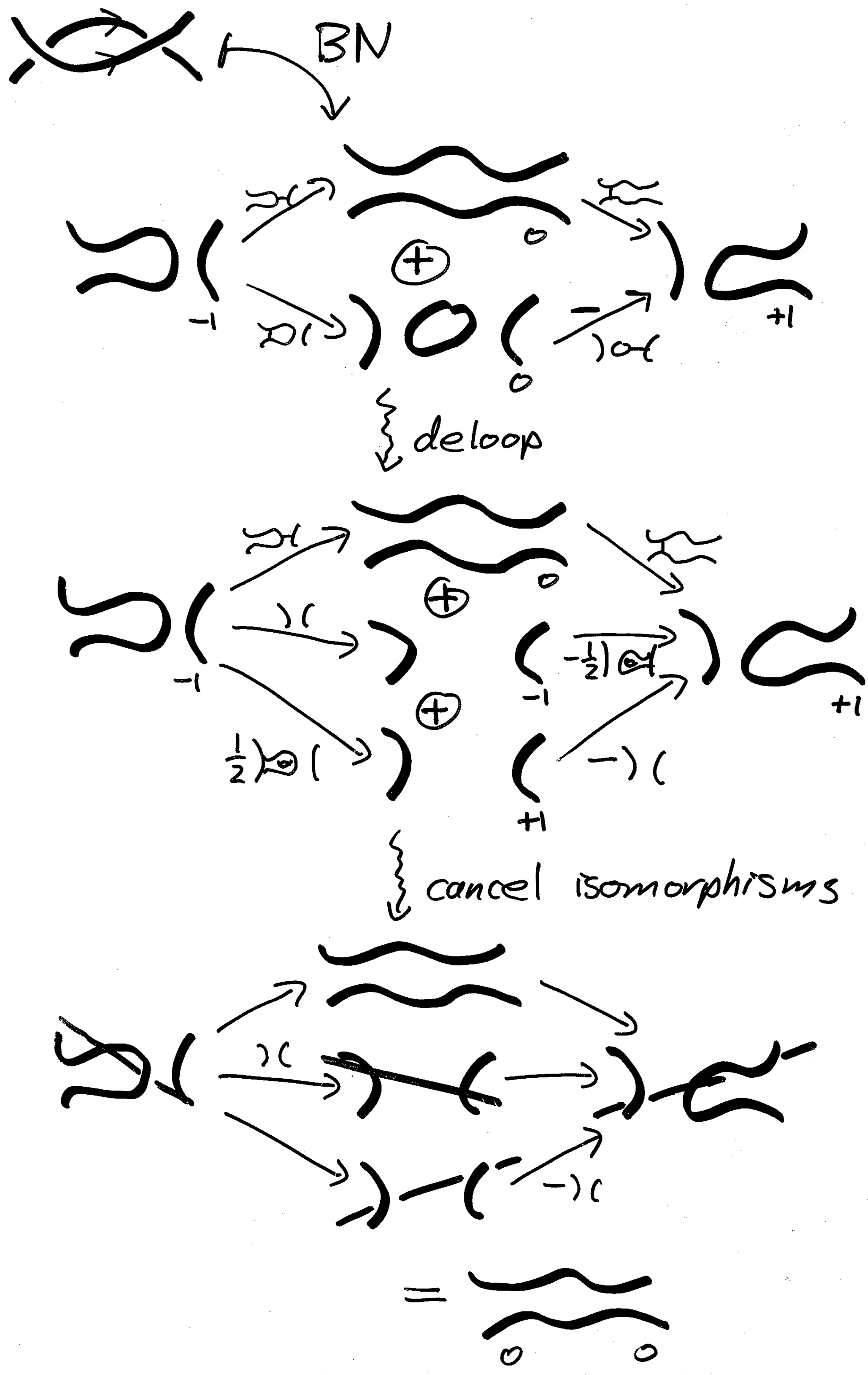
This trick will make our up-to-homotopy complexes almost as manageable as the Jones polynomial itself!

Okay — now we're ready for the 'modern' isotopy invariance proofs.



Neat, huh?

Make sure you understand the morphisms in the complex in the middle line!



BN
−1
+1
0
0
deloop
−1
−1
+1
+1
½
−½
cancel isomorphisms
=
0
0

And something more interesting: