Disoriented and confused: fixing the functoriality of Khovanov homology.

Scott Morrison
scott@math.berkeley.edu
joint work with Kevin Walker

U.C. Berkeley

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What's wrong with Khovanov homology?
It's almost functorial
... but not quite
... and it ought to be!

How do we fix it?
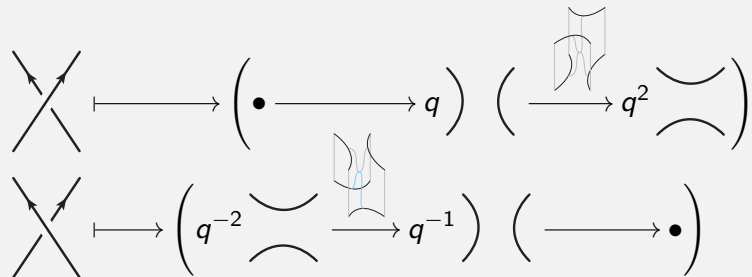Disorientations
Movie moves
Calculations
Confusions

Odds and ends.
Recovering the original theory
Decategorifying
Conclusions

What is Khovanov homology?

Bar-Natan's model of Khovanov homology is a map from tangles to up-to-homotopy complexes of cobordisms.

- On single crossings it is given by

- It is a map of planar algebras: to find the invariant of a tangle composed out of two smaller tangles using a 'planar operation', apply that same planar operation to the two smaller complexes...


Planar composition of complexes

- Given a quadratic tangle, and a pair of complexes associated to the inner discs,
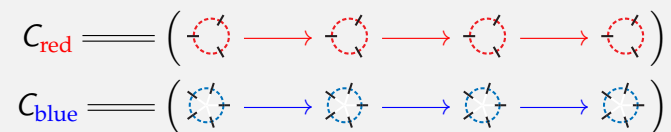
we need to define a new complex associated to the outer disc.

- We'll imitate the usual construction for tensor product of complexes, but use the quadratic tangle to combine objects and morphisms.

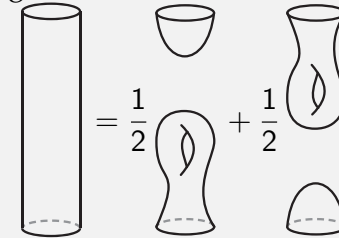Form a double complex then collapse along the anti-diagonal.
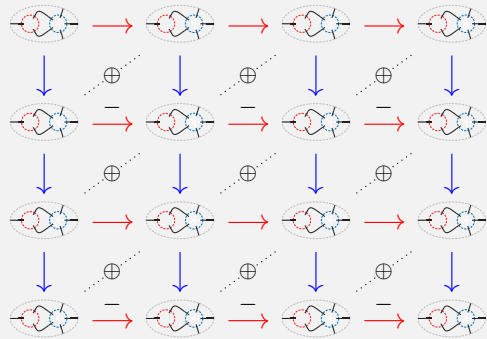
We need to impose some relations on cobordisms in order to make this a tangle invariant.

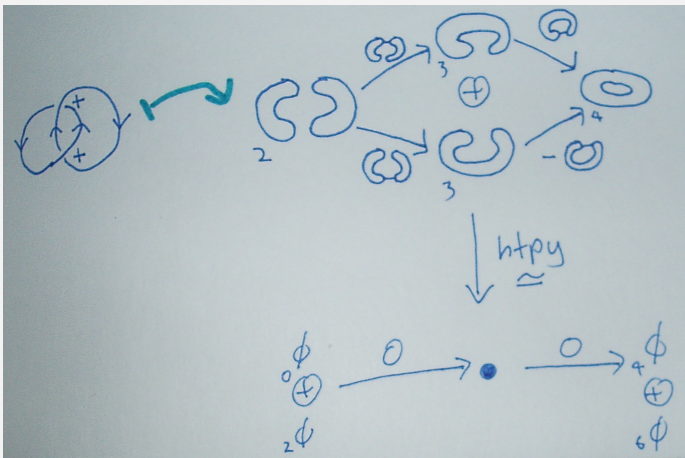- Closed surface relations:
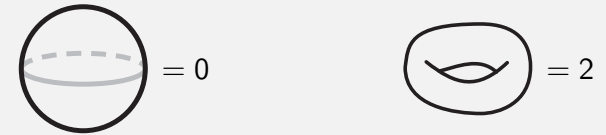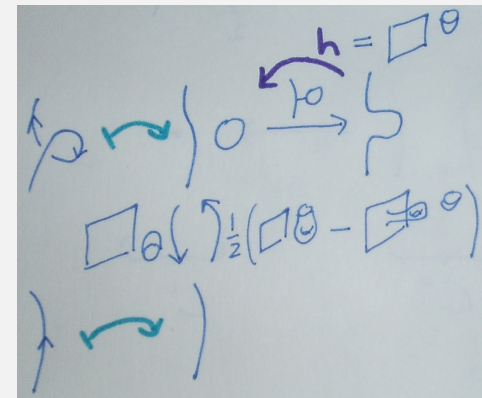

- The "neck cutting" relation:

Example

The hopf link.

Why is it actually an invariant of tangles?

We need to construct homotopy equivalences between the complexes on either side of each Reidemeister move

Example

Khovanov homology is almost functorial

So far, I've described an invariant associated to tangles.
We can try to make Khovanov homology **functorial**,
associating to a cobordism between two tangles some chain
map between the associated complexes.
Link cobordisms can be given presentations as 'movies'. Each
frame of a movie is a tangle diagram. Between each pair of
frames, one of the 'elementary movies' takes place:

- ▶ a Reidemeister move, in either direction
- ▶ the birth or death of a circle
- ▶ a 'saddle' move between two parallel arcs

We need to assign chain maps to each of the elementary
movies.

- ▶ The birth, death and saddle moves are easy; there are
  obvious cobordisms implementing them.
- ▶ To each Reidemeister move, we assign the chain map we
  constructed when showing that the two sides of the
  Reidemeister move were homotopically equivalent
  complexes.

To assign a chain map to an arbitrary link cobordism, we
choose a movie presentation, and compose the chain maps
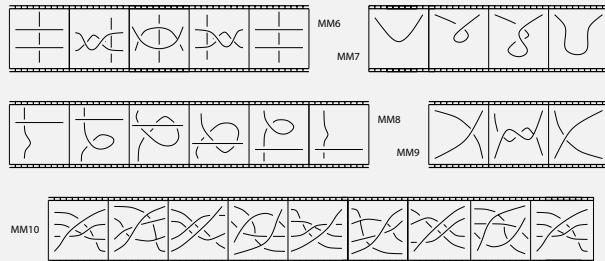associated to each elementary piece. Is this well defined?

... but not quite

Theorem (Carter and Saito)

*Two movies are presentations of the same link cobordism exactly if
they are related by a sequence of 'movie moves'.*

Example (Movie moves 6-10)

Each movie here is equivalent to the 'do nothing' movie.

Thus to check our proposed invariant of link cobordisms is
well defined, we 'only' need to check that we assign the same
chain map (up to homotopy equivalence!) to either side of each
movie move.

Theorem (Bar-Natan, 2004)

The two sides of a movie move agree up to sign!

Theorem (Jacobsson, 2002)

*The signs don't come out right. (You can shuffle them around, but not
make them go away.)*

... and it ought to be!

It would be nice if Khovanov homology really were functorial.

- ► Functors are good!
- ► You could identify generators in the Khovanov homologies calculated from two different presentations of a knot.
- ► Khovanov's construction of a categorification of the coloured Jones polynomial would be easier.
- ► It may help Bar-Natan's 're-embeddability' argument for mutation invariance work.
- ► You can define a nice 4-category, and define Khovanov homology for a link in the boundary of an arbitrary 4-manifold.

How do we fix it?

To fix the sign problems in Khovanov homology, we'll make two modifications to the 'target category' of cobordisms.

disorientations Objects and cobordisms will be 'piecewise oriented', with 'disorientations' where the orientations disagree.

confusions Extra morphisms called 'confusions' fix some defects in the category, and make proofs manageable. They are 'spinorial' objects.

Note that we don't need to modify the original tangles or 4 dimensional cobordisms between them; these are still just oriented. Disorientations only appear on the (abstract) cobordisms appearing as differentials.
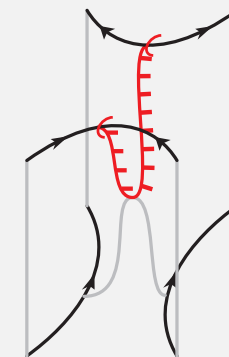
Disorientations

We'll replace the unoriented cobordism category previously used with a category of 'disoriented cobordisms'.

Objects Non-crossing arcs embedded in a disc, each piecewise oriented. Each 'disorientation mark' separating oppositely oriented intervals also has a preferred direction.

Morphisms Surfaces are piecewise oriented, with 'disorientation seams' marking the boundaries between regions with opposite orientations. Each disorientation seam has a 'fringe', indicating a preferred side.

Example

In the oriented regions, we impose the usual cobordism relations. We also need some rules for removing closed disorientation seams, and reconnecting parallel disorientation seams.

Disorientation relations

Fix a parameter ω, such that $\omega^4 = 1$.

- ▶ At $\omega = 1$, we recover the old theory by forgetting all orientation data. (We also recover the sign problems!)
- ▶ At $\omega = i$, we'll have functoriality!

Introduce some relations on disorientations:

$$\bigcirc = \omega$$

$$\bigcirc = \omega^{-1}$$

$$\asymp = \omega^{-1} \,\big)\big(\quad \big)\big($$

- ▸ These are consistent!

Motivation

- ▶ Our original motivation was to find a suitable modification of Bar-Natan's cobordism category which 'decategorified' to the disoriented $\mathfrak{su}_2$ skein theory. (See, for example, Kirby and Melvin.)
- ▶ There we have the relation

reflecting the fact that the standard representation of $\mathfrak{su}_2$ is *anti-symmetrically* self-dual.

Modifying the tangle invariant

Now tangles are mapped to complexes of disoriented cobordisms. It's obvious where to put the seams in, if we want to preserve orientation data away from crossings.

Disorientation marks face to the right, relative to the direction of the crossing.

Theorem (M&W)

This is still an invariant of tangles. We'll see all the homotopy equivalences for Reidemeister moves soon!

Movie moves

Now we need to check 15 movie moves. These come in several types.

Inverses These almost trivial moves insist that the time reverse of a Reidemeister move is also its inverse.

Circular clips These 'circular' clips should be equivalent to the identity. These include the 3 'hard' clips that involve a type III Reidemeister move.

Non-reversible clips These pairs of clips should be equivalent, when read either up or down.
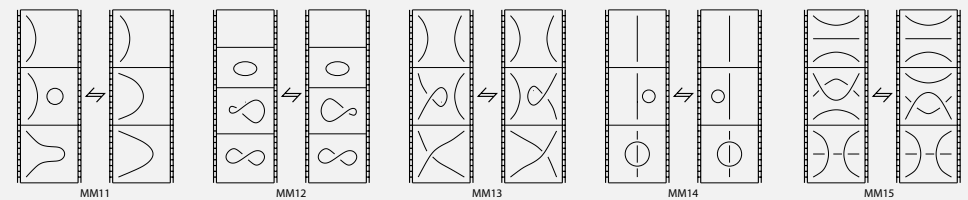
Inverse moves

MM1   MM2   MM3   MM4   MM5
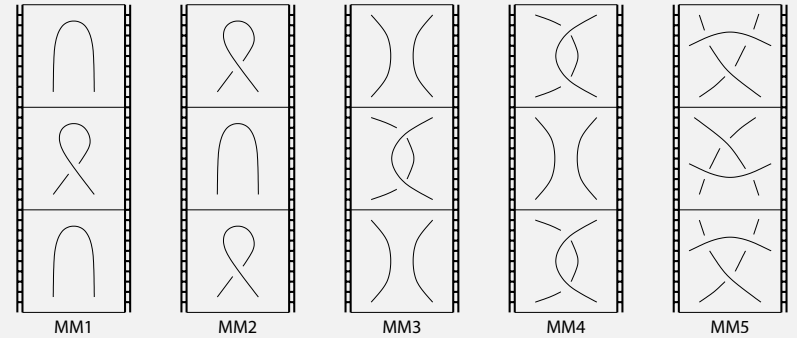
These are boring; we know these are identities, because the two successive steps are a homotopy equivalence and its inverse.

MM6
MM7
MM8
MM9
MM10
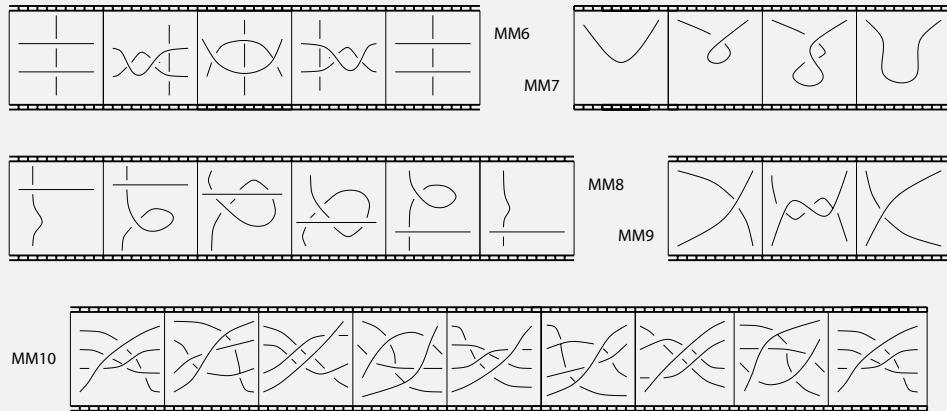
These are 'hard'; moves 6, 8 and 10 involve the third Reidemeister move.

MM11   MM12   MM13   MM14   MM15

▶ Each pair of clips should give the same map, whether read up or down.

▶ These ones don't seem so bad (there are no $R3$ moves), but there are lots of sign problems lurking here!

▶ Often there's a sign problem one way but not the other.

Jacobsson's sign tables

Jacobsson reported sign problems in almost every move!

MM	J♯	±
6	15	+
7	13	+
7 (mirror)	13	-
8	6	-
8 (mirror)	6	+
9	14	-
9 (mirror)	14	+
10	7	+

MM	J♯	↓	↑
11	9	+	+
12	11	-	+
12 (mirror)	11	+	−
13	12	-	+
13 (mirror)	12	+	-
14	8	+	-
15	10	-	+

We can calculate the corresponding table for the disoriented theory, as a function of ω.

- At $\omega = 1$, we recover the tables on the previous slide
  - These include two disagreements with Jacobsson's calculations.
  - David Clark, from UC San Diego, also reports finding these discrepancies, having used Lee homology to simplify calculations.
- At $\omega = i$, all the signs agree.

What about all the orientations!?

- At this point it appears we need to check many orientations of each of these movie moves; up to 16 in the worst case.
- For now, we'll ignore this, and just check the signs for one oriented representative of each movie move.
- Later, the introduction of 'confusions' will deal with the rest.

## Calculations

Now it's time to do the real work! We need to

- calculate explicit chain maps corresponding to Reidemeister moves,
  - These are unique up to a unit.
  - We can easily write these down for the R1 and R2 moves, but R3 will take some work; we used Bar-Natan's cone construction to organise this.
- detect the signs for each movie move, in at least one orientation,
- and explain away all the other orientations!

Twist maps

The twist maps implement the Reidemeister I moves. There are two variations.

Positive twist

Negative twist

The positive twist map is
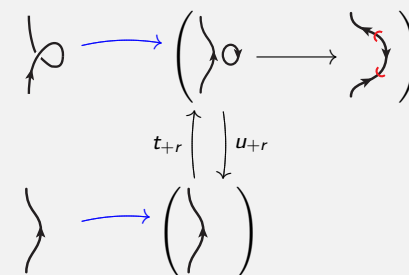
where t_{+r} and u_{+r} are given by

$$t_{+r} = \frac{1}{2}\left(\text{▱} - \omega^{-2}\, \text{▱} \right)$$
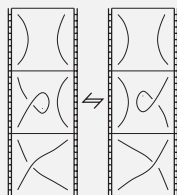
$$u_{+r} = \text{▱}$$

MM13: Our first example!

Each side of MM13 consists of a twist move followed by a morse move. Reading down the left side, we get

$$\frac{1}{2}\left(\text{▱} - \omega^{-2}\, \text{▱} \right)$$

and on the right

$$\frac{1}{2}\left(-\omega^{2}\, \text{▱} + \text{▱} \right)$$

Thus we see the two sides of MM13 differ by a sign of $-\omega^2$!

Bar-Natan's argument

Bar-Natan gave a simple proof that Khovanov homology is well-defined up-to-sign:

▶ Certain tangles are *simple*, in that the automorphism group of the associated complex consists only of multiples of the identity.

▶ Each of movie moves 1-10 starts and ends with a 'simple tangle', and so must be a multiple of the identity.

▶ (Movie moves 11-15 can be done easily by hand.)

In our situation, many small tangles are still 'simple' in this sense, although now there are more units in our coefficient ring: $\pm 1, \pm i$. We'll make use of this often.
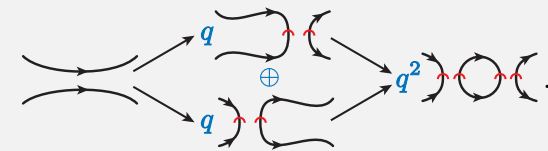
Detecting the sign

We can now easily detect the sign associated to a movie move.

- ▶ Cobordisms between loopless diagrams are all in non-positive grading.
- ▶ Because of the grading shifts in the definition of the complex associated to a tangle diagram, homotopies must be in strictly positive grading.
- ▶ This means not many homotopies are possible. We call a direct summand of an object in a complex *homotopically isolated* if there are no possible homotopies in or out.

Example

The complex associated to ⤫⤫ is

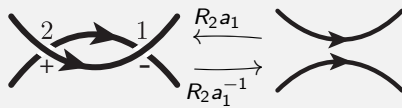Only the first object is isolated. If $f : \text{⤫⤫} \to \text{⤫⤫}$ is homotopic to, say, z times identity, it must act by $z\mathbf{1}$ when restricted to the object ⤝⤞, since there $dh + hd = 0$.

We can often detect the sign associated to a movie move by choosing an isolated summand in the complex, and observing its image under the movie move.
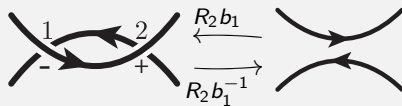
Tuck maps

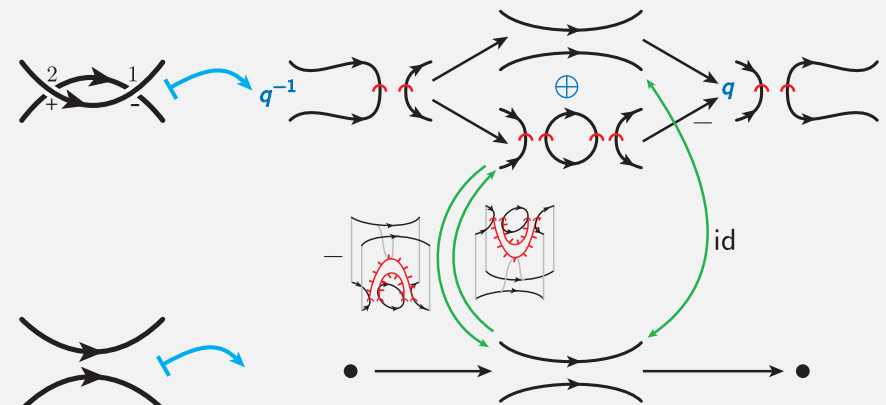We have to distinguish between the braid-like and non-braid-like R2 moves.

R2a

R2b

R2a maps

⤇ Skip other moves…

R2b maps

q^{-1} $\oplus$ q

$-\omega^2$ ω^2

▸ Skip R3 moves...

The cone construction, and R3

Obtaining the R3 map takes some work! We used Bar-Natan's 'categorified Kauffman trick' ▶, but for now will just state a lemma encapsulating what we need to now.

The complexes appearing on either side of the R3 move can be realised as the cone over the morphism resolving the central crossing:

$$-\!\!\!\times\!\!\!- = C\left(-|\!\!\leftarrow\!\!|- \to -\!\!\times\!\!-\right)$$
$$-\!\!\!\times\!\!\!- = C\left(-|\!\!\leftarrow\!\!|- \to -\!\!\times\!\!-\right)$$

Lemma (for MM6, 8 and 10)

The chain equivalence between the two R3 complexes

$$1 \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \star & \star & 0 \\ 0 & \star & \star & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- ▸ *is 'triangular',*
- ▸ *the identity on the 'source' part of the cone,*
- ▸ *and on the 'target' part of the cone kills the objects in which both remaining crossings have been resolved the same way.*

MM10

Look at the initial frame. The associated complex has one object in homological degree 0; the object we obtain from the 'positive smoothing' of each of the four crossings, and it's homotopically isolated:

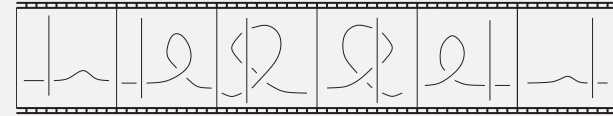

We just need to calculate its image under the movie.

Happily, the cone construction tells us that the 'all positive smoothings' diagram on one side of a Reidemeister III move is taken, with coefficient one, to the 'all positive smoothings' diagram on the other side.
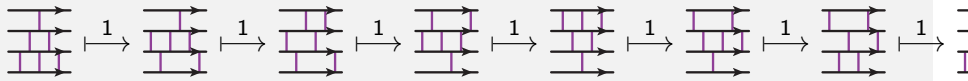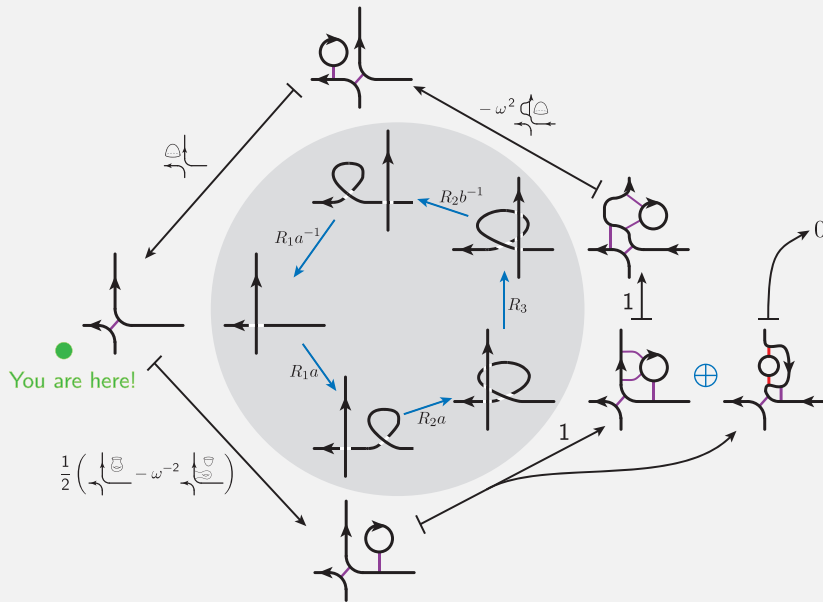

Thus the sign of MM10 is $1^8 = 1$.

MM8 is the second hardest of the movie moves involving R3, but it turns out to barely depend on the details of the the R3 map. We calculate the image of a homotopically isolated element of the initial complex.

Following the maps around the circular movie, starting at the left, we obtain the following composition:

$$\includegraphics{} \circ -\omega^2 \includegraphics{} \circ \frac{1}{2}\left(\includegraphics{} - \omega^{-2}\includegraphics{}\right) = -\omega^{-2}\frac{1}{2}\includegraphics{}$$

$$= -\omega^{-2}\includegraphics{}$$

Again, the disoriented theory gets the sign right!

Theorem (M&W)

All the movie moves come out right in at least one orientation. At $\omega = 1$ we see pretty much the same sign problems as Jacobsson observed, but at $\omega = i$ movie moves really are equivalences.
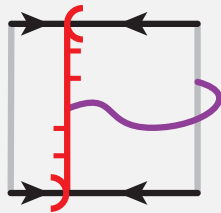
Question

What about all the other orientations?

Confusions

The disoriented cobordism category has some defects.

▶ There are no cobordisms from the empty diagram to a circle with two clockwise disorientation marks, for example.

▶ The categorified Kauffman trick for constructing the $R3$ map doesn't actually work!

▶ If we want to extend the invariant to disoriented tangles, there's no nice equivalence allowing us to slide a disorientation mark past a crossing.

▶ We have to deal with all the orientations of movie moves separately.

Introducing some new morphisms called 'confusions' solves all of these problems.

Definition

Confusions are points on a disorientation seam at which the fringe changes side. They have a spin framing, recorded with a (possibly twisted) ribbon attached to the confusion.

Thus the simplest appearance of a confusion is

This is a map between two disoriented strands, which changes the preferred direction of the disorientation mark.

Ribbon rules

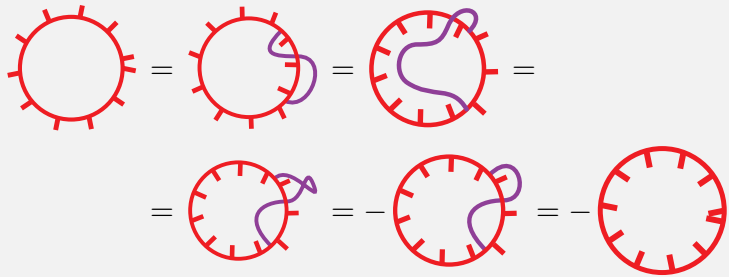We can create and annihilate confusion pairs, according to the following rules.

We think of the relation
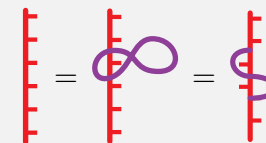
as "creating a pair of confusions".

Example

We can create a confusion pair, slide one around a circle, and then annihilate them.

You can see here that the spinorial nature of confusions is forced upon us, for consistency with the disorientation relations.
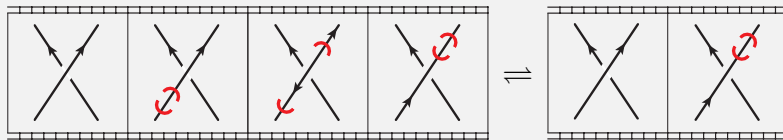
There is now an isomorphism of complexes which allows us to slide a disorientation mark through a crossing. (At the expense of an overall grading and degree shift.)
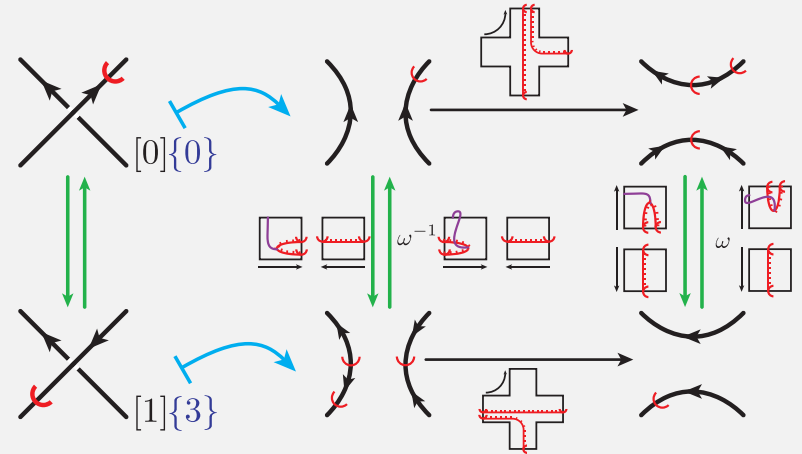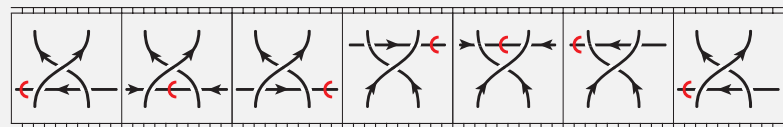

There are various 'new movie moves', involving disorientations

Example

► We can create a pair of disorientations, then slide them past a crossing, or just create them on the other side:
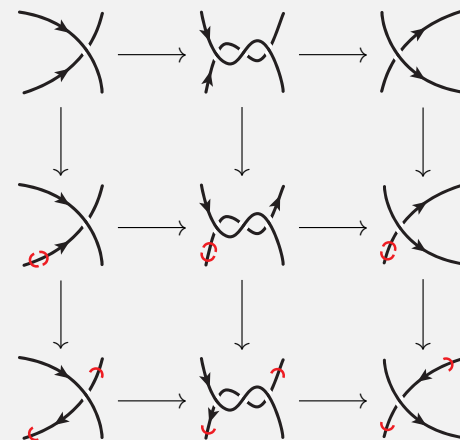

► The disorientation slide isomorphisms commute with Reidemeister moves:

Now it's easy to show all the other orientations of movie moves are equivalent to the ones we've checked.

Example

Recovering the original theory

Theorem

The complex associated to a knot diagram is isomorphic to the complex constructed in the original theory.

Proof.

It's always possible to put orientations on the unoriented diagrams appearing in the old complex for a knot: orient the circles alternately clockwise and counterclockwise, according to their nesting depth.

At the same time, we can remove all disorientation marks on the disoriented diagrams in the new complex, by choosing (non-canonical) isomorphisms.

These two complexes are 'cube complexes', which can only differ by a sprinkling of signs, and so must be isomorphic. $\square$

Decategorifying

▶ We can "decategorify" a category, (not quite the ▸ usual way) obtaining an abelian group:

$$\mathcal{C} \rightsquigarrow \left\langle \mathrm{Obj}(\mathcal{C}) \;\middle|\; \begin{matrix} A = B + C \text{ whenever} \\ A \cong B \oplus C \end{matrix} \right\rangle$$

▶ What do we get? Just as with the unoriented cobordism category, we have the relation

$$= q + q^{-1}$$

due to the neck cutting relation.

▶ To see the relation

appearing in our theory, we need to introduce an additional $\mathbb{Z}/2\mathbb{Z}$ grading on morphisms; the parity of the number of confusions. Now the decategorification consists of $\mathbb{Z}[q, q^{-1}, \sigma]$ modules, with $\sigma^2 = 1$.

▶ A confusion provides an isomorphism between the two diagrams above, but it is in the non-zero $\mathbb{Z}/2\mathbb{Z}$ grading, so in the decategorification we have the equation

.

Conclusions

We've described a new model for Khovanov homology, using a category of *disoriented cobordisms*.

▶ It is properly functorial with respect to link cobordisms.

▶ The complex associated to a knot diagram is isomorphic to the usual Khovanov complex, but not canonically so.

▶ Disoriented cobordisms decategorify to the disoriented $\mathfrak{su}_2$ skein theory.

Appendix: Disorientation relations are consistent

Example

We can create a disorientation seam, split it in two, then annihilate both parts:

$$\omega = \;\bigcirc\; = \omega^{-1}\;\bigcirc\;\bigcirc\; = \omega^{-1}\omega^2$$

Alternatively, we could create a pair, join them, and then annihilate:

$$1 = \;\bigcirc\; = \omega^{-1}\;\bigcirc\; = \omega^{-1}\omega$$

Appendix: Frobenius Reciprocity

We actually prove a slightly stronger result than Bar-Natan's 'simplicity' result.

Theorem
For appropriate tangles P, Q and R, there are natural isomorphisms between the spaces of chain maps

$$\mathsf{Hom}_{[PQ]}\left([R], \cong\right)\mathsf{Hom}_{[P]}\left(\llbracket R\overline{Q}\rrbracket, .\right)$$

Moreover, these isomorphisms are compatible with the natural isomorphisms between the spaces of 4-dimensional cobordisms between the tangles themselves.

Why is that the correct R1 chain map?

We want to observe $t_{+r}u_{+r} - \mathbf{1} = dh + hd$.

▶ Using neck cutting, we obtain

$$t_{+r}u_{+r} - \mathbf{1} = \frac{1}{2}\left(\Box - \omega^2 \right) - \Box = -\frac{1}{2}\left(\Box + \omega^2 \right).$$

▶ The only possible homotopies are $z\;\square$ for some z, and

so $hd = z\;\square$.

▶ Using neck cutting, then removing the resulting bounding disorientation seams, we see that $z = -\omega^{-1}$ works, as long as $\omega^4 = 1$.

Categorified Kauffman trick

We follow through Bar-Natan's proof of R3 invariance, keeping track of the explicit homotopy equivalence being constructed.

Lemma
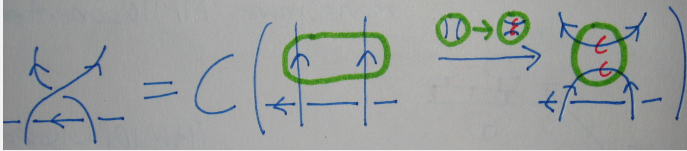*The R2 untuck moves are strong deformation retracts.*

Lemma
If $f : A^\bullet \to B^\bullet$ is a chain map, and $r : B^\bullet \to C^\bullet$ is a strong deformation retract, then $C(rf) \simeq C(f)$.

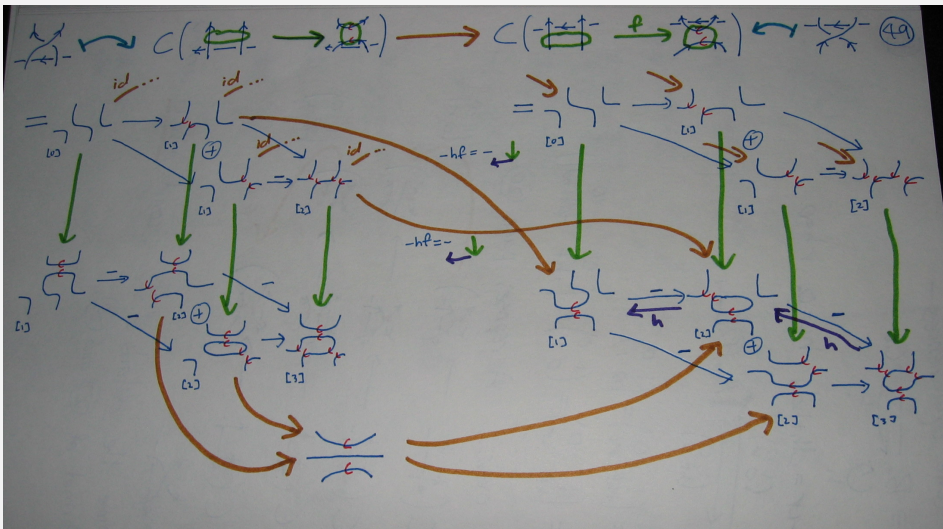## Lemma ('Categorified Kauffman trick')

Each side of the R3 move can be realised as a cone over the morphism switching between two smoothings of the 'central' crossing.

We can then compose this morphism with the 'untuck' move, a strong deformation retract. Doing this to either side of the R3 move, we obtain the same cone!

Putting this together, we have

$$\simeq \quad \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$$

$$= \quad \text{same morphism!}$$

$$\simeq \quad \begin{pmatrix} 1 & 0 \\ -hf & i \end{pmatrix}$$

Appendix: Deformation retracts

Definition
A chain map $r : B^\bullet \to C^\bullet$ is a *strong deformation retract* if there is a chain map $i : C^\bullet \to B^\bullet$ and a homotopy $h : B^\bullet \to B^{\bullet-1}$ such that $ri = \mathbf{1}_C$, $ir - \mathbf{1}_B = dh + hd$, and $rh = hi = 0$.

Lemma
If $f : A^\bullet \to B^\bullet$ is a chain map, and $r : B^\bullet \to C^\bullet$ is a strong deformation retract, then the cone $C(rf)$ is homotopic to the cone $C(f)$, via

$$C(f)^\bullet = A^{\bullet+1} \oplus B^\bullet \underset{\begin{pmatrix} 1 & 0 \\ -hf & i \end{pmatrix}}{\overset{\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}{\rightleftarrows}} A^{\bullet+1} \oplus C^\bullet = C(rf)^\bullet$$